

Loop equation in Lattice gauge theories and bootstrap-like methods

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Summary

- AdS/CFT motivates the formulation of (large N) gauge theories purely in terms of gauge invariant quantities.

Wilson loops  Loop equation (Migdal-Makeenko)

- Lattice regularization gives well defined equations for a discrete set of quantities. Numerical solutions?
- Easy for large coupling. At small coupling we had to reformulate the problem (bootstrap-like method): impose positivity constraints + loop equation to put bounds on the plaquette average $u(\lambda)$.
- Apply to 2,3,4D pure YM and discussion of $\mathcal{N} = 4$.

Lattice gauge theory, pure YM, large-N, cubic lattice

Action

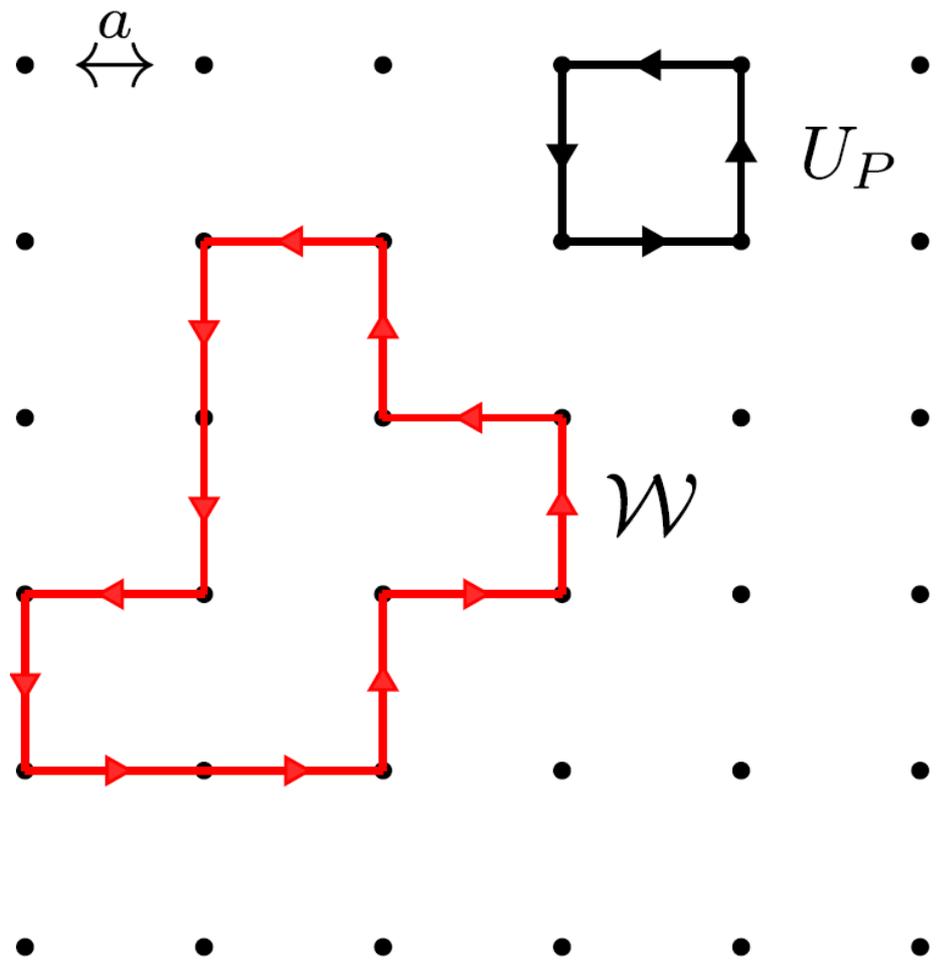
$$S = -\frac{N}{2\lambda} \sum_P \text{Tr} U_P$$

$$Z = \int \prod_{\vec{x}, \mu} dU_\mu(\vec{x}) e^{-S}$$

λ : 't Hooft coupling

Energy (action) density

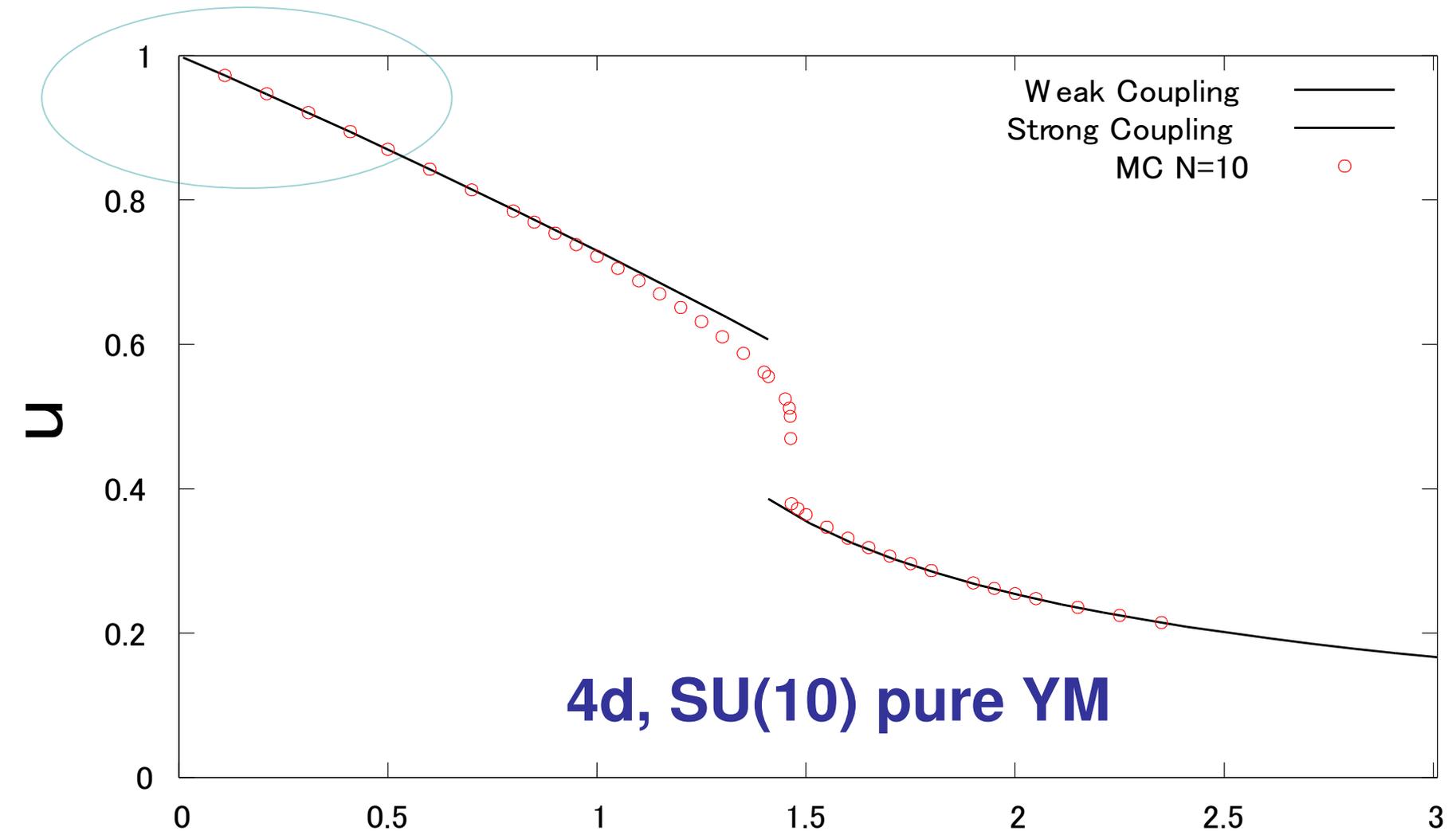
$$\frac{E}{V} = -\frac{d(d-1)}{2} \frac{N^2}{\lambda} u, \quad u = \frac{1}{N} \langle \text{Tr} U_P \rangle$$



