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ISOQUANT SFB1225

Improved real-time dynamics from imaginary frequency lattice simulations

Alexander Rothkopf
Institute for Theoretical Physics
Heidelberg University

in collaboration with Jan Pawlowski

Reference:

J.M. Pawlowski, A.R. arXiv:1610.09531

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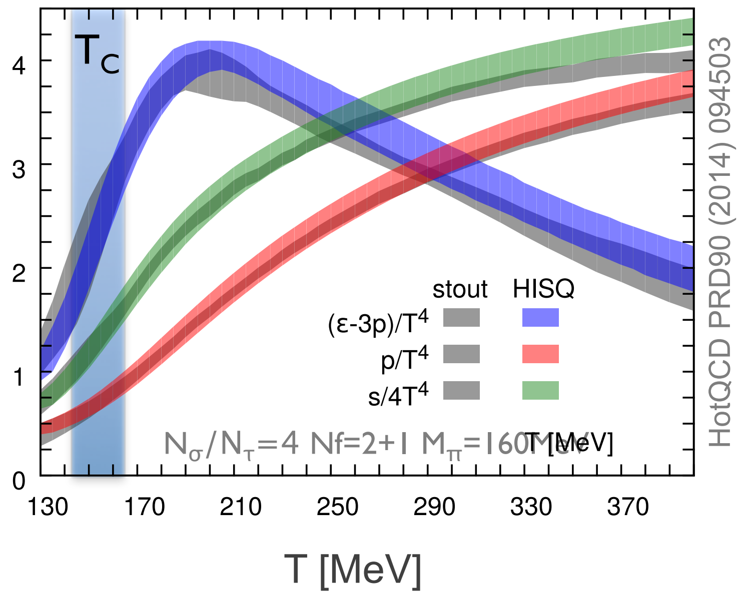
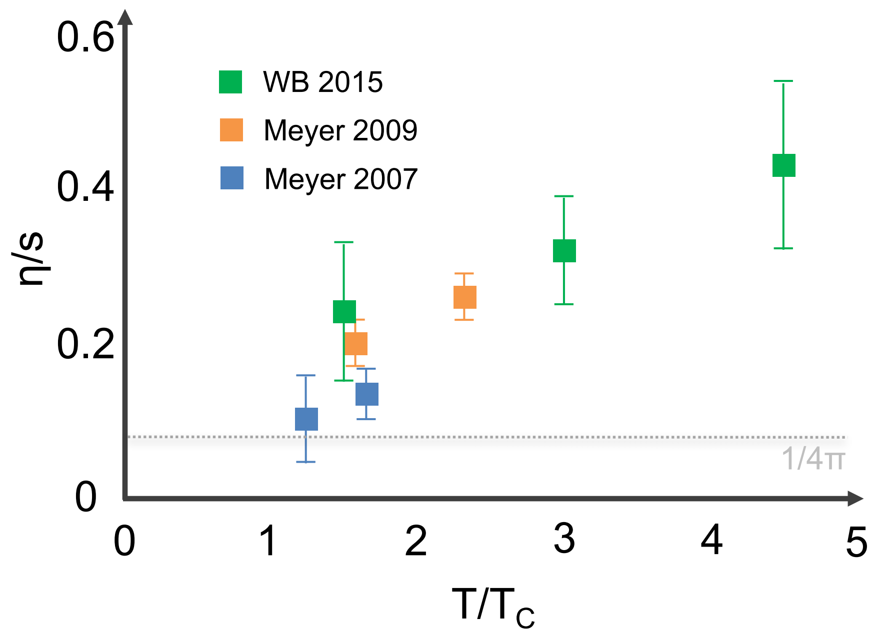
Static vs. Real-time Observables

- LQCD highly successful for QCD E.O.S., challenged for e.g. transport coefficients





- Example: transport coefficients only accessible in LQCD via spectral functions

$$\eta = \frac{1}{20} \lim_{\mathbf{p}\to 0} \lim_{\omega \to 0} \frac{1}{\omega} \int d^4x \, e^{ipx} \, \langle [T_{12}(0), T^{12}(x)] \rangle$$

