The ξ/ξ_{2nd} ratio as a test for Effective Polyakov line actions

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Granada 21/06/2017

Summary:

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Introduction and motivations: EPL models

In the Effective Polyakov Loop (EPL) models the original theory is mapped to a three-dimensional, centre-symmetric, effective Polyakov loop spin model obtained by integration over the gauge and matter degrees of freedom.
The resulting theory in general involves couplings between more than two loops which in principle may be far apart and of any possible representation of the gauge group.
A widely used simplified version is:

$$S_{eff} = \sum_{p} \sum_{|\mathbf{r}|\geq 1} \sum_{|\mathbf{x}-\mathbf{y}|=\mathbf{r}} \lambda_{p,\mathbf{r}} \, \chi_p(\mathbf{x}) \, \chi_{\mathbf{p}}(\mathbf{y})$$

where $\chi_p(\mathbf{x})$ is the character in the p representation of the loop in the spatial site $\mathbf{x}$ and $\lambda_{p,\mathbf{r}}$ is the coupling between the effective spins.
They are very useful for several reasons:

- They capture the physics of the underlying models but ar much simpler to simulate
- They allow to better understand the relevant symmetries and degrees of freedom of the underlying theory
- The sign problem is milder and can be treated with known methods, or it can be avoided entirely.

EPL models

- EPL models can be obtained with different strategies: strong coupling expansions, inverse Montecarlo methods, matching of correlators....
- All of them are in principle correct as fas as they capture some relevant feature of the underlying theory.
- It is easy to test if the critical properties (order of the phase transition or universality class) of the original theory near the deconfinement transition are recovered by the effective model. All the existing proposals fulfill this requirement.
- It would be nice to have a tool to test if the complex spectrum of the theory is recovered also outside the critical point and if the $1/R$ term in the potential (the "Lüscher term") is effectively recovered by the model.

The ξ/ξ_{2nd} is a simple, easy to simulate, tool to address this issue.

ξ versus ξ_{2nd} in spin models

In a d-dimensional spin model the exponential correlation length ξ describes the long distance behavior of the connected two point function.

$$\frac{1}{\xi} = -\lim_{|\vec{n}|\to\infty} \frac{1}{|\vec{n}|} \log\langle s_{\vec{0}} s_{\vec{n}}\rangle_c \quad .$$

where

$$\langle s_{\vec{m}} s_{\vec{n}}\rangle_c = \langle s_{\vec{m}} s_{\vec{n}}\rangle - \langle s_{\vec{m}}\rangle^2$$

The square of the second moment correlation length ξ_{2nd} is defined as:

$$\xi^2_{2nd} = \frac{\mu_2}{2d\mu_0} \ ,$$

where

$$\mu_0 = \lim_{L\to\infty} \frac{1}{V} \sum_{\vec{m},\vec{n}} \langle s_{\vec{m}} s_{\vec{n}}\rangle_c$$

and

$$\mu_2 = \lim_{L\to\infty} \frac{1}{V} \sum_{\vec{m},\vec{n}} |\vec{m}-\vec{n}|^2 \langle s_{\vec{m}} \, s_{\vec{n}}\rangle_c \ ,$$

where $V = L^d$ is the lattice volume.

ξ versus ξ_{2nd} in spin models

ξ_{2nd} is not exactly equivalent to ξ. The difference is in general very small, but it carries important information on the spectrum of the underlying theory.

The relation between ξ and ξ_{2nd} can be understood introducing the "time slice" variables

$$S_{n_0} = \frac{1}{L^2} \sum_{n_1,n_2} s_{(n_0,n_1,n_2)}$$

and the "time-slice" correlation function

$$G(\tau) = \sum_{n_0} \left\{ \langle S_{n_0} S_{n_0+\tau} \rangle - \langle S_{n_0} \rangle^2 \right\} \ .$$

whose large distance behaviour is controlled by ξ

$$G(\tau) \sim \ \exp(-\tau/\xi) \ .$$

ξ versus ξ_{2nd} in spin models

Using time slice variables, μ_2 and μ_0 can be rewritten as:

$$\mu_2 \quad = \quad \frac{d}{V} \sum_{\vec{m},\vec{n}} (n_0 - m_0)^2 \; \langle s_{\vec{m}} \, s_{\vec{n}} \rangle_c \quad .$$

i.e.