Numerically (Creutz, Moriarty '82, Meyer-Teper '04)

Strong coupling expansion (Drouffe Zuber '83)

Weak coupling expansion (Heller, Karsch, '85)

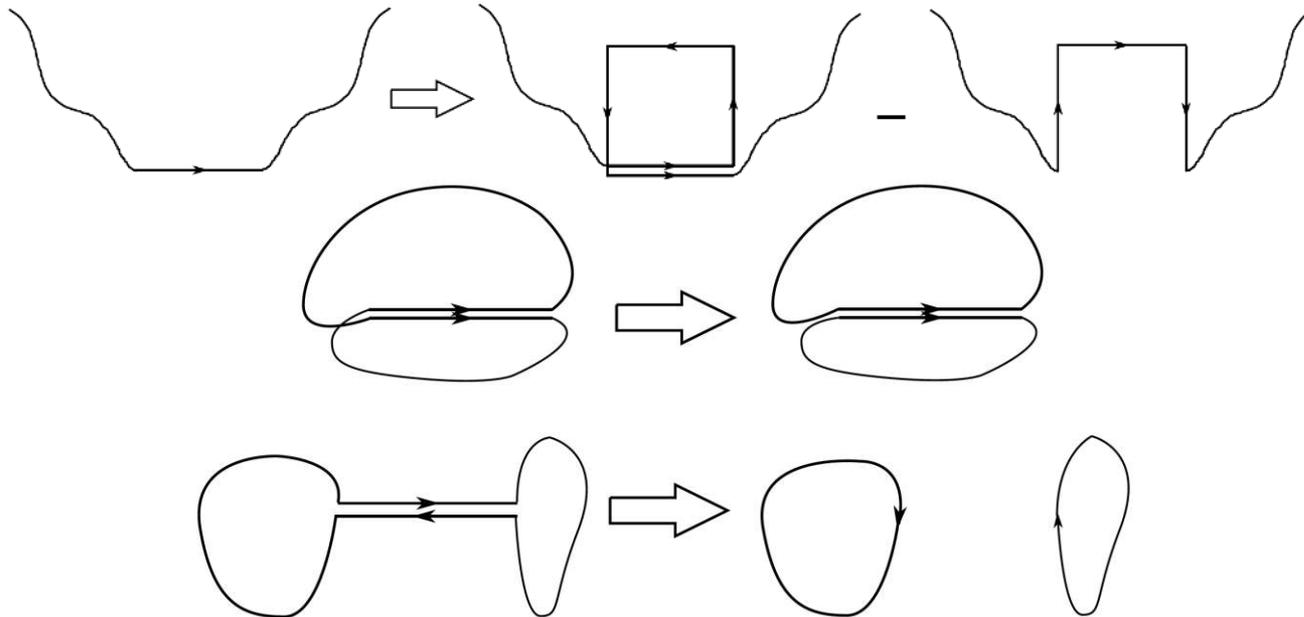


Loop equation (Migdal-Makeenko, Eguchi, Foerster,...)

Derivation

$$U_\mu \rightarrow (1 + i\epsilon)U_\mu, \quad U_\mu^\dagger \rightarrow U_\mu^\dagger(1 - i\epsilon), \quad \epsilon^\dagger = \epsilon, \quad \text{Tr}\epsilon = 0$$