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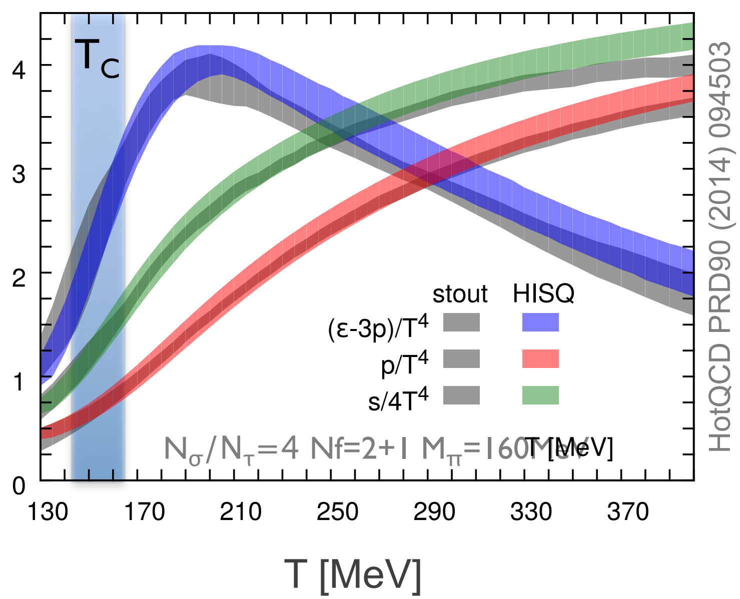

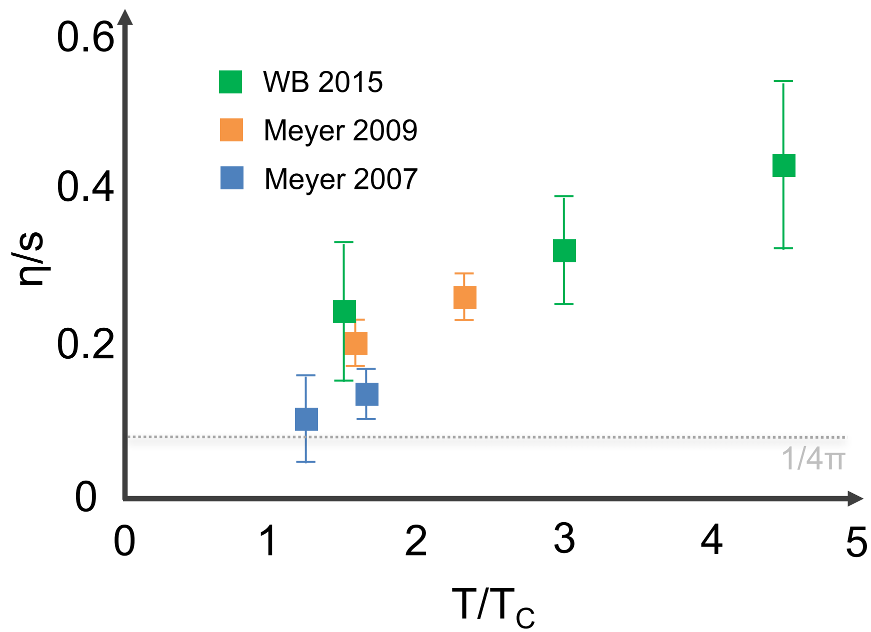

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The underlying difficulty

- Intrinsic problem of standard spectral reconstruction: exponential information loss

$$D(\tau) = \int_0^\infty d\omega \, \frac{\cosh[\omega(\tau - \beta/2)]}{\sinh[\omega\beta/2]} \rho(\omega)$$

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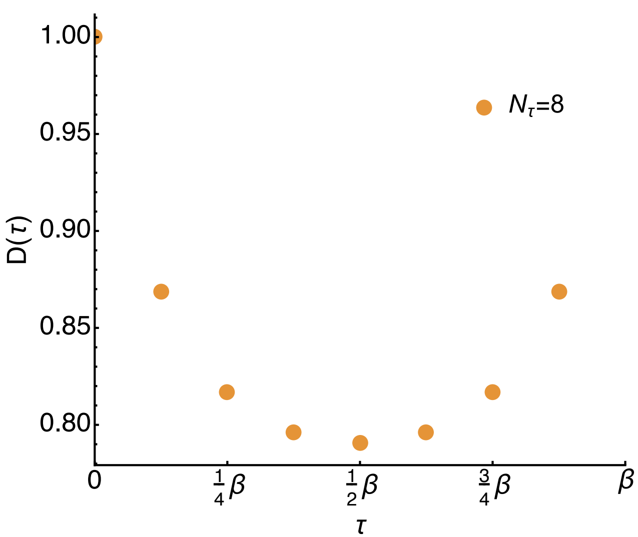