$$\mu_2 = dL^2 \sum_{\tau=-\infty}^{\infty} \tau^2 \; \langle S_0 \; S_\tau \rangle_c$$

and

$$\mu_0 = L^2 \sum_{\tau=-\infty}^{\infty} \langle S_0 \; S_\tau \rangle_c \quad .$$

From which we have

$$\xi_{2nd}^2 = \frac{\sum_{\tau=-\infty}^{\infty} \tau^2 \; G(\tau)}{2 \sum_{\tau=-\infty}^{\infty} G(\tau)} \quad .$$

ξ versus ξ_{2nd} in spin models

Assuming a multiple exponential decay for $G(\tau)$,

$$\langle S_0 \; S_\tau \rangle_c \propto \sum_i \; c_i \; \exp(-|\tau|/\xi_i) \quad ,$$

and replacing the summation by an integration over τ we get

$$\xi_{2nd}^2 = \frac{1}{2} \; \frac{\int_{\tau=0}^{\infty} \mathrm{d}\tau \; \tau^2 \sum_i c_i \; \exp(-\tau/\xi_i)}{\int_{\tau=0}^{\infty} \mathrm{d}\tau \; \sum_i c_i \exp(-\tau/\xi_i)} = \frac{\sum_i c_i \xi_i^3}{\sum_i c_i \xi_i} \; ,$$

which is equal to ξ^2 if only one state contributes. It is thus clear that we can use the ξ/ξ_{2nd} to have some insight on the spectrum of the theory and on the amplitude c_i of these states.

Example: the Ising case

	d	ξ/ξ_{2nd}	Method
High T phase	2	1.00040...	
	3	1.00016(2)	strong-coupling + ϵ-expansion [1]
	3	1.00021(3)	perturbative $d = 3$ calculation[2]
	3	1.000200(3)	strong-coupling expansion [1]
Low T phase	2	1.58188...	
	3	1.031(6)	Monte Carlo simulations[3]
	3	1.032(4)	strong-coupling expansion [1]
critical isotherm	2	1.07868...	
$(t = 0, \lvert H \rvert \neq 0)$	3	1.024(4)	strong-coupling + ϵ-expansion [1]

Table : Values of the ξ/ξ_{2nd} ratio for an Ising spin system in three different conditions: in the high-temperature symmetric phase, in the low-temperature broken symmetry phase and along the critical isotherm.

[1] M. Campostrini et al. Phys. Rev.E60 (1999) 3526-3563 cond-mat/9905078

[2] M. Campostrini et al. Phys. Rev.E57 (1998) 184-210 cond-mat/9705086

[3] M. Caselle et al. Nucl. Phys. B556 (1999) 575-600 hep-lat/9903011

These values have a natural interpretation:

- In the high-T symmetric phase, where the spectrum is composed by a single massive state, we would expect that $\xi/\xi_{2nd} = 1$: the small but not negligible difference from 1 is due to the cut above the pair production threshold at momentum p equal twice the lowest mass.
- In the low-T broken symememtry phase in $d = 3$ the spectrum is more complex, most likely it is composed by an infinite tower of bound states and one of them lies below the two particles threshold: $m_{bound} = 1.83(3)\, m_{ph}$ and in fact $\xi/\xi_{2nd} \sim 1.03$
- In the $d = 2$, $T = T_c$, $H \neq 0$ thanks to the exact solution S-matrix solution of Zamolodchikov we know that there are three particles in the spectrum below the two-particle threshold and accordingly we find $\frac{\xi}{\xi_{2nd}} = 1.07868...$
- Finally, in the $d = 2$, low-T case, the Fourier transform of the correlators starts with a cut whcih can be shown to be due to the coalescence of an infinite number of states. Accordingly we find $\frac{\xi}{\xi_{2nd}} = 1.58188...$

The $d = 3 + 1$ SU(2) LGT

To test the behaviour of ξ/ξ_{2nd} in Lattice Gauge Theories we performed a set of simulations in the $d = 3 + 1$ $SU(2)$ model looking at correlators of Polyakov loops in the confined phase: the temperature $T = 1/(a(\beta)N_t)$ is varied using the inverse coupling β and the temporal extent N_t.