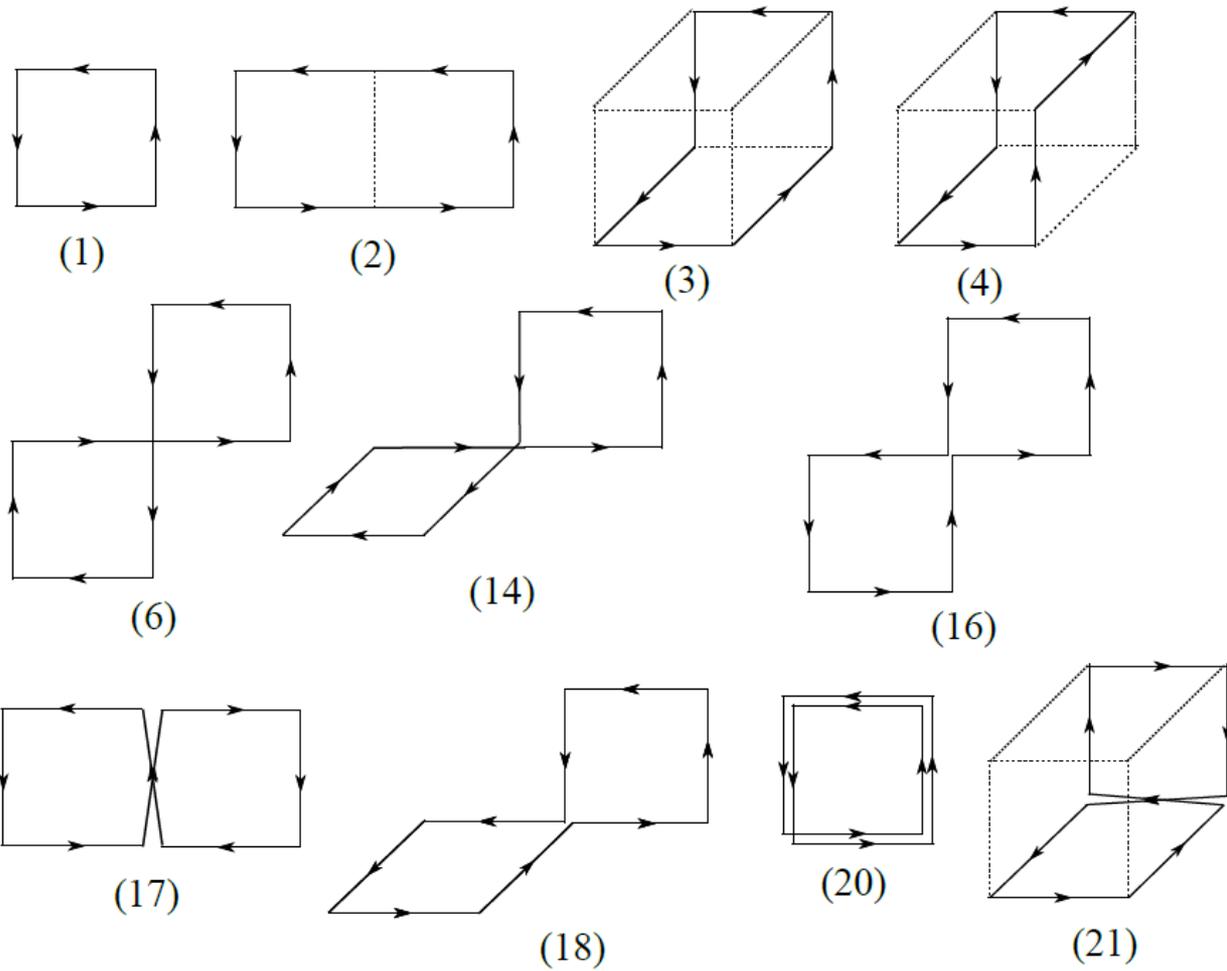
Graphic form of the equation:



Algebraic form of the equations (sum over links) :

$$\mathbb{K}_{i \rightarrow j} \mathcal{W}_j + 2\lambda \mathcal{W}_i + 2\lambda \mathbb{C}_{i \rightarrow jk} \mathcal{W}_j \mathcal{W}_k = \delta_{i1}$$

$$- \frac{1}{NL} S * \mathcal{W} + \mathcal{W} + \frac{1}{L} \sum_i \sigma_i \mathcal{W}_{1i} \mathcal{W}_{2i} = 0$$



L	# WL(4d)
10	268
12	5,324
14	142,105
16	4,483,136
18	152,322,746

$$-\mathcal{W}_0 - \mathcal{W}_2 - 4\mathcal{W}_3 + \mathcal{W}_{17} + \mathcal{W}_{20} + 4\mathcal{W}_{21} + 2\lambda\mathcal{W}_1 = 0$$

$$\mathcal{W}_2 + \mathcal{W}_6 + 4\mathcal{W}_{14} - \mathcal{W}_{16} - \mathcal{W}_{17} - 4\mathcal{W}_{18} = 0$$

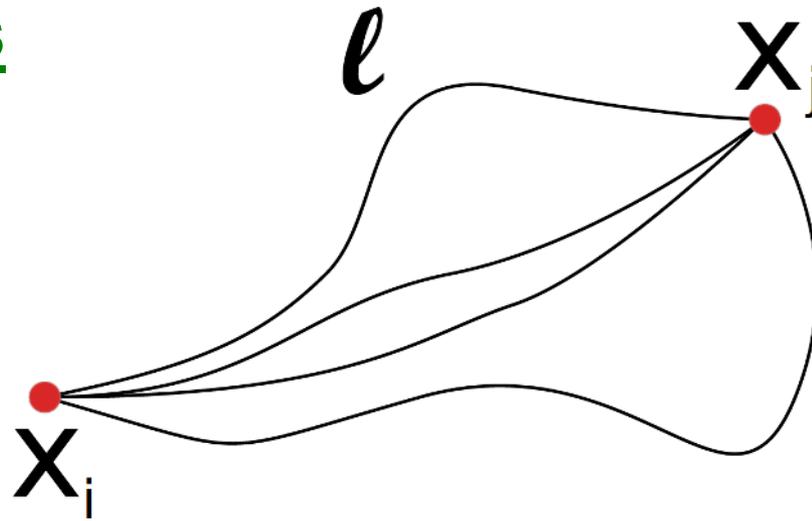
In trying to solve the equations one faces a problem:

- To define the equations properly we have to cut the set of loops, e.g. $\text{length} \leq L$, and then consider $L \rightarrow \infty$.
- The equations for length L have loops of length $L+4$. The number of loops increases exponentially with L .
- The limit $L \rightarrow \infty$ **does not seem well defined**, except at strong coupling where we set the unknown loops to 0.

We argue that including a certain set of positivity constraints gives a well defined limit $L \rightarrow \infty$ at any coupling. The reason is that the constraints put bounds on the energy density that improve as $L \rightarrow \infty$

Positivity constraints

$$A = \sum_{\ell=1}^L c_{\ell} U^{(\ell)}$$



$$\text{Tr} A^{\dagger} A \geq 0 \Rightarrow \sum_{\ell \ell'} c_{\ell}^* c_{\ell'} \text{Tr} \left[\left(U^{(\ell)} \right)^{\dagger} U^{(\ell')} \right] \geq 0 \quad \forall c_{\ell}$$

$$\hat{\rho}_{\ell \ell'}^{(L)} = \frac{1}{NL} \langle \text{Tr} \left[\left(U^{(\ell)} \right)^{\dagger} U^{(\ell')} \right] \rangle \succeq 0$$

$$\hat{\rho}_{\ell \ell}^{(L)} = \frac{1}{L} \Rightarrow \text{Tr} \left(\hat{\rho}^{(L)} \right) = 1$$

Closed loop:
goes along ℓ'
comes back along ℓ

ρ can be thought as a reduced density matrix obtained by tracing over color indices

