

# Two-dimensional antiferromagnetic Ising model as a toy model for usual $\theta$ physics

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<sup>1</sup> [arXiv:1612.08598](https://arxiv.org/abs/1612.08598)

## 1 Motivation

## 2 Method

- Cumulant expansion
- Computing the observables

## 3 Results

- Comparing with analytic results
- Staggered magnetization and critical line

## 4 Summary and conclusions

# A brief introduction

- The Sign Problem: a major difficulty
  - ▶  $\theta$ -QCD
  - ▶  $\mu > 0$  QCD
  - ▶ Chains of quantum spins with AF interactions  $\leftrightarrow O(3)_\theta NL\sigma M$
  - ▶ Hubbard model
- Usual importance sampling techniques fail, other methods needed
- It is convenient to have a set of benchmark calculations

# Motivation

- Analytical solution in few systems
  - ▶ 1d Ising model within an imaginary magnetic field
  - ▶ 2d compact  $U(1)$ -model with a topological term
  - ▶ 2d Ising model with  $H/KT = i\pi/2$
- Reformulation of variables, overcoming the sign problem
  - ▶  $U(1) / Z_3$  gauge-Higgs model,  $\mu > 0^2$

**Our motivation:** to provide a benchmark calculation for a system both without an analytical solution or a known reformulation, the **2d AF Ising model within a pure imaginary field**.

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<sup>2</sup>[Mercado, Gattringer and Schmidt, 2013]

# What we have done

A study of the 2d AF Ising model within a pure imaginary field, which partition function is

$$\mathcal{Z} = \sum_{\{s_i\}} \exp \left( i\theta \frac{1}{2} \sum_i s_i + F \sum_{\langle ij \rangle} s_i s_j \right). \quad (1)$$

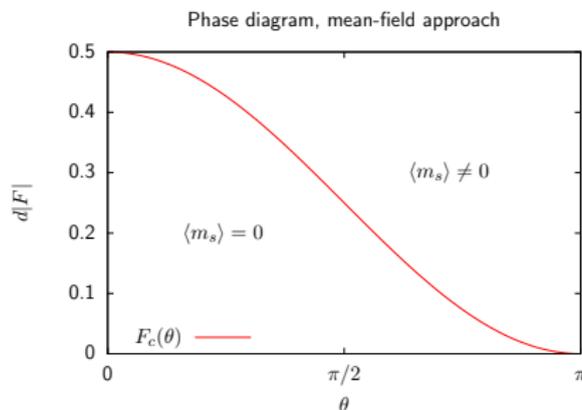
- Exact cumulant expansion to 8th order
- Analytic computation of  $\mathcal{Z}$  and other observables for a large number of DOFs
- Analyze the phase structure of the model

# What we expect

A mean-field study<sup>3</sup> of the 2d AF Ising model within a pure imaginary field was performed. The resulting partition function was

$$\mathcal{Z}_{MF}(F, \theta) = \sum_{\{s_i\}} \exp \left( i\theta \frac{M_1 + M_2}{2} - \frac{Fd}{N} (M_1 - M_2)^2 \right), \quad (2)$$

Applying saddle-point techniques, the critical line is  $dF_c = \cos^2(\theta_c/2)/2$ .



<sup>3</sup>[Azcoiti et al, NPB 851, 2011]

# Reformulating $\mathcal{Z}$

- First, we divide the lattice into two sublattices  $\Omega_1, \Omega_2$ , and define the magnetization densities  $m_1$  and  $m_2$  as

$$m_j \equiv \frac{M_j}{N/2} \equiv \frac{\sum_{i \in \Omega_j} s_i}{N/2} \quad j = 1, 2, \quad (3)$$

and the density of states  $g(m_1, m_2)$  as the number of microstates with magnetization densities  $m_1, m_2$ .

- Then, we reformulate  $\mathcal{Z}$  as

$$\mathcal{Z} = \sum_{m_1, m_2} g(m_1, m_2) \left\langle \exp \left( i \frac{\theta}{2} \sum_i s_i + F \sum_{\langle ij \rangle} s_i s_j \right) \right\rangle_{m_1, m_2}. \quad (4)$$

# Cumulant expansion

$$\mathcal{Z} = \sum_{m_1, m_2} g(m_1, m_2) e^{\frac{1}{4} N i \theta (m_1 + m_2)} \left\langle \exp \left( F \sum_{\langle ij \rangle} s_i s_j \right) \right\rangle_{m_1, m_2}. \quad (5)$$

- We recall the definition of the cumulants  $\kappa_n$ ,

$$\langle e^{tX} \rangle \equiv \exp \left( \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} \right), \quad (6)$$

- The  $n$ -th cumulant is an  $n$ -th degree polynomial in the first  $n$  non-central moments of  $X$ ,

$$\kappa_n = \mu'_n - \sum_{m=1}^{n-1} \binom{n-1}{m-1} \kappa_m \mu'_{n-m}, \quad \mu'_n \equiv \langle X^n \rangle. \quad (7)$$