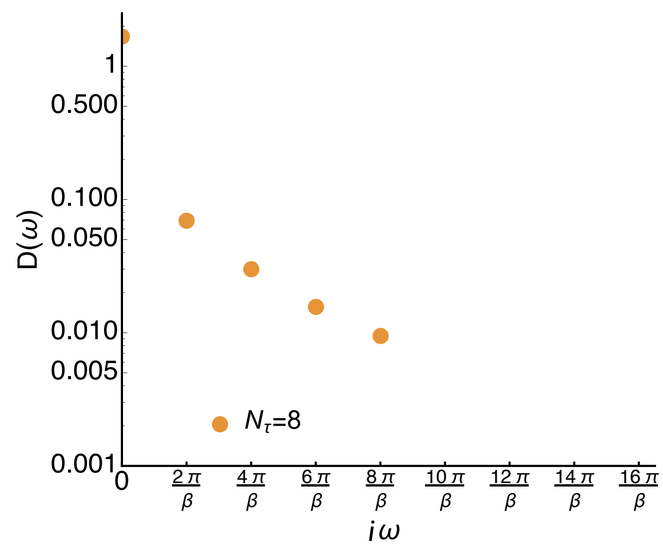

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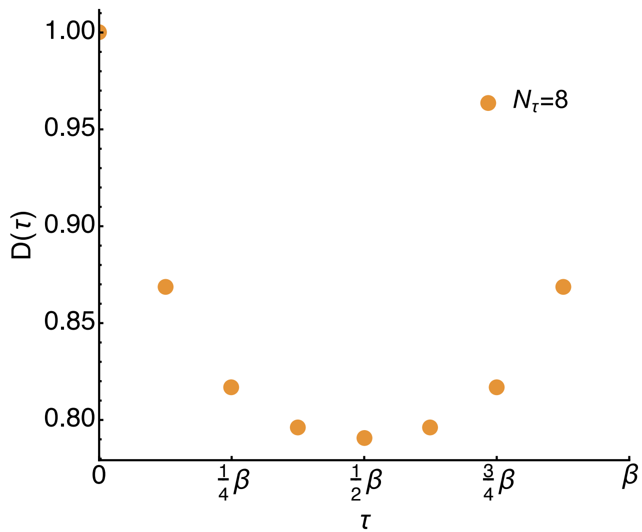