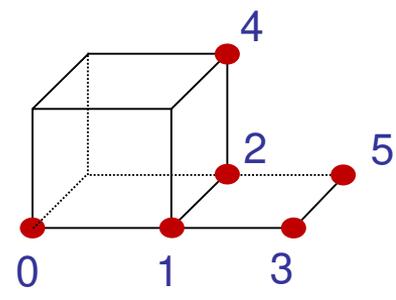
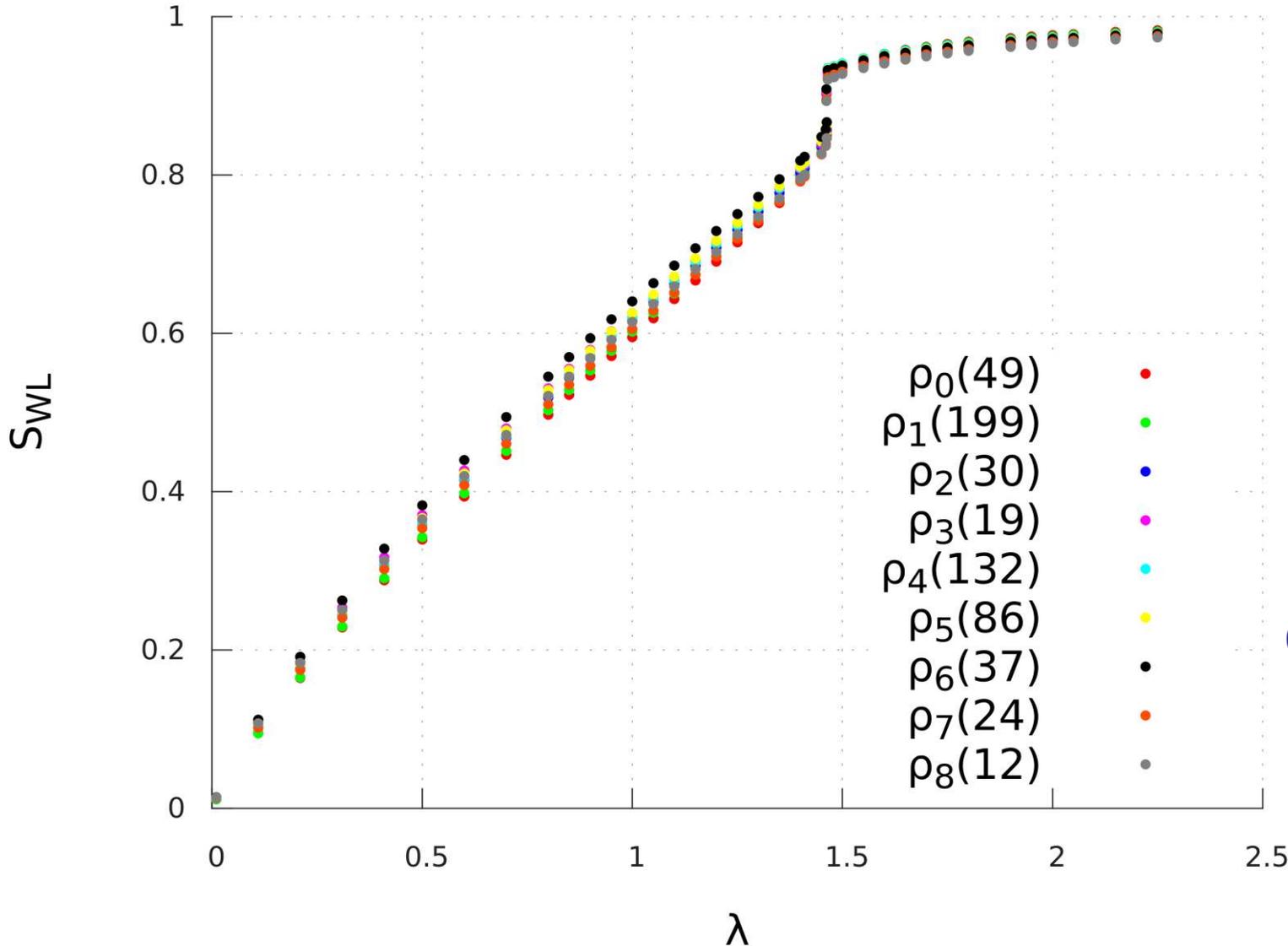
$$\hat{\rho}_{\ell\ell'}^{(L)} = \frac{1}{NL} \langle \text{Tr} \left[\left(U_{ab}^{(\ell)} \right)^* U_{ab}^{(\ell')} \right] \rangle$$

Its entropy computes the information loss due to tracing:

$$S_{WL} = -\text{Tr} \hat{\rho}^{(L)} \log_L \hat{\rho}^{(L)}$$

When $\lambda=0$ all loops are 1, $S=0$, when $\lambda \rightarrow \infty$, all loops are zero, $\rho=\mathbf{I}$, S is maximal. Behaves as system entropy.

Numerically S_{WL} is approx. independent of the choice of ρ



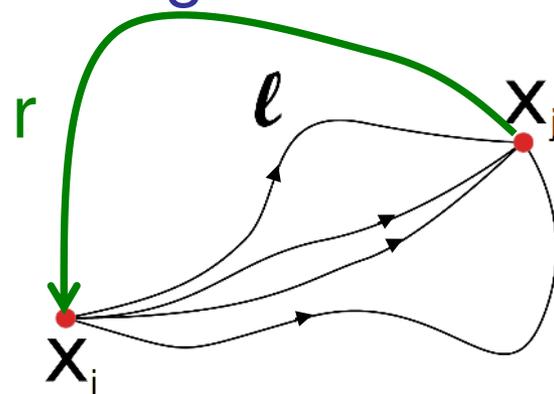
If ρ has a zero eigenvector \mathbf{c}_0 (boundary of the domain):

$$\rho_{\ell\ell'} \mathbf{c}_{0\ell'} = 0 \Rightarrow \mathbf{c}_{0\ell} \rho_{\ell\ell'} \mathbf{c}_{0\ell'} = 0 \Rightarrow \langle \text{Tr}(A_0^\dagger A_0) \rangle = 0$$