## Final $\mathcal{Z}$ and cumulant expressions

$$\mathcal{Z} = \sum_{m_1, m_2} g(m_1, m_2) \exp \left( \frac{1}{4} N i \theta (m_1 + m_2) + \sum_{n=1}^{\infty} \kappa_n(m_1, m_2) \frac{F^n}{n!} \right) \quad (8)$$

$$\kappa_1 = 2N m_1 m_2,$$

$$\kappa_2 = 2N(m_1^2 - 1)(m_2^2 - 1),$$

$$\kappa_3 = 8N m_1 m_2 (m_1^2 - 1)(m_2^2 - 1),$$

$$\begin{aligned} \kappa_4 &= 4N(21m_1^2 m_2^2 - 9(m_1^2 + m_2^2) + 5) \\ &\quad \times (m_1^2 - 1)(m_2^2 - 1), \\ &(\dots) \end{aligned} \quad (9)$$

# Computing the observables

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \sum_{m_1, m_2} \mathcal{O}(m_1, m_2) g(m_1, m_2) \times \exp \left\{ i\theta \frac{M_1 + M_2}{2} + \sum_{n=1}^{n_{max}} \frac{F^n}{n!} \kappa_n(m_1, m_2) \right\} \quad (10)$$

- The complex valued exponentials in (8) and (10) give rise to a SSP.
- Multiprecision libraries (GMP, GNU MPFR, GNU MPC, gmpy2) are used to sum over  $m_1, m_2$ .
- $\mathcal{Z} > 0$ ?<sup>4</sup>

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<sup>4</sup>[Seung-Yeon Kim, 2004]

# Observables of interest

- We have computed several observables, including the density of free energy  $\phi$ , the density of internal energy  $e$ , the specific heat  $c_v$  and both usual and staggered magnetizations  $\langle m \rangle, \langle m_s \rangle$ .

$$\phi \equiv -\frac{1}{NF} \log \mathcal{Z}, \quad (11)$$

$$e \equiv -\frac{1}{2N} \frac{d \log \mathcal{Z}}{dF}, \quad c_v \equiv -F^2 \frac{d}{dF} e, \quad (12)$$

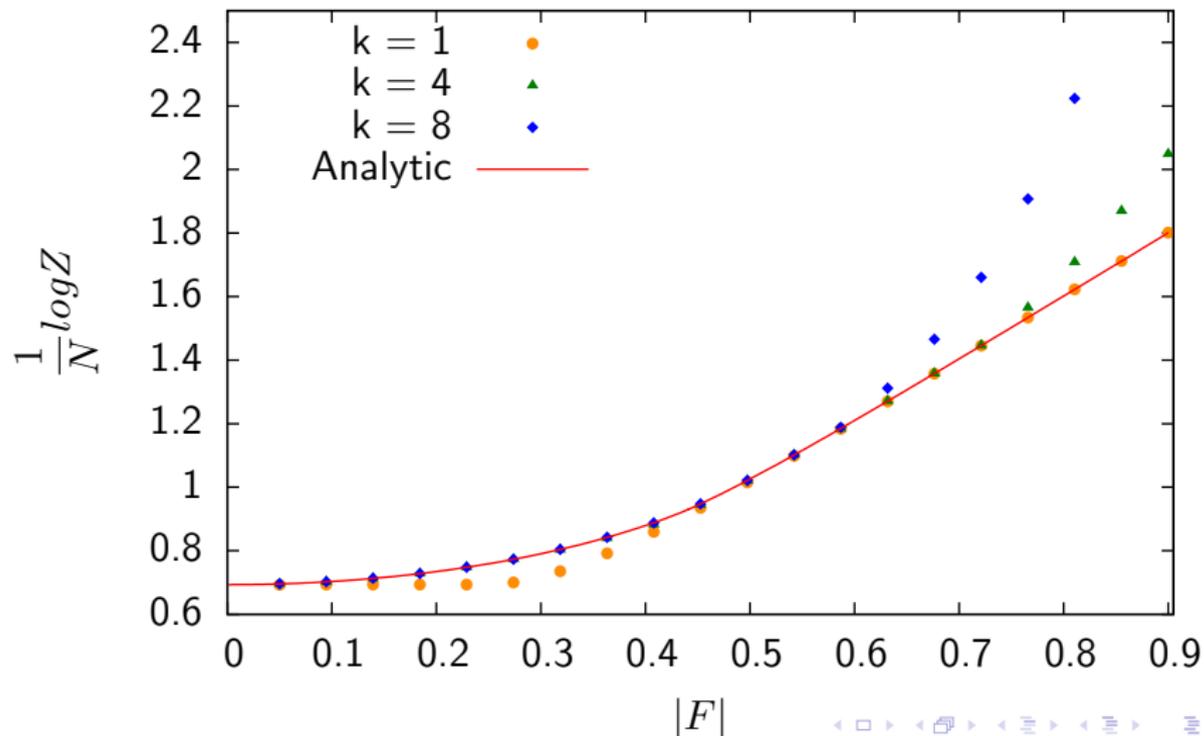
$$\langle m \rangle \equiv \left\langle \frac{m_1 + m_2}{2} \right\rangle, \quad \langle m_s \rangle \equiv \left\langle \frac{m_1 - m_2}{2} \right\rangle. \quad (13)$$