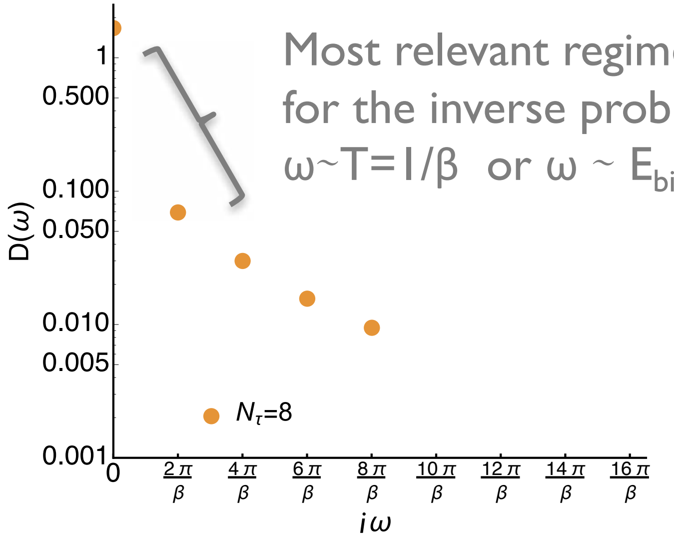

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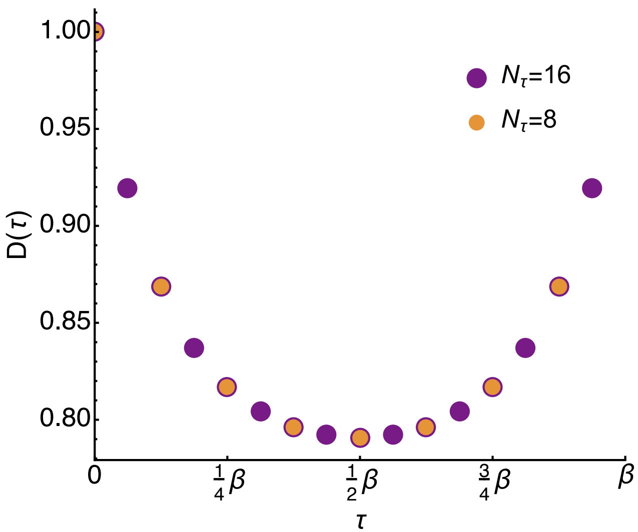

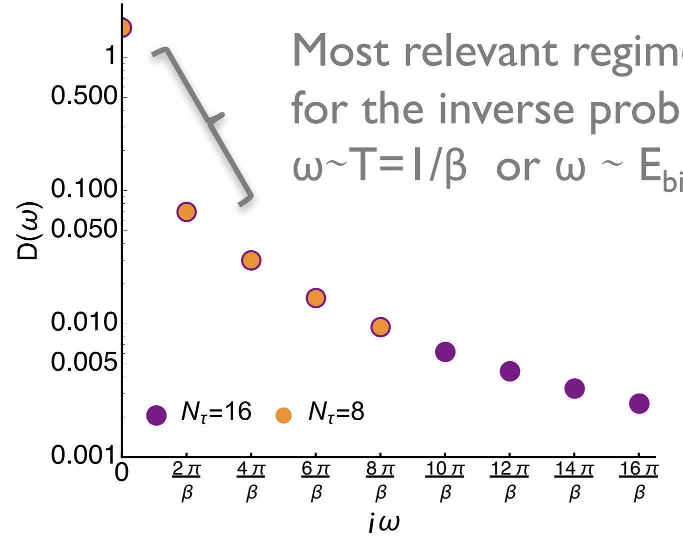

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Our proposal

Simulating thermal fields in imaginary frequencies

A thermal initial value problem

$$\mathcal{Z} = \int_{\varphi_E(0)=\varphi_E(\beta)} \mathcal{D}\varphi_E e^{-S_E[\varphi_E]} \int_{\varphi^+(t_0,\mathbf{x})=\varphi_E(0)}^{\varphi^-(t_0,\mathbf{x})=\varphi_E(\beta)} \mathcal{D}\varphi e^{iS_M[\varphi^+]-iS_M[\varphi^-]}$$

- Setup on the Schwinger-Keldysh contour
  - For finite t φ^+ and φ^- path coupled at t
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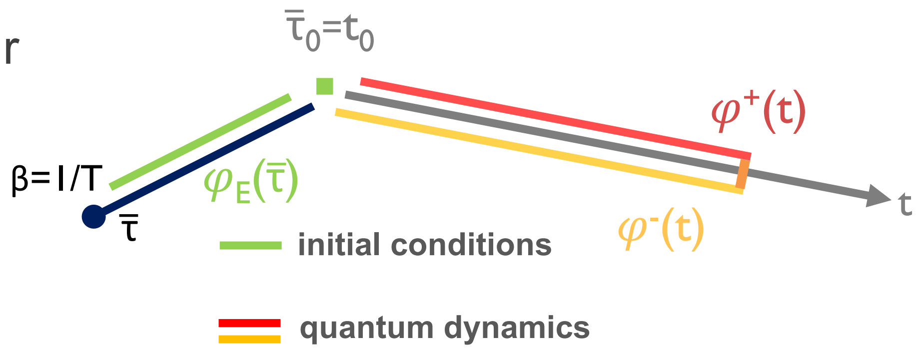

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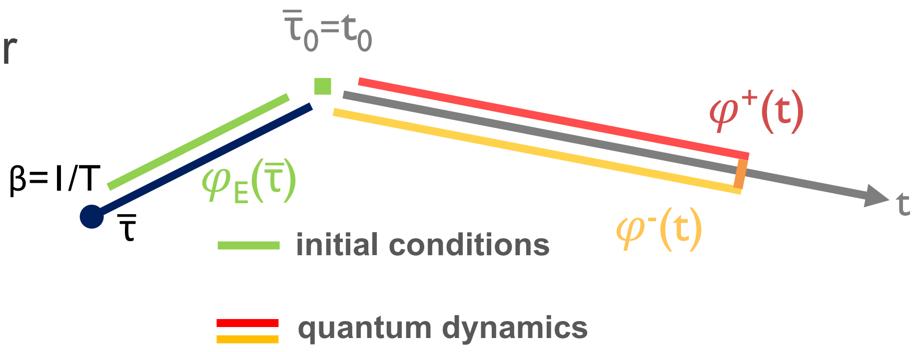