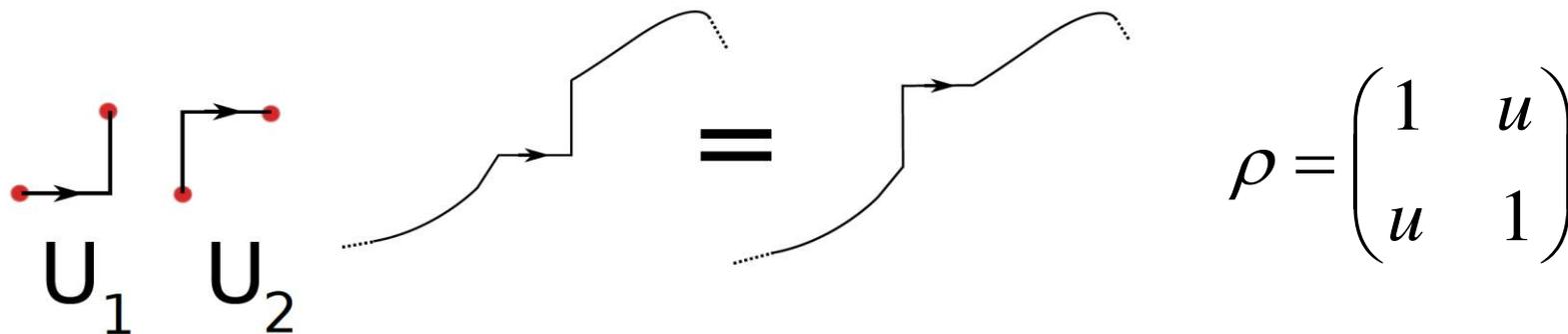
Thus $A_0 = \sum \mathbf{c}_{0\ell} U^{(\ell)} = 0$

Closing with an arbitrary path r we get linear equations

$$\left\langle \sum_{\ell=1}^L \mathbf{c}_{0\ell} \text{Tr} \left[U_r^\dagger U^{(\ell)} \right] \right\rangle = 0$$



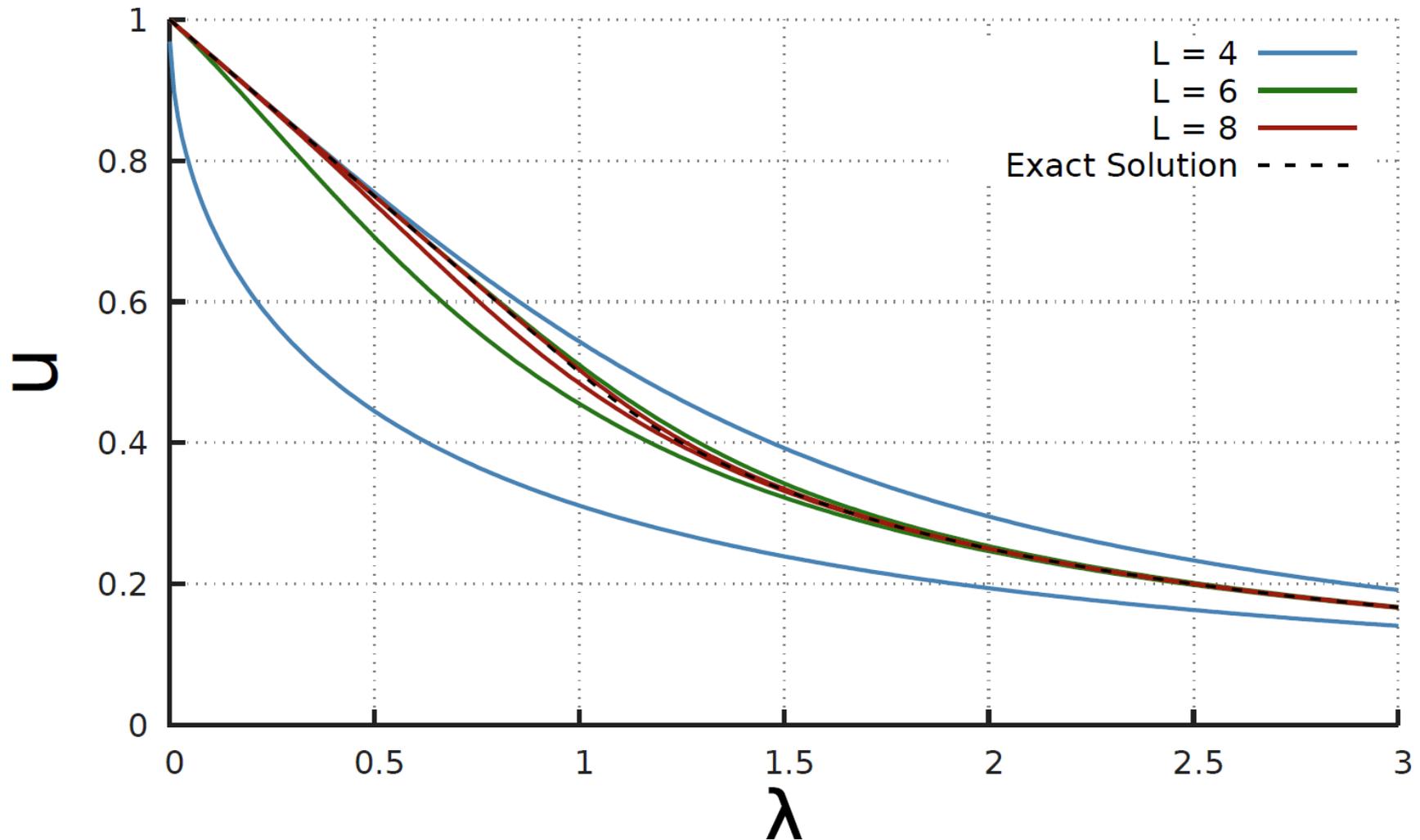
valid for arbitrary long loops. In particular, if $u=1$ then all loops are 1.