Here are a few info on the simulations

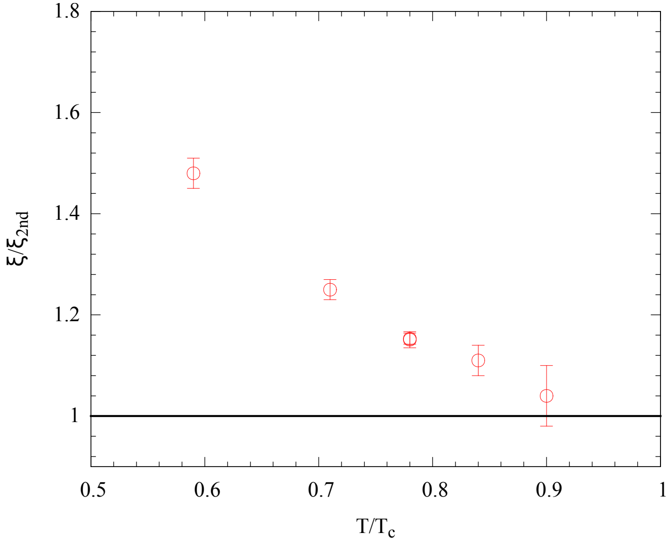
β	$N_s^3 \times N_t$	T/T_c	n_{conf}
2.27	$32^3 \times 6$	0.59	4.5×10^5
2.33	$32^3 \times 6$	0.71	2.25×10^5
2.3	$32^3 \times 5$	0.78	5.5×10^5
2.357	$32^3 \times 6$	0.78	2.25×10^5
2.25	$64^3 \times 4$	0.84	3×10^4
2.4	$64^3 \times 6$	0.90	2×10^4

The $d = 3 + 1$ SU(2) LGT

Results:

T/T_c	L	ξ/a	ξ_{2nd}/a	$\frac{\xi}{\xi_{2nd}}$
0.59	32	1.31(2)	0.887(8)	1.48(3)
0.71	32	2.31(4)	1.842(15)	1.25(2)
0.78	32	2.56(2)	2.22(1)	1.153(11)
0.78	32	3.08(4)	2.67(2)	1.151(16)
0.84	64	3.05(6)	2.74(4)	1.11(3)
0.90	64	6.9(2)	6.6(3)	1.04(6)

The $d = 3 + 1$ SU(2) LGT



The $d = 3 + 1$ SU(2) LGT

We see two main features

- for $T/T_c \to 1$ ξ/ξ_{2nd} is very close to 1.
  Same as what happens in the high-T phase of the 3d Ising model, in agreement with Svetitsky-Yaffe conjecture
- ξ/ξ_{2nd} increases dramatically as T/T_c decreases.
  This increase is due to the combination of two non-trivial features of the gauge theory spectrum:
  - first, the fact that as T/T_c decreases the states of the spectrum coalesce toward the ground state, exactly as it happens in the $d = 2$ Ising model below T_c;
  - second, the fact that the overlap constants c_i increase exponentially with the energy of the states.

A very useful tool to understand both these features is the effective string description of the Polyakov loop correlators which indeed predicts, as a consequence of the "string" nature of the color flux tube, a rich spectrum of excitations.

Comparison with the effective string description

A very good approximation of this effective string model is the Nambu-Gotō action which is simple enough to be exactly solvable. The large distance expansion of the Polyakov loop correlator in D space-time dimensions is[1]:

$$\langle P(x)^* P(y) \rangle = \sum_{n=0}^{\infty} w_n \frac{2r\sigma N_t}{E_n} \left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}(D-2)} \left(\frac{E_n}{2\pi r}\right)^{\frac{1}{2}(D-1)} K_{\frac{1}{2}(D-3)}(E_n r)$$

where w_n denotes the multiplicity of the state, N_t the size of the lattice in the compactified time direction and E_n the closed-string energies which are given by

$$E_n = \sigma N_t \sqrt{1 + \frac{8\pi}{\sigma N_t^2}\left[-\frac{1}{24}(D-2) + n\right]}.$$

In the $D = 3 + 1$ case, thanks to the identity $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ we have:

$$\langle P(x)^* P(y) \rangle = \sum_{n=0}^{\infty} \frac{N_t}{2r} w_n e^{-E_n r}$$

which represents a collection of free particles of mass E_n.