- For symmetry reasons, the *natural* order parameter  $\langle m_s \rangle$  is always zero, and we compute instead the expected value of its square,  $\langle m_s^2 \rangle$ .

- We show results keeping one, four and eight cumulants
- For clarity we plot just the largest N calculated
- At  $\theta = 0$  and  $\theta = \pi$  we have analytical solutions

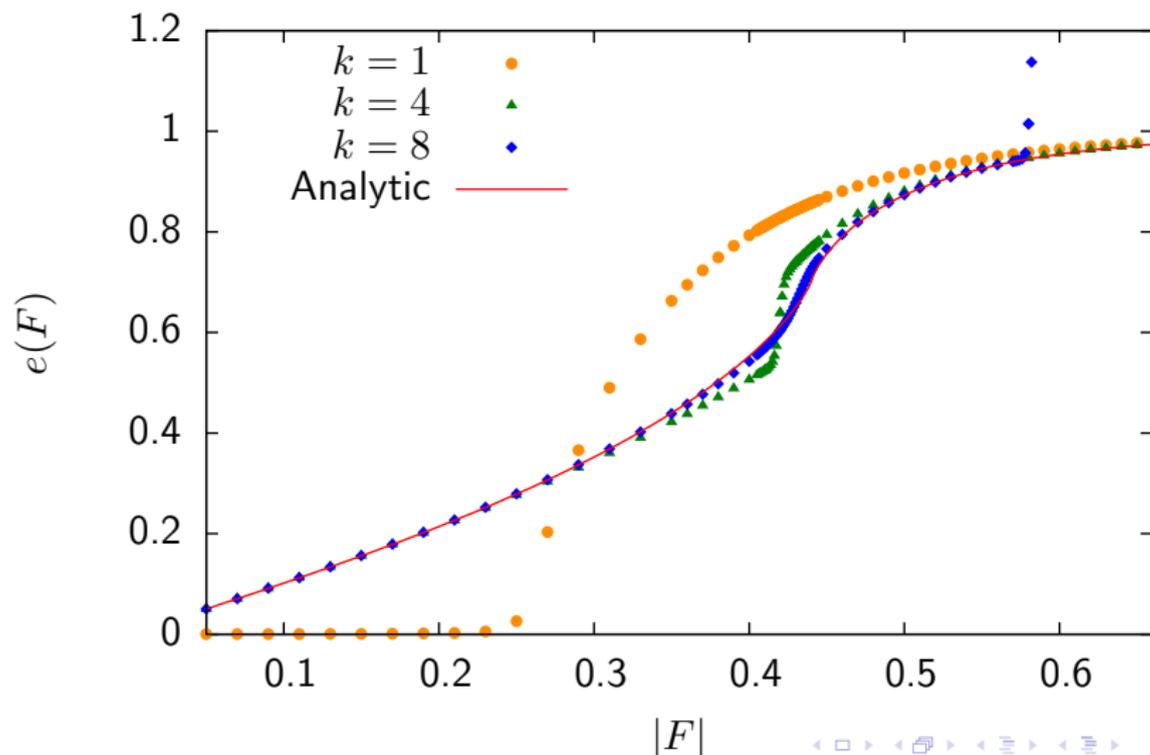
# Results

Free energy at  $\theta = 0$ ,  $N = 2000$



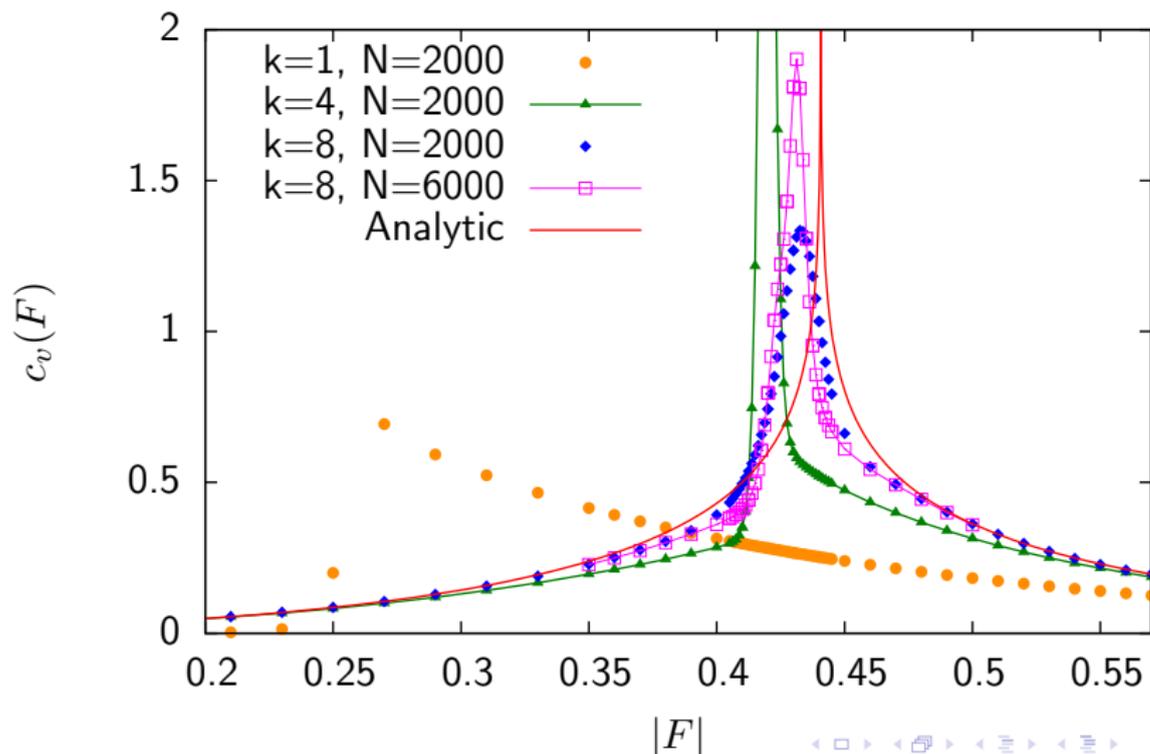