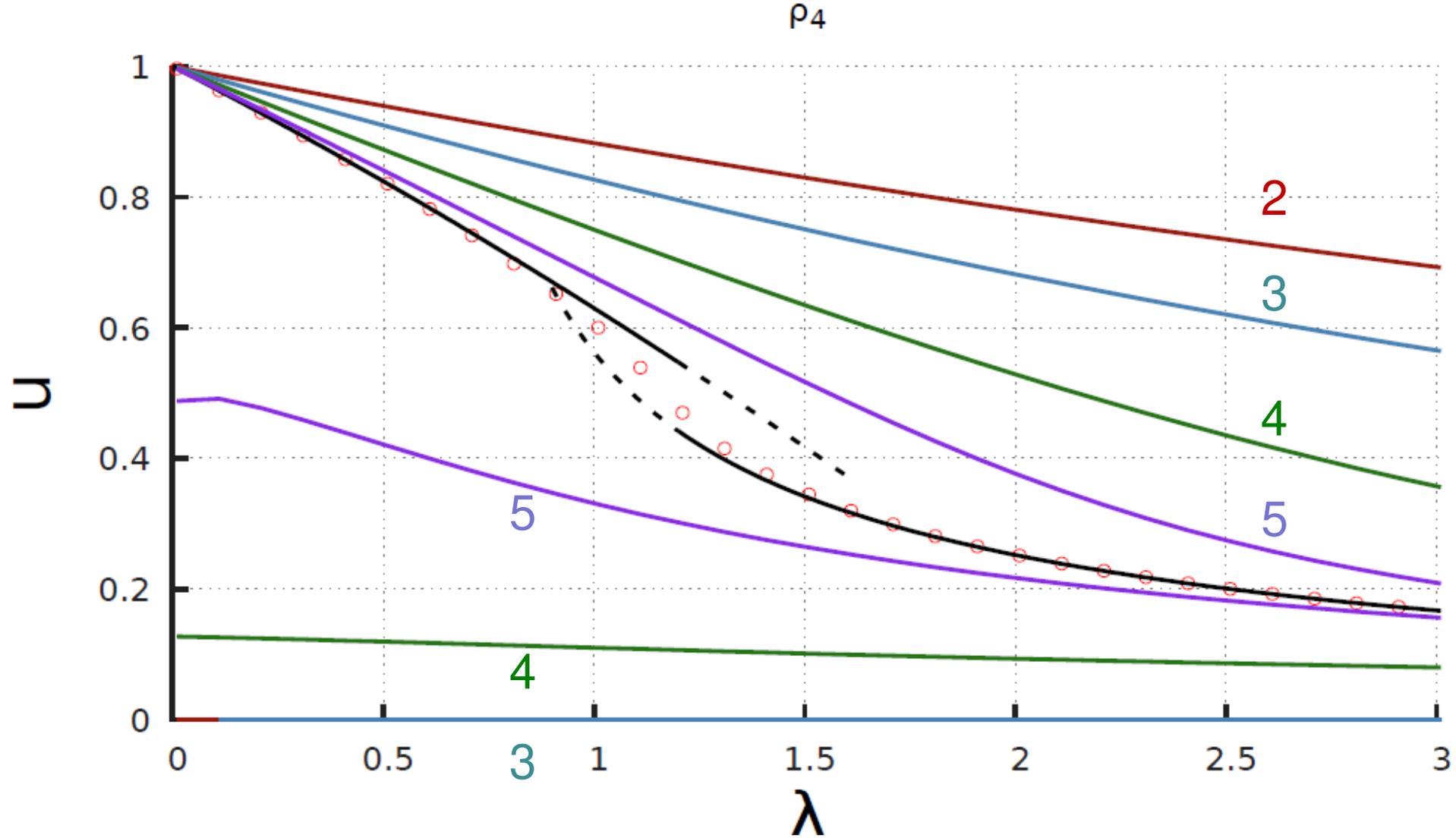
2D case

$$\mathcal{W}_1 = u = \begin{cases} 1 - \frac{\lambda}{2} & \lambda \leq 1 \\ \frac{1}{2\lambda} & \lambda \geq 1 \end{cases}$$

Exact solution
(Gross and Witten)



3D case



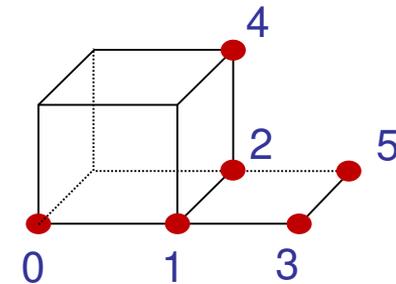
3D case “bootstrap-like” solution

Numerical values at weak coupling

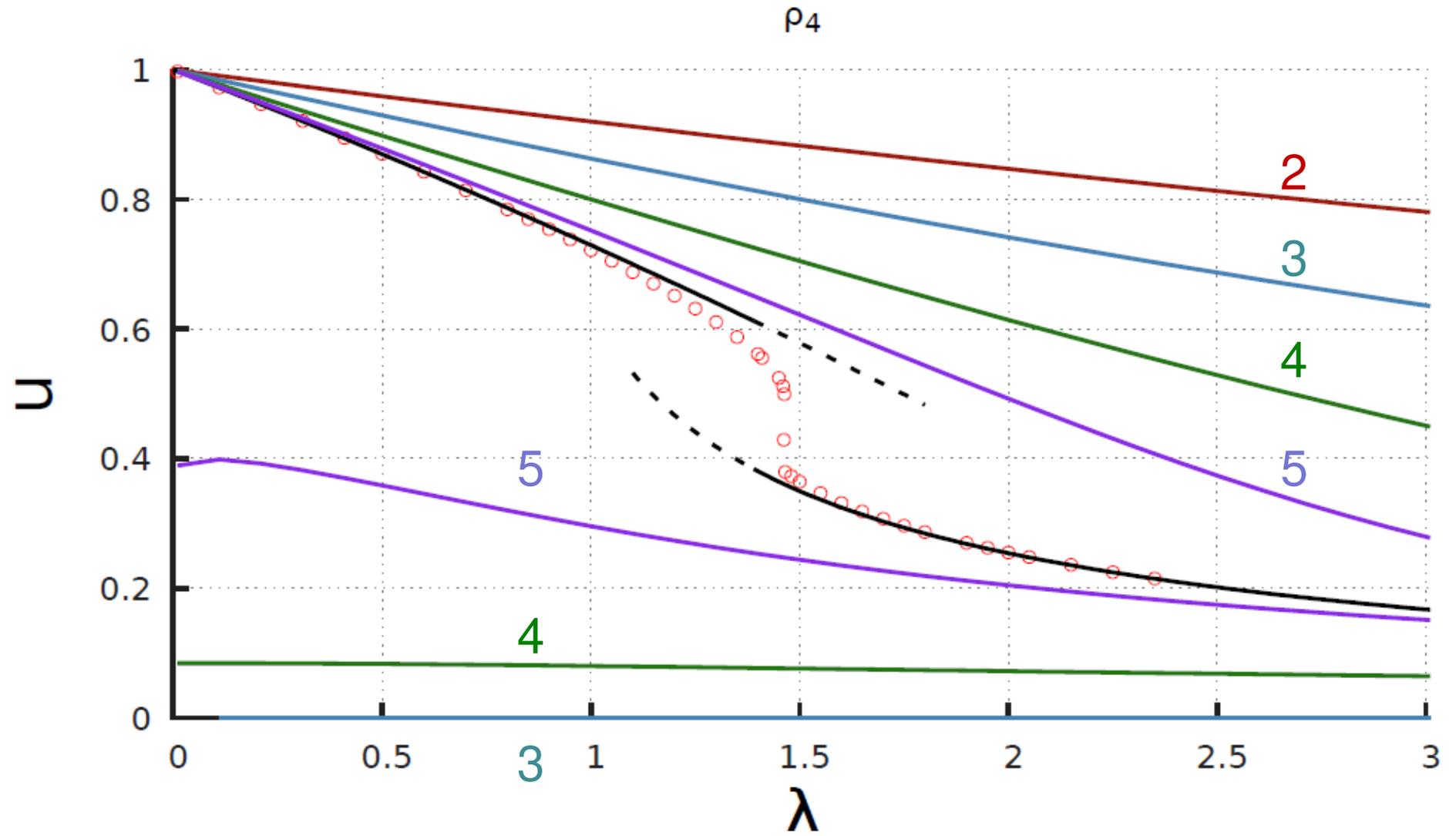
λ	u_{\max}	Pert .Th.	MC(N=10)
0.01	.9968	.9967	.9967
0.11	.9647	.9629	.9633
0.21	.9324	.9284	.9292
0.31	.8997	.8931	.8941
0.41	.8667	.8571	.8580
0.51	.8336	.8204	.8207

