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Beyond complex Langevin equations: positive representation of a class of complex measures

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I. The Langevin method - real and complex cases

Real case

$$S(x) \quad \xrightarrow{\dot{x}(\tau)=-\partial_x S+\eta(\tau)} \quad x(\tau) \quad \longrightarrow \quad P(x,\tau) \quad \xrightarrow{\tau\to\infty} \quad P(x)=e^{-S(x)}$$

$$\int f(x)e^{-S(x)}dx \,/\!\int e^{-S(x)}dx = \int f(x)P(x)dx \,/\!\int P(x)dx \,.$$

Complex case, e.g. $\rho(x) \equiv e^{-S(x)} = e^{-\sigma x^2/2}, \sigma \in \mathcal{C}$,

G. Parisi, 1983
J.Klauder, 1984

$$S(x) \quad \xrightarrow{\dot{z}(\tau)=-\partial_z S+\eta(\tau)} \quad z(\tau) \quad \longrightarrow \quad P(x,y,\tau)$$

But $P(x,y,\tau)$ **does not** $\xrightarrow{\tau\to\infty} \quad e^{-S}$

Nevertheless in some cases $P(x,y,\infty)$ **exists, and indeed**

$$\frac{\int f(x)e^{-S(x)}dx}{\int e^{-S(x)}dx} = \frac{\int\int f(x+iy)P(x,y,\infty)dxdy}{\int\int P(x,y,\infty)dxdy}. \qquad (1)$$

In general, however

- $P(x, y, \tau)$ not known
- not connected with S

Moreover

- $z(\tau) \to \infty$, i.e. diverges
- $z(\tau) \to$ wrong answer

G. Aarts et al. (2 - 6), 2005 - present

II. Avoiding the trouble (Beyond CL ...)

- Construct $\rho(x) \leftrightarrow P(x,y)$ without any reference to the stochastic process.

The only requirement (matching conditions)

$$\langle f(x)\rangle_{\rho(x)} = \langle f(x+iy)\rangle_{P(x,y)}$$

Done for the non-compact case JW, 2016

1 DOF (gaussian and quartic)

∞ DOF - quantum mechanical path integrals (gaussian) in the Minkowski time

earlier works: Weingarten 2002, Salcedo 2007

see also: Ruba, Wyrzykowski talks

III. This work: periodic weights

Matching equations (ME) for Fourier components

$$\int_{-\pi}^{\pi} e^{-inx}\rho(x)dx = \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} e^{-in(x+iy)}P(x,y)dxdy.$$

That is

$$a_n = \int_{-\infty}^{\infty} e^{ny}P_n(y)dy$$

where

$$\rho(x) = \Sigma_n a_n e^{inx} \quad and \quad P(x,y) = \Sigma_n P_n(y)e^{inx}$$

$\Longrightarrow$ freedom in choosing y - dependence

$\Longrightarrow$ Ansatz

$$P_n(y) = \lambda_n \delta(y - y_s) + \mu_n \delta(y + y_s)$$

then ME imply

$$\lambda_n = \frac{\mathrm{e}^{\,ny_s} a_n - \mathrm{e}^{\,-ny_s} a^*_{-n}}{2\sinh(2ny_s)}$$
$$\mu_n = \frac{\mathrm{e}^{\,ny_s} a^*_{-n} - \mathrm{e}^{\,-ny_s} a_n}{2\sinh(2ny_s)}\,.$$

- **P(x,y) is real from the construction.**
- **Positivity achieved by the dominance of the lowest mode for large y_s**
- **Other Ansätze possible, e.g. gaussians.**
- **Many DOFs: $x, n \longrightarrow \vec{x}, \vec{n}$ etc.**

Example 1: one DOF – Polyakov line (a prototype)

$$\rho_P(x) = \frac{1}{I_1(\beta)} e^{ix} \exp\left(\beta \cos(x)\right) = \Sigma_n \frac{\overbrace{I_{n-1}(\beta)}^{a_n}}{I_1(\beta)} e^{inx},$$

$$P_P(x,y) = P^+(x)\delta(y - y_s) + P^-(x)\delta(y + y_s)$$

$$P^{\pm}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \cos(nx) C_n^{\pm}.$$

$$C_n^{(\sigma)} = \frac{e^{n\sigma y_s} a_n - e^{-n\sigma y_s} a_{-n}}{\sinh(2n\sigma y_s)}, \qquad 0 < n.$$

$< \sin^2(x) >$.

$$\int_{-\pi}^{\pi} \frac{dx}{2\pi} \sin^2(x) \rho_P(x) = \frac{1}{4I_1(\beta)} \left(I_1(\beta) - I_3(\beta)\right)$$

$$\int_{-\pi}^{\pi} \frac{dx}{2\pi} dy \sin^2(x + iy) P(x,y) = \int \frac{dx}{2\pi} \sin^2(x + iy_s) \left\{\frac{1}{2} + \sum_{n=1}^{\infty} \cos(nx) C_n^+\right\}$$
$$+ \int \frac{dx}{2\pi} \sin^2(x - iy_s) \left\{\frac{1}{2} + \sum_{n=1}^{\infty} \cos(nx) C_n^-\right\} =$$
$$\frac{1}{2} - \frac{1}{4} \cosh(2y_s) \left(C_2^+ + C_2^-\right) = \frac{1}{4I_1(\beta)} \left(I_1(\beta) - I_3(\beta)\right),$$