# Results

$e(F)$  curves for  $\theta = 0, N = 2000$



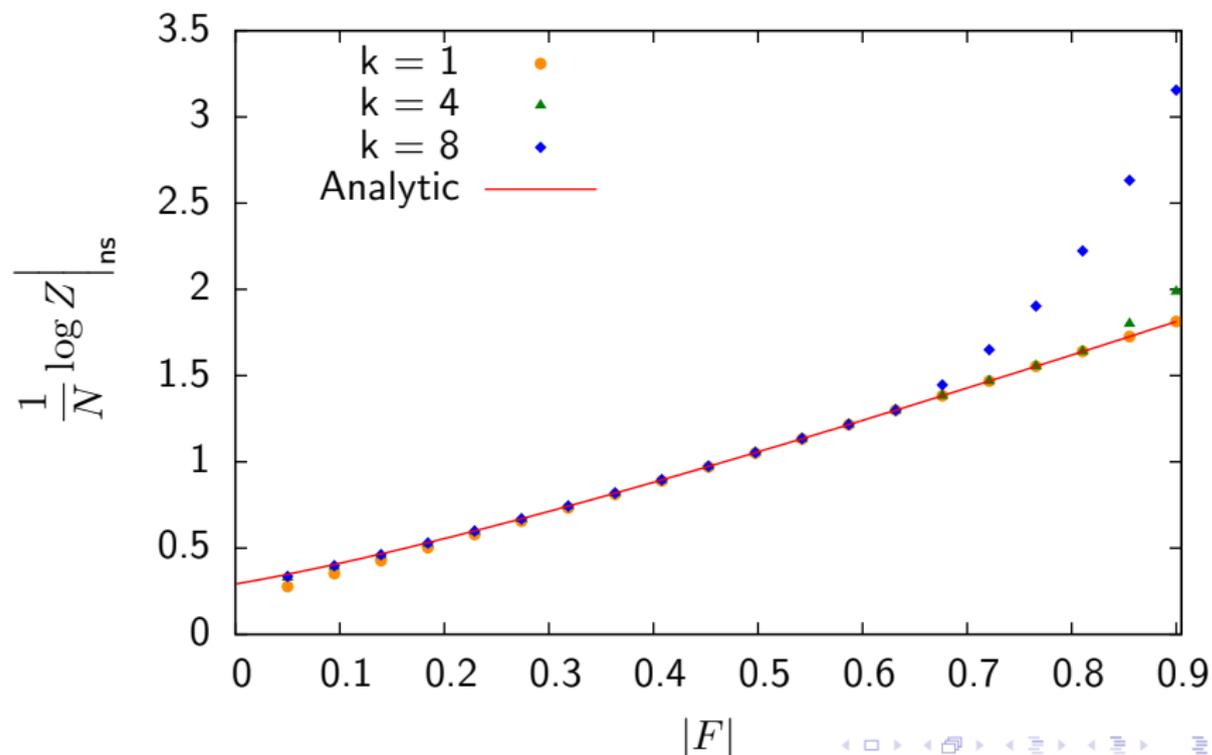
# Results

$c_v(F)$  curves for  $\theta = 0$



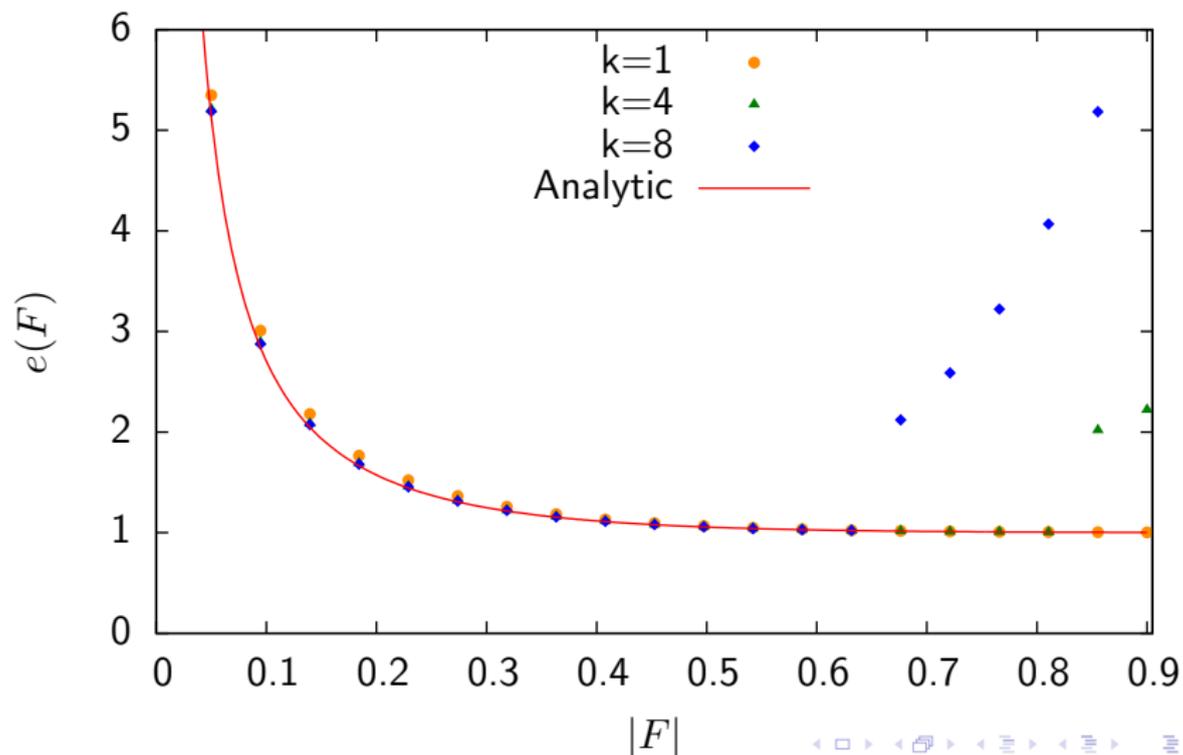
# Results

Free energy at  $\theta = \pi$ ,  $N = 2000$



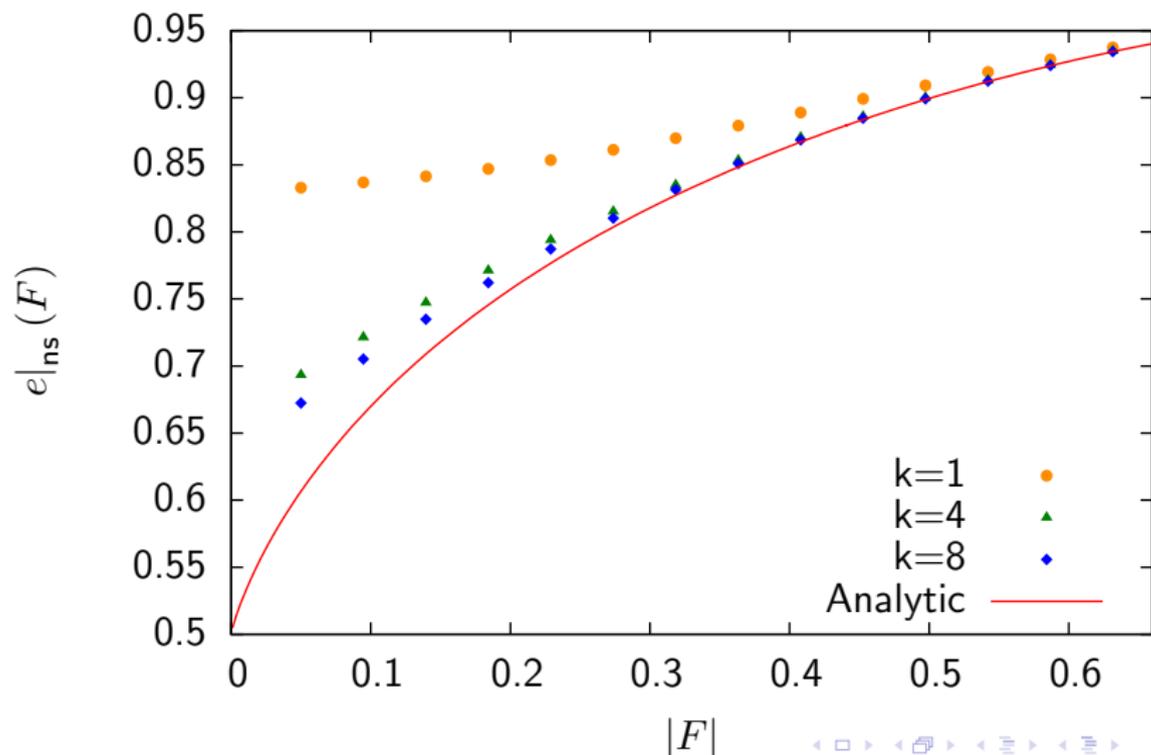