$L_{\max}=20$.

Using matrix ρ_4 size 330x330
involving 5,299 variables.



4D case



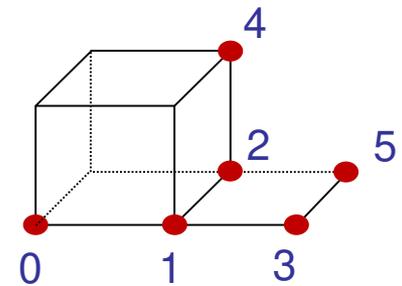
4D case “bootstrap-like” solution

Numerical values at weak coupling

λ	u_{\max}	Pert .Th.	MC (N=10)
0.01	0.9976	0.9975	0.9975
0.11	0.9737	0.9723	0.9725
0.21	0.9497	0.9466	0.9471
0.31	0.9253	0.9205	0.9212
0.41	0.9008	0.8941	0.8947
0.51	0.8762	0.8672	0.8676

$L_{\max}=20$.

Using matrix ρ_4 size 786x786
involving 11302 variables.

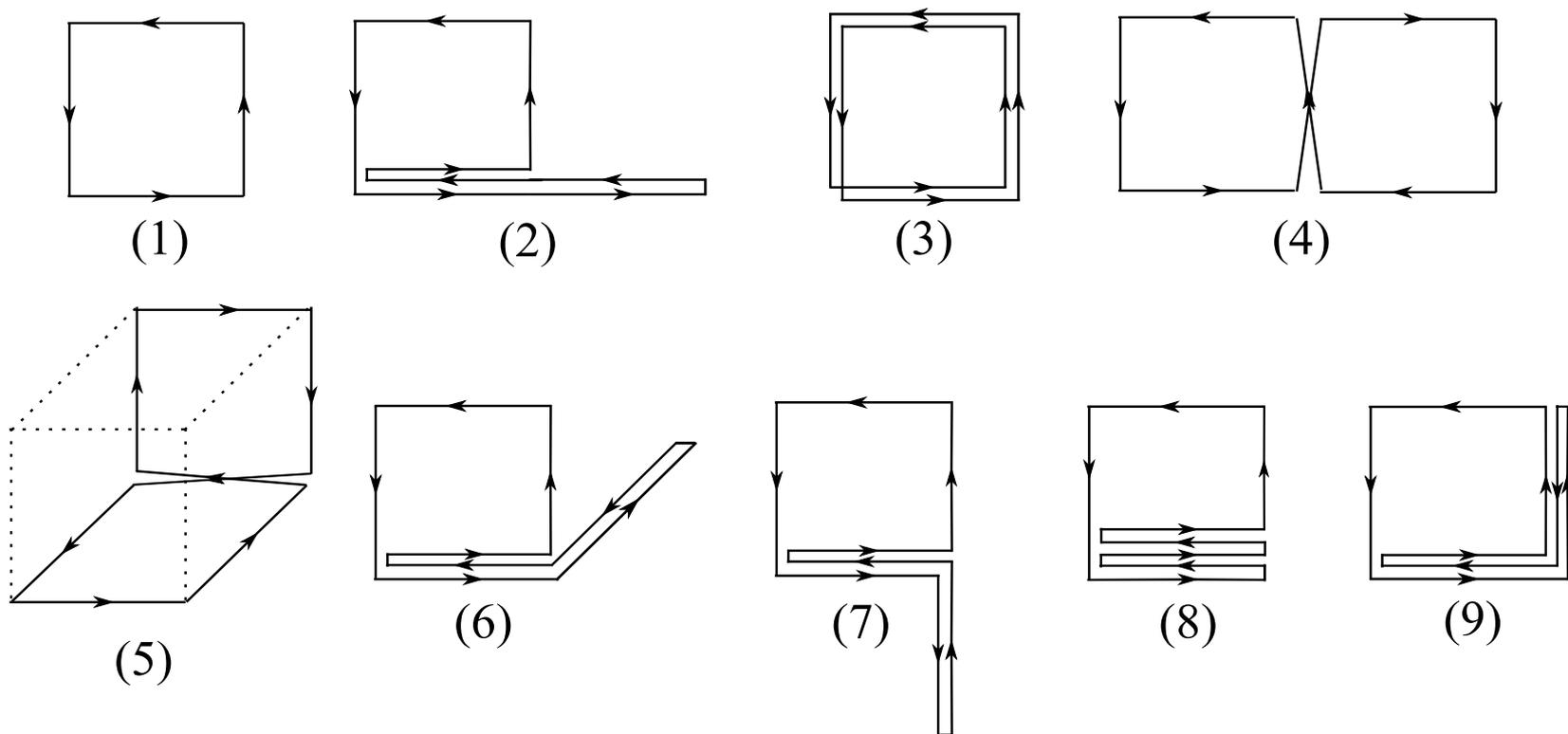


$\mathcal{N} = 4$ SYM, bosonic sector $U(N)$ gauge group

Action:

As long as it is written in terms of Wilson loops we can immediately write the loop equation.