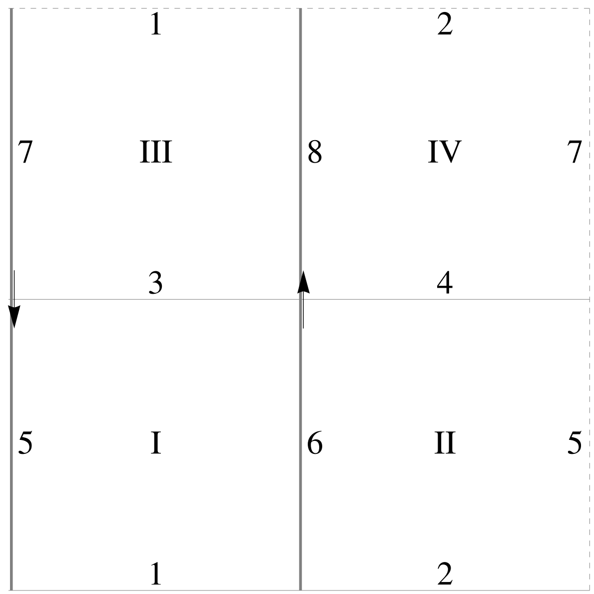
Example 2: four DOF with gauge invariance – Wilson loop

$$\rho_P(x_1, x_2, x_3, x_4) = e^{i(x_1+x_2-x_3-x_4)} \exp\left(\beta \cos(x_1 + x_2 - x_3 - x_4)\right) = \sum_{\vec{n}} a_{\vec{n}} e^{i\vec{n}\cdot\vec{x}}$$

$$a_{\vec{n}} = \sum_m I_{m-1} \delta_{m,n_1} \delta_{m,n_2} \delta_{m,-n_3} \delta_{m,-n_4} \tag{2}$$

$$P^\sigma(\vec{x}) = \frac{I_1}{2} + \sum_{m,m\neq 0} \frac{e^{4my_s} I_{m-\sigma}}{\sinh(8my_s)} \cos(m(x_1 + x_2 - x_3 - x_4). \tag{3}$$

IV. Tiny 2D abelian lattice



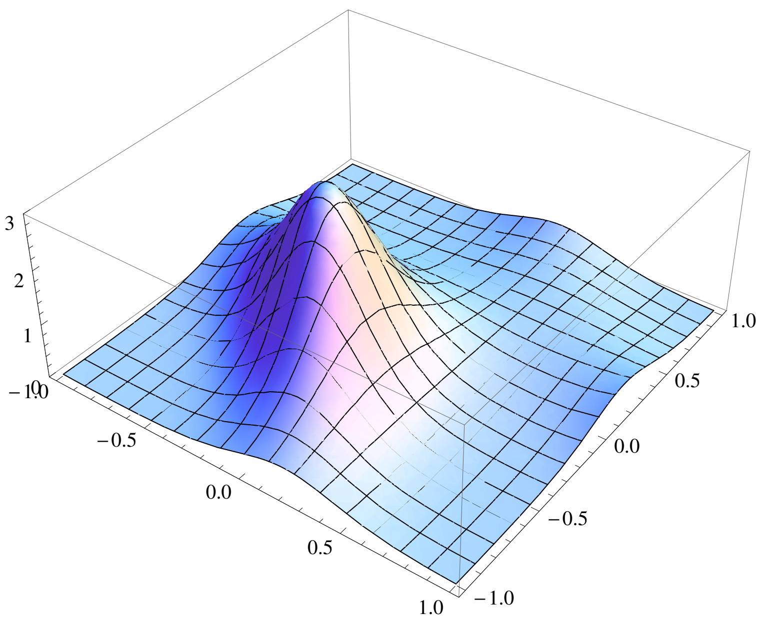
$$\begin{aligned}\rho(\vec{\theta}) = \; & B(3+8-1-7)B(4+7-2-8) \\ & B(1+6-3-5)B(2+5-4-6) \\ & U(-5-7)U(6+8)\end{aligned}$$