- Thermal equilibrium is special: $G^{++}=\langle\varphi^+\varphi^+\rangle$ correlator alone suffices to compute ρ

$$G^{++}(p^0,\mathbf{p}) = \int \frac{dq^0}{2\pi i} \frac{\rho(q^0,\mathbf{p})}{p^0-q^0+i\epsilon} - n(p^0)\rho(p^0,\mathbf{p})$$

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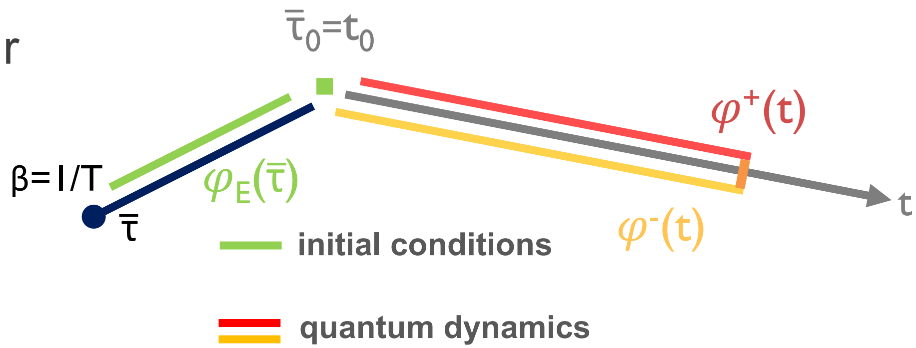

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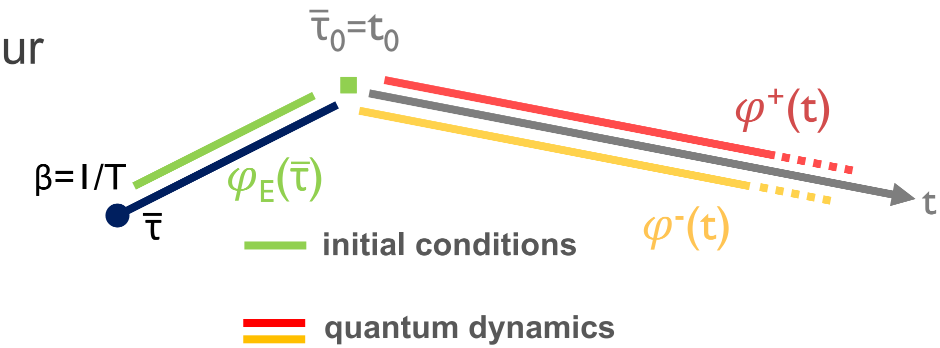

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Analytic continuation

- Our idea: rotate branches of the real-time contour into noncompact Euclidean time

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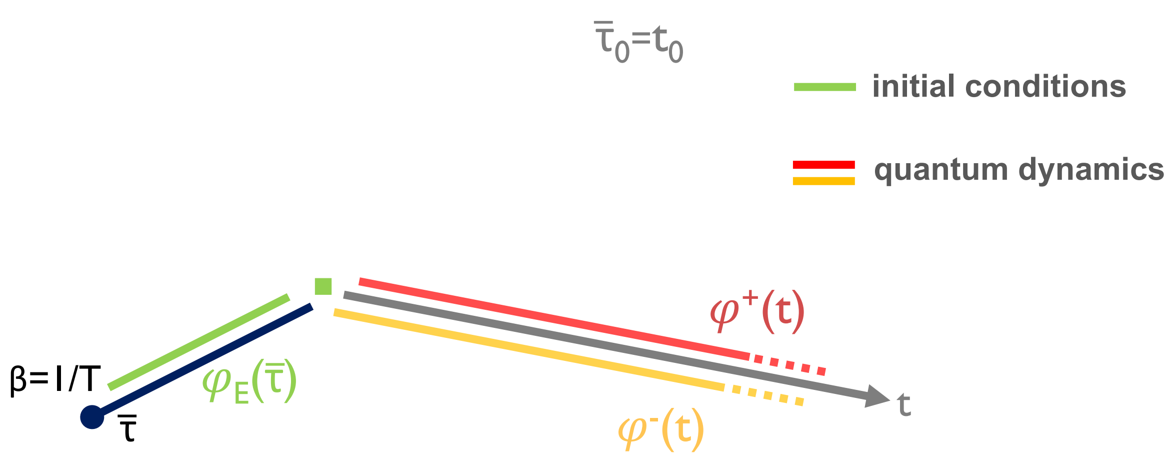