$$S = \frac{N}{2\lambda_{\text{lat}}} \left\{ \sum_{\mu \neq \nu} \left[-2 \left(\begin{array}{c} \text{square loop } \mu \rightarrow \nu \\ \text{square loop } \nu \rightarrow \mu \end{array} \right) + \left(\begin{array}{c} \text{L-shaped loop } \mu \rightarrow \nu \\ \text{L-shaped loop } \nu \rightarrow \mu \end{array} \right) + \left(\begin{array}{c} \text{J-shaped loop } \mu \rightarrow \nu \\ \text{J-shaped loop } \nu \rightarrow \mu \end{array} \right) + \left(\begin{array}{c} \text{corner loop } \mu \rightarrow \nu \\ \text{corner loop } \nu \rightarrow \mu \end{array} \right) \right] + \sum_{\mu} \left[\left(\begin{array}{c} \text{double line } \mu \\ \text{double line } \mu \end{array} \right) - \left(\begin{array}{c} \text{crossed line } \mu \\ \text{crossed line } \mu \end{array} \right) \right] \right\}$$



$$\frac{1}{2\lambda_{lat}} (-\mathcal{W}_2 - \mathcal{W}_3 - \mathcal{W}_4 - 6\mathcal{W}_5 + 6\mathcal{W}_6 + \mathcal{W}_7 + \mathcal{W}_8 + \mathcal{W}_9) - \mathcal{W}_1 = 0$$

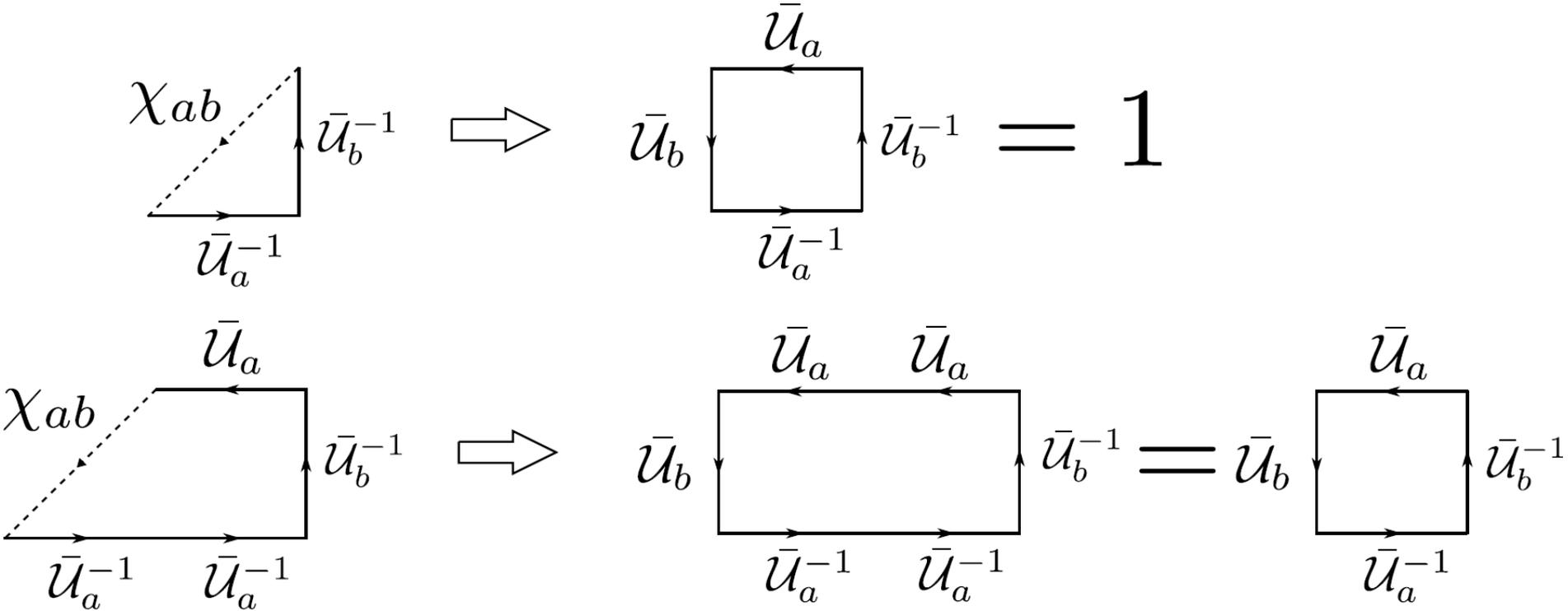
$$U = e^{iA+\Phi}, \quad U^\dagger = e^{-iA+\Phi}$$

Monte Carlo simulation

(using **Schaich-Catterall** code)

	N=2			N=3		
$\lambda =$	0.8	1.0	1.2	0.8	1.0	1.2
\mathcal{W}_1	0.01158	0.01451	0.01740	0.01151	0.01440	0.01724
\mathcal{W}_2	0.00241	0.00377	0.00543	0.00241	0.00377	0.00540
\mathcal{W}_3	0.00052	0.00082	0.00118	0.00048	0.00075	0.00109
\mathcal{W}_4	0.00026	0.00041	0.00059	0.00026	0.00040	0.00057
\mathcal{W}_5	0.00023	0.00036	0.00051	0.00023	0.00036	0.00051
\mathcal{W}_6	0.00205	0.00321	0.00462	0.00206	0.00321	0.00462
\mathcal{W}_7	0.00218	0.00343	0.00494	0.00219	0.00341	0.00489
\mathcal{W}_8	0.00525	0.00823	0.01184	0.00520	0.00812	0.01164
\mathcal{W}_9	0.00330	0.00518	0.00745	0.00322	0.00503	0.00723
Eq.	-0.00002	-0.000006	-0.00001	0.00002	-0.000003	-0.00002

$\mathcal{N} = 4$ SYM, adding fermions, BPS loops.



We can use supersymmetry to prove a set of loops are equal to 1.

Conclusions

-) We constructed a matrix $\rho_{\geq 0}$ with WLs as entries and use it to correctly formulate the problem of solving the loop equations (especially at small coupling).
-) This numerically reproduces (in 2,3,4d) the $\lambda \rightarrow 0$ result

$$\mathcal{W}_1 = u = 1 - \frac{\lambda}{d} + \mathcal{O}(\lambda^2)$$

-) In the weak coupling phase ρ saturates the bounds, it has zero eigenvalues whose number increases as $\lambda \rightarrow 0$ (relevant for the continuum limit?).
-) We defined an off-shell Wilson loop entropy as the entropy associated with ρ (\sim indep. of particular ρ).
-) For the $\mathcal{N}=4$ case, we checked the loop eq. in the bosonic sector and BPS condition in the full theory.