# Results

$e(F)$  curves for  $\theta = \pi, N = 2000$



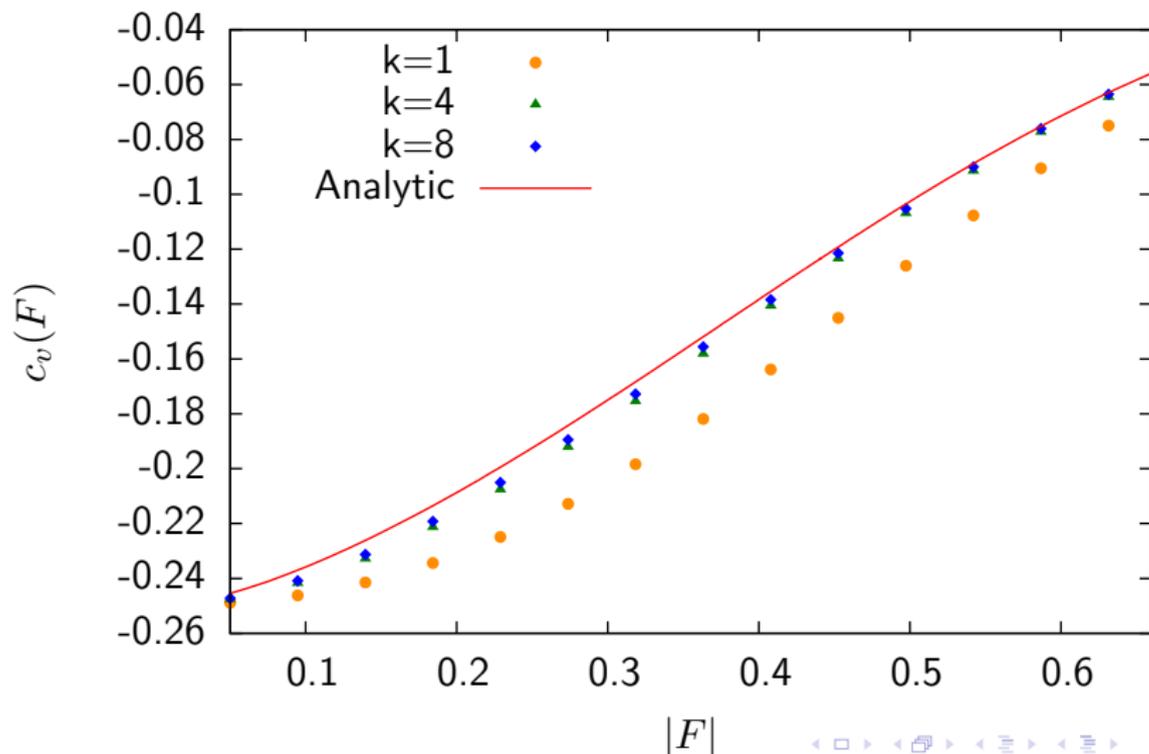
# Results

$e|_{\text{ns}}(F)$  curves for  $\theta = \pi, N = 2000$



# Results

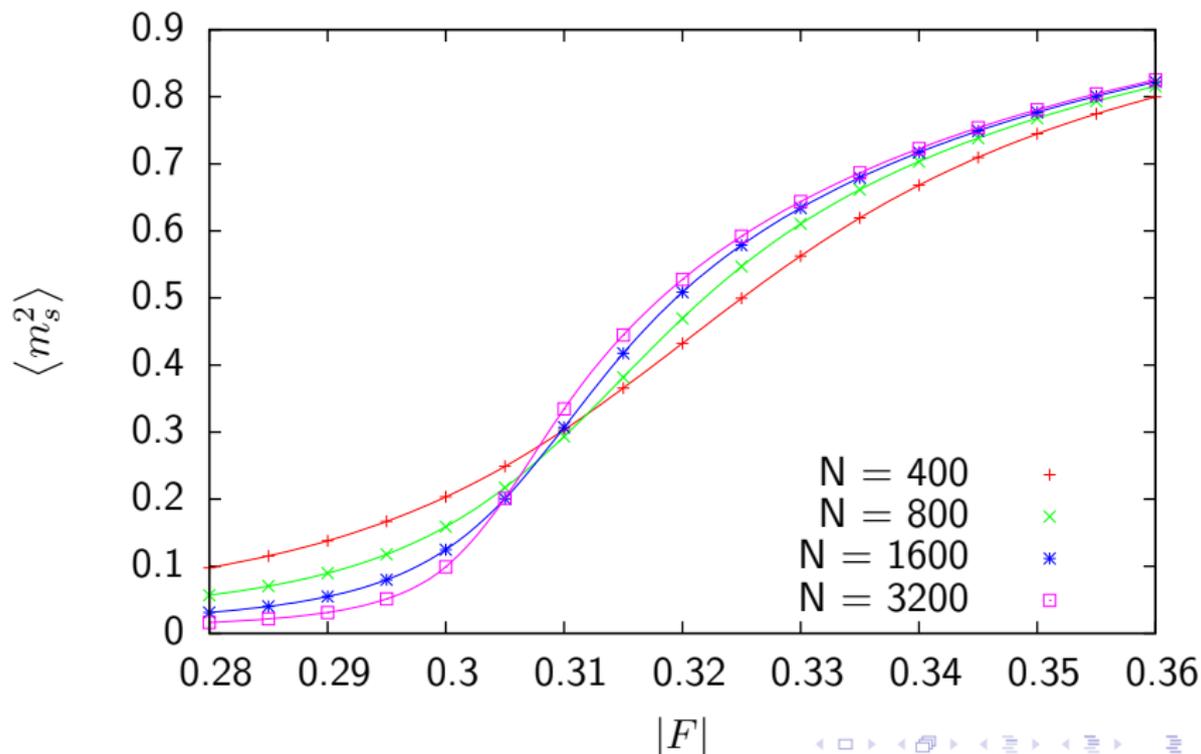
$c_v(F)$  curves for  $\theta = \pi, N = 2000$



- The agreement with the exact results both at  $\theta = 0$  and  $\theta = \pi$  suggests that the cumulant expansion can be trusted at all values of  $\theta$ , as long as  $|F| \lesssim 0.57$
- Now with the results for  $\theta = 2\pi$ ...