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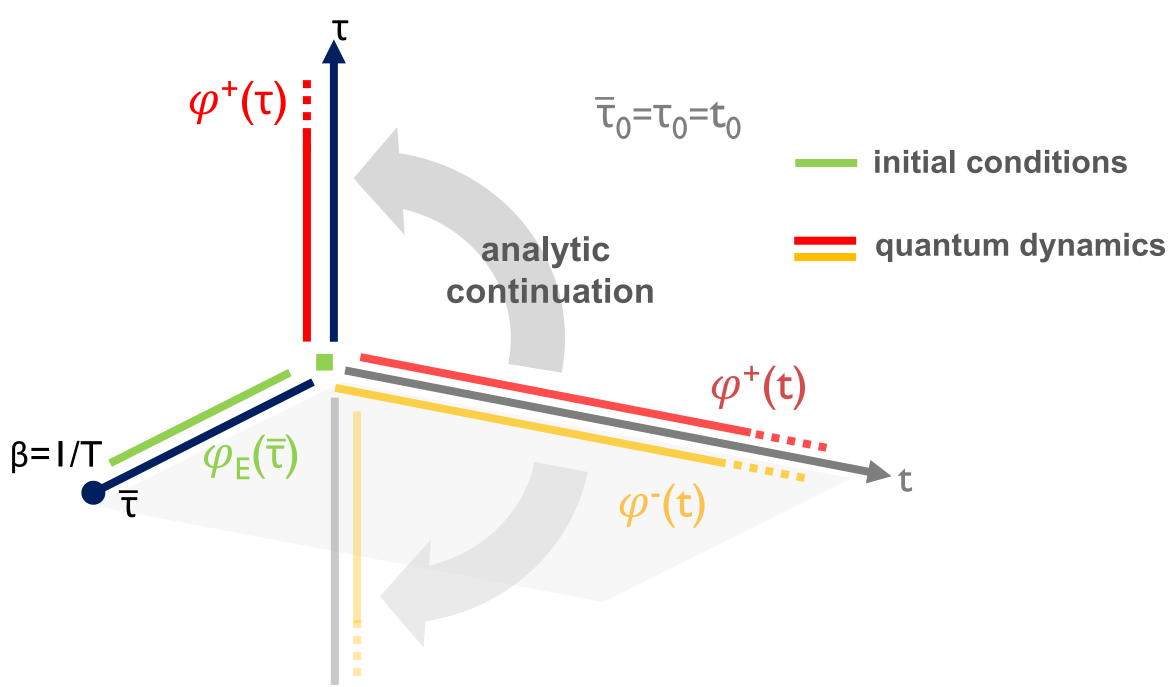

