Explicit positive representation for complex weights on $\mathbb{R}^d$

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Outline

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Statement of the problem

Task
Given a complex weight ρ on $\mathbb{R}^d$ find a probability distribution function P on $\mathbb{R}^{2d}$ such that for analytic observables $\mathcal{O}$ we have

$$\langle \mathcal{O} \rangle_\rho \equiv \frac{\int_{\mathbb{R}^d} \mathrm{d}^d t \rho(t) \mathcal{O}(t)}{\int_{\mathbb{R}^d} \mathrm{d}^d t \rho(t)} = \int_{\mathbb{R}^{2d}} \mathrm{d}^d x \mathrm{d}^d y P(x,y) \mathcal{O}(x+iy). \quad (1)$$

- Formally polynomial $\mathcal{O}$ are sufficient.
- Relation to stochastic processes not required.
- We start from $d = 1$ case.

Previous studies

An incomplete list

- Parisi (1983)
- Klauder (1984)
- Ambjorn, Yang (1985)
- Salcedo (1996, 1997)
- Weingarten (2002)
- Aarts, James, Seiler, Stamatescu (2010, 2011)
- Wosiek (2016)
- Seiler, Wosiek (2017)

Moment matching approach

Idea

Impose equality of moments. Solve for P using Fourier series.

$$P(r,\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} P_k(r) e^{ik\theta}, \tag{2a}$$

$$P_k(r) = P_{-k}(r)^*, \tag{2b}$$

$$\int_0^{\infty} \mathrm{d}r \; P_{-k}(r) r^{k+1} = \left\langle t^k \right\rangle_{\rho(t)}, \quad k \geq 0. \tag{2c}$$

Observation

This system is greatly underdetermined.

Ensuring positivity

Problem
Deciding whether given Fourier series is positive is difficult.

Idea
The lowest (i.e. angle independent) Fourier mode should be the largest.

Necessary condition $P_0(r) \geq |P_k(r)|$.

Sufficient condition $P_0(r) \geq \sum_{k \neq 0} |P_k(r)|$.

Explicit construction

Ansatz

We take P_k functions to be

$$P_0(r) = \frac{\sigma_0}{\pi} \exp(-\sigma_0 r^2), \tag{3a}$$

$$P_k(r) = c_k r^{|k|} \exp(-\sigma r^2), \quad k \neq 0, \tag{3b}$$

- $0 < \sigma_0 < \sigma$ - free parameters,
- c_k - undetermined coefficients.
- Moment matching condition yields

$$c_k = \frac{2\sigma^{k+1}}{k!} \left\langle t^k \right\rangle^*_{\rho(t)}, \quad k \geq 0. \tag{4}$$

Explicit construction

Result

It turns out that Fourier series can be summed for **any** $\rho(t)$,

$$P(r,\theta) = \frac{\sigma_0}{\pi} e^{-\sigma_0 r^2} + \frac{2\sigma}{\pi} e^{-\sigma r^2} \mathrm{Re} \left\langle \exp\left(\sigma t r e^{-i\theta}\right) - 1 \right\rangle_{\rho(t)}. \quad (5)$$

Probability P is expressed in terms of Fourier transform of ρ.

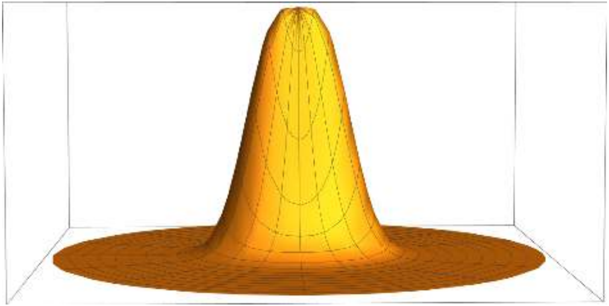
Theorem

Suppose that $|\rho(t)| \leq C e^{-at^2}$, for some $a > 0,\ C > 0$. Then:

1. The average in (5) converges and defines an analytic function,
2. σ_0, σ can be chosen so that P is positive and decays quickly: $P(r,\theta) \leq C' e^{-a' r^2}$ for some $a' > 0,\ C' > 0$.

Explicit construction

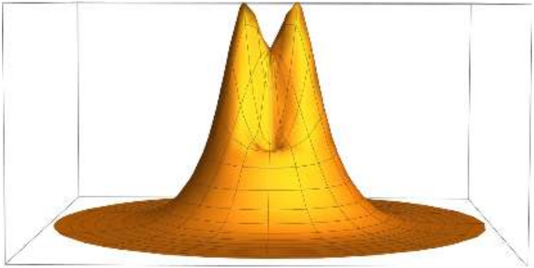
Positive representation for $\rho(t) = e^{-\lambda t^2}$.



$$P(r,\theta) = \frac{\sigma_0}{\pi} e^{-\sigma_0 r^2} + \frac{2\sigma}{\pi} e^{-\sigma r^2} \mathrm{Re}\left[\exp\left(\frac{\sigma^2 r^2 e^{2i\theta}}{4\lambda}\right) - 1\right].$$

Explicit construction

Positive representation for $\rho(t) = e^{-\lambda t^4}$

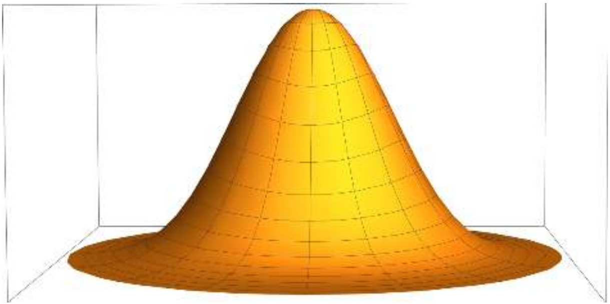


$$P(r,\theta) = \frac{\sigma_0}{\pi} e^{-\sigma_0 r^2} + \frac{2\sigma}{\pi} e^{-\sigma r^2} \mathrm{Re}\left({}_0F_2\left[\tfrac{1}{2}, \tfrac{3}{4}; \frac{\zeta^2}{256}\right] - 1 + \frac{\Gamma\left(\frac{3}{4}\right)\zeta}{2\Gamma\left(\frac{1}{4}\right)} {}_0F_2\left[\tfrac{5}{4}, \tfrac{3}{2}; \frac{\zeta^2}{256}\right] \right),$$

where $\zeta = \dfrac{\sigma^2 r^2 e^{2i\theta}}{\sqrt{\lambda}}$.

Explicit construction

Positive representation for $\rho(t) = \exp\left(-it - \frac{1}{2}t^2 + it^3 - t^4\right)$



- Here $\rho(t) = \exp\left(-it - \frac{1}{2}t^2 + it^3 - t^4\right)$,
- Distribution P was obtained by numerical integration.

Generalizations

- Other choices of P_k possible,
- Extension to higher dimensions straightforward:

$$P\left(\vec{r},\vec{\theta}\right) = \left(\frac{\sigma_0}{\pi}\right)^d e^{-\sigma_0 r^2} + 2\left(\frac{\sigma}{\pi}\right)^d e^{-\sigma r^2}\left\langle \exp\left(\sigma \vec{t}\cdot\vec{z}^*\right) - 1\right\rangle_{\rho(\vec{t})}, \tag{6}$$

where $\vec{z}^* = (r_1 e^{-i\theta_1}, ..., r_d e^{-i\theta_d})$.

Summary

- New construction of positive representations was found.
- Regularity (smoothness and decay at infinity) was proven.
- Examples were investigated analytically and numerically.
- Non-uniqueness of the problem was explicitly confirmed.