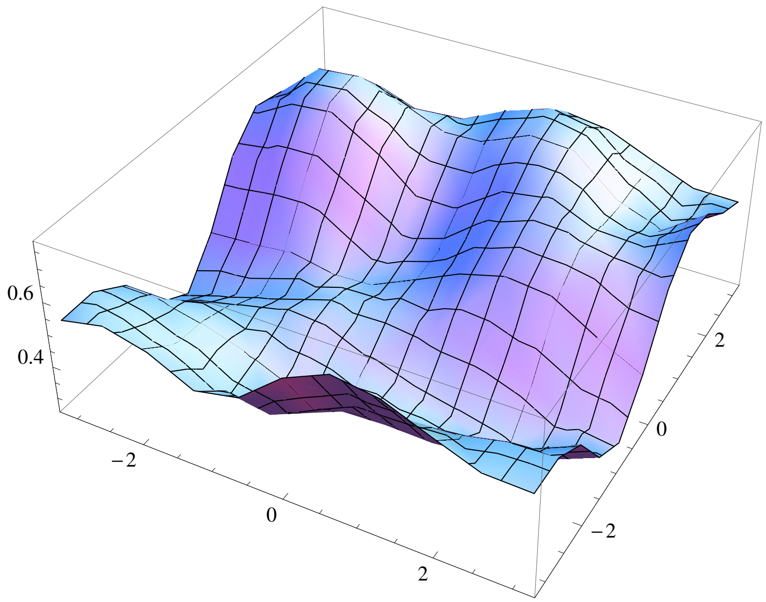
$$\theta_i \to i, \quad B(\phi) = \exp(\beta\cos(\phi)), \quad U(\phi) = \exp(i\phi),$$

Or in plaquette variables $(\phi_I, \phi_{II}, \phi_{III}) \to (\phi_1, \phi_2, \phi_3)$

$$\rho(\vec{\phi}) = B(\phi_1)B(\phi_2)B(\phi_3)B(\phi_1+\phi_2+\phi_3)U(\phi_1)U(\phi_3)$$

$$a_{\vec{n}} = \sum_m I_m I_{m-n_2} I_{m-n_1+1} I_{m-n_3+1}$$

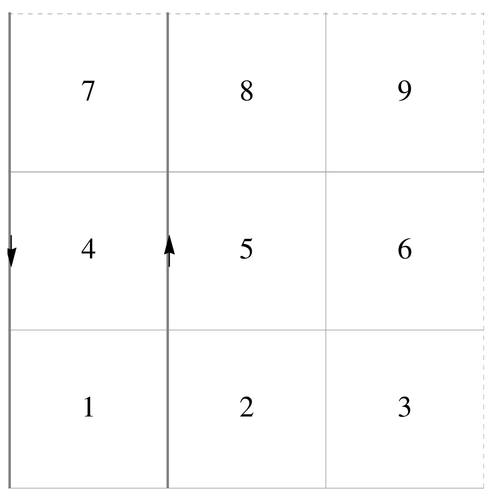

3
2
1
0
−1.0
−0.5
0.0
0.5
1.0
1.0
0.5
0.0
−0.5
−1.0

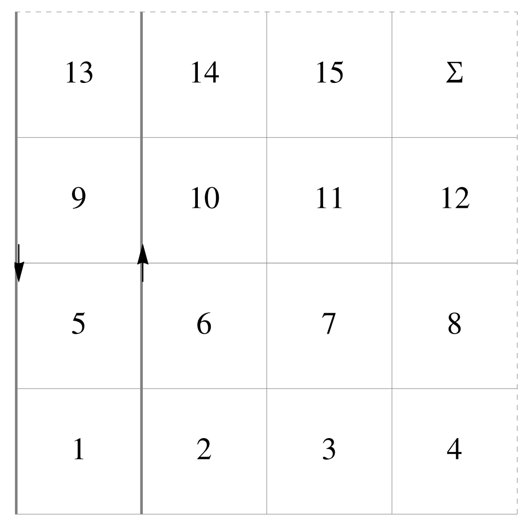

0.6
0.4
−2
0
2
2
0
−2

V. Larger abelian lattices: locality vs separability

$$\rho(1...8) = B(1)B(2)...B(8)B(1+2+3+...+8)U(1)U(4)U(7)$$

$$a_{\vec{n}} = \sum_m (I_{m-n_7+1} I_{m-n_8} I_m)\,(I_{m-n_4+1} I_{m-n_5} I_{m-n_6})\,(I_{m-n_1+1} I_{m-n_2} I_{m-n_3})$$





Arbitrary lattices

$$\rho(\vec{\phi}) = \left(\prod_{exterior\ of\ W} B(\phi_{ext}) \right) \left(\prod_{interior\ of\ W} B(\phi_{int}) U(\phi_{int}) \right)$$

For separable $\rho(\vec{x})$, $P(\vec{x}, \vec{y})$ will be also separable $\Rightarrow$ local algorithms.

In general positive densities will not be local.

VI. Summary

- Construct $\rho(x) \leftrightarrow P(x, y)$ without stochastic process
- Compact $-\pi < x < \pi$, positivity $\Leftarrow$ dominance of the lowest mode
- Applications - positive representations for Polyakov loops
- 1 DOF, 4 DOF with gauge inv., tiny U(1) lattices, arbitrary 2D lattices
  $\Rightarrow$ very simple systems, but many variables
- Local P(x,y) only for above simple examples
- Use a freedom in constructing P(x,y) to reduce non-locality ??