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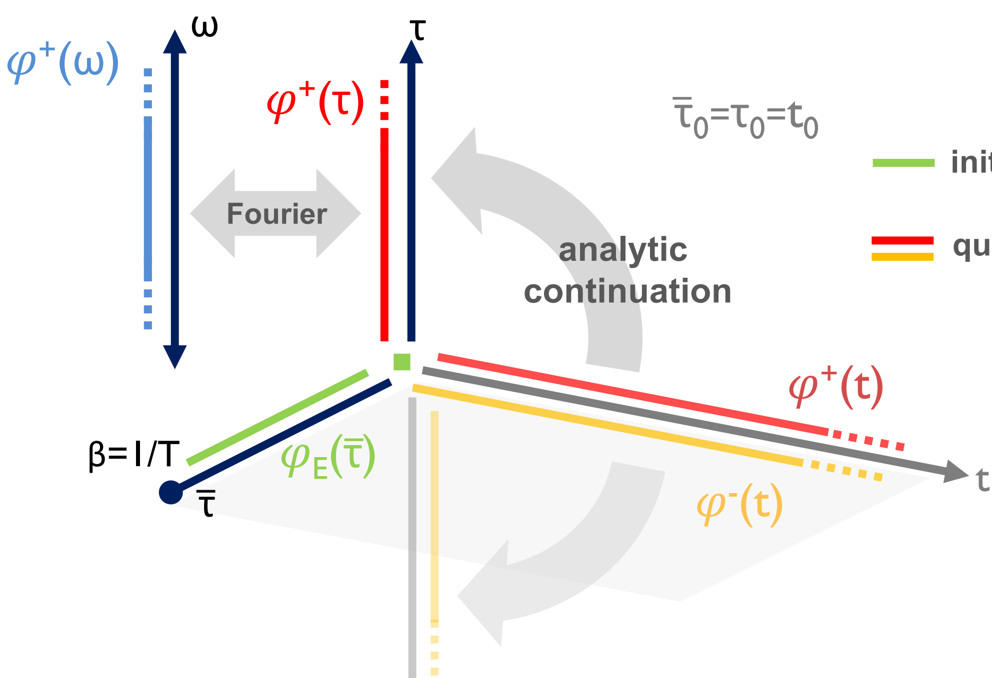

Simulating φ^+ in (0+1) dimensions

- Combine standard simulation of $\varphi_E(\bar{\tau})$ with additional simulation of $\varphi^+(\tau)$

$$S_E = \int d\tau \Big(\underbrace{\frac{1}{2}(\partial_\tau \varphi_E)^2 + \frac{1}{2} m^2 \varphi_E^2}_{S_E^0} + \underbrace{\frac{\lambda}{4!}\varphi_E^4}_{S_E^{int}} \Big)$$

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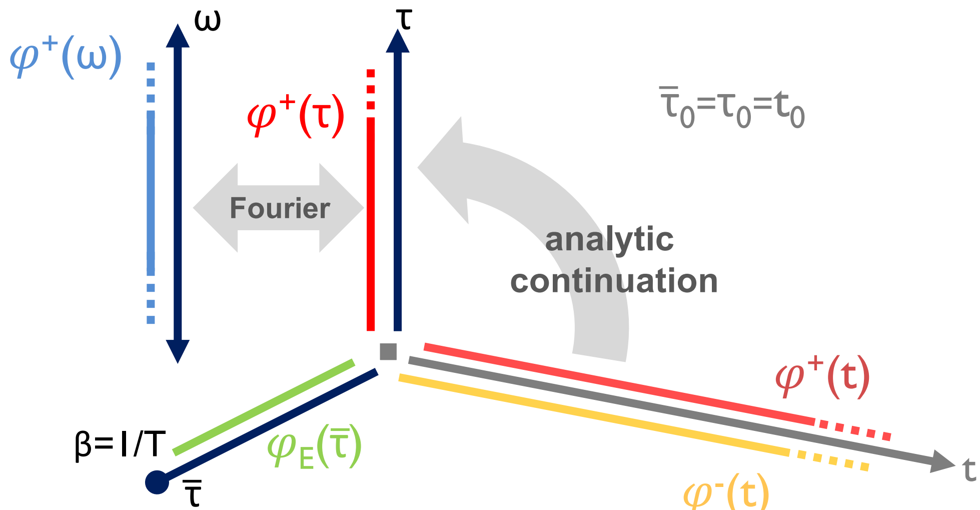