[1] M. Lüscher and P. Weisz, JHEP 07 (2004) 014 hep-th/0406205
M. Billó and M. Caselle, JHEP 07 (2005) 038, hep-th/0505201

Comparison with the effective string description: fixing T/T_c

In the framework of the Nambu-Gotō approximation one can also derive an estimate of the critical temperature T_c measured in units of the square root of the string tension $\sqrt{\sigma}$

$$\frac{T_c}{\sqrt{\sigma}} = \sqrt{\frac{3}{\pi(D-2)}}$$

given by the value of the ratio $\frac{T_c}{\sqrt{\sigma}}$ for which the lowest mass E_0 vanishes. We can thus rewrite the energy levels as a function of T/T_c; setting $D = 3+1$ we find

$$E_n = \frac{2\pi T_c^2}{3T} \left\{ 1 + 12 \frac{T^2}{T_c^2} \left[n - \frac{1}{12} \right] \right\}^{1/2} .$$

The gap between the different states, *decreases* as T/T_c decreases and all the states tend to accumulate toward the lowest state!

It is exactly this behaviour which leads in the other string channel to the appearence of the $1/R$ Lüscher term in the potential.

Comparison with the effective string description

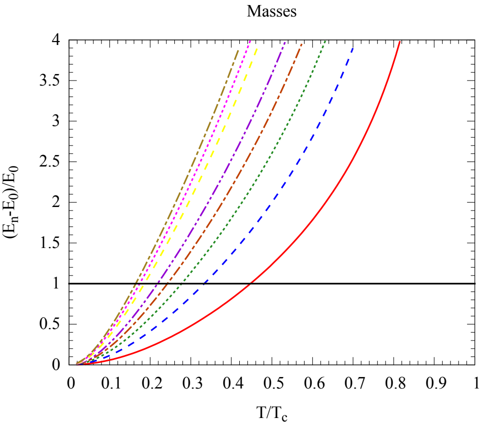


Figure : $(E_n - E_0)/E_0$ as a function of T/T_c for the first ten states. The black horizontal line represents the two particle threshold.

Comparison with the effective string description: the weights

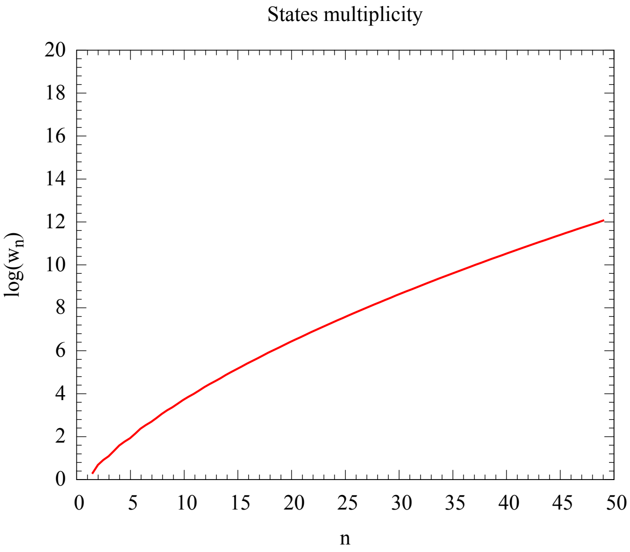
The weights w_n are given by:

$$\left(\prod_{r=1}^{\infty} \frac{1}{1-q^r}\right)^{D-2} = \sum_{k=0}^{\infty} w_k q^k .$$

These weights diverge exponentially as n increasese:

$$w_n \sim e^{\pi \sqrt{\frac{2(D-2)n}{3}}} .$$

Comparison with the effective string description



Conclusions

- The ξ/ξ_{2nd} is a simple and "easy to evaluate" observable to test existing proposal for Effective Polyakov loop actions
- In $d = 3 + 1$ $SU(2)$ LGT we find values of the ratio larger and larger as the temperature decreases.
- These values have a natural interpretation in terms of effective string decsription of the Polyakov loop correlators. They are due to the presence of an infinite number of excited states in the spectrum, which become denser and denser as T decreases and to the exponential increase of the weights.
- This behaviour is typical of spin models in the broken symmery phase, but not of the symmetric phase
- Since the Effective Polyakov loop action in the confining phase behaves as a $d = 3$ spin model in the symmetric phase, some further ingredient should be added to the action in order to generate the complex spectrum that we observe and justify such a large value of ξ/ξ_{2nd}.