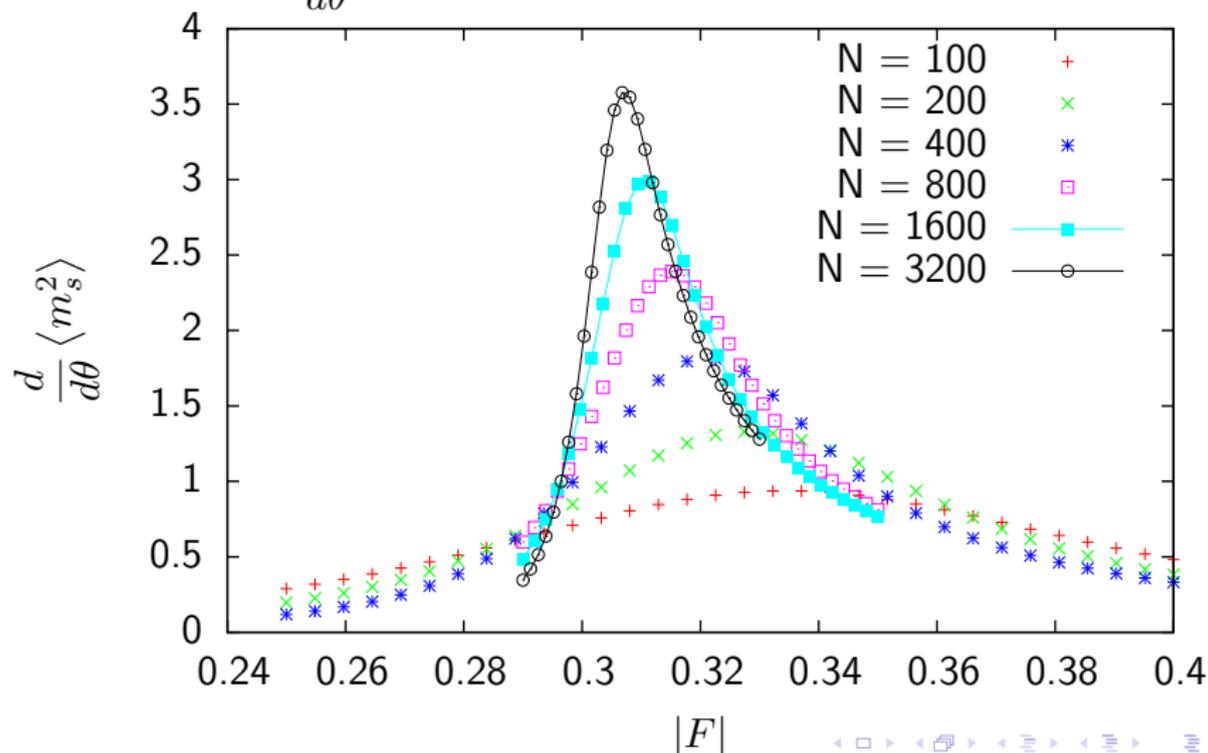
# Results

$\langle m_s^2 \rangle$  as a function of  $|F|$  for  $\theta = 2$ ,  $k = 8$



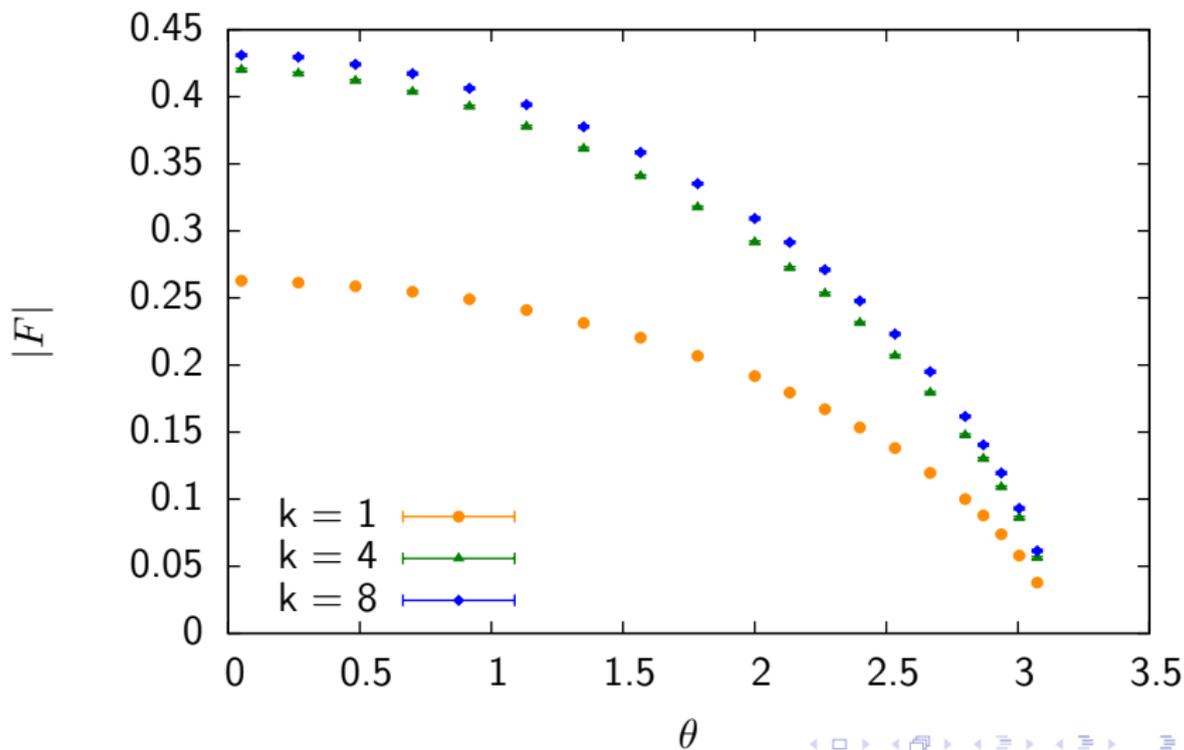
# Results

$\frac{d}{d\theta} \langle m_s^2 \rangle$  as a function of  $|F|$  for  $\theta = 2, k = 8$



# Results

$F_c$  as a function of  $\theta$ ,  $N = 2000$



# Summary and conclusions

- We have analyzed the two-dimensional antiferromagnetic Ising model within an imaginary magnetic field by analytical techniques.
- By means of a reformulation of  $\mathcal{Z}$  and a cumulant expansion to 8-th order in  $F$ , we have computed physical quantities for a large number of DOFs with the help of multiprecision algorithms.
- Comparisons with the analytical results at  $\theta = 0, \pi$ , give reliability to our results.
- The qualitative picture described in previous works, predicting the existence of two phases with the staggered magnetization as an order parameter.
- This model could be a good laboratory to check proposals to simulate physical systems afflicted by a SSP.