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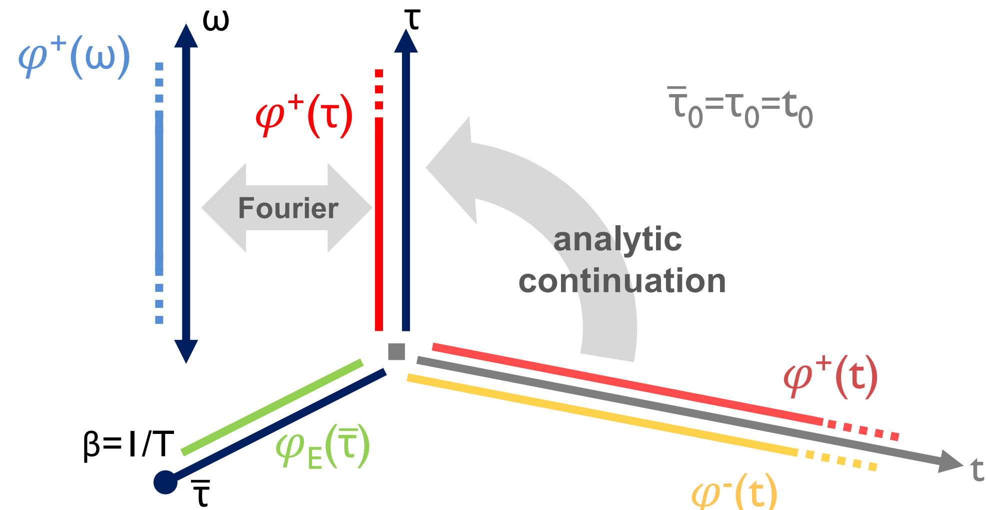

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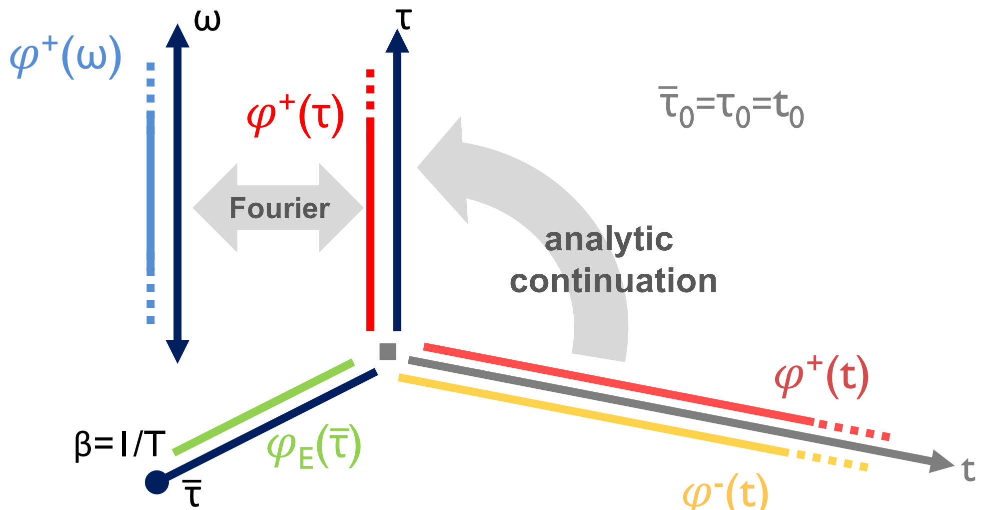
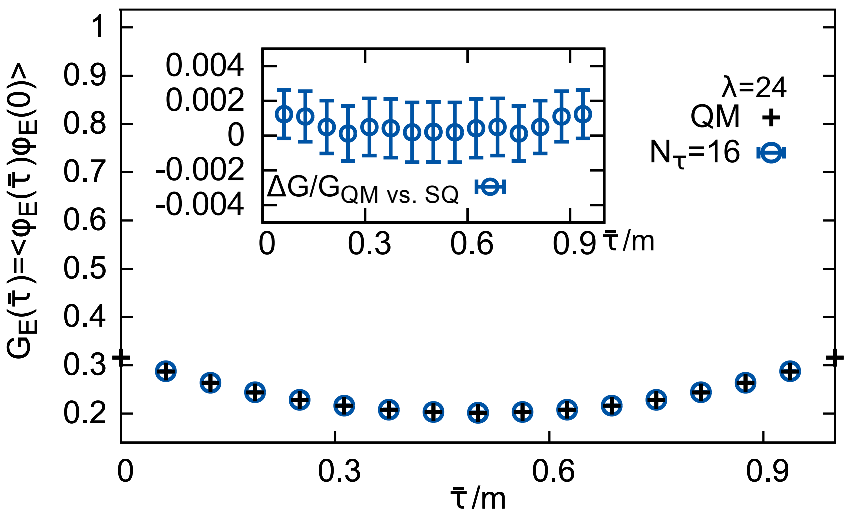
  - T enters only via interaction term, which is local in τ
  - Strategy: when evaluating $\delta S_E^{int}/\delta\varphi^+(\tau)$ replace $\varphi^+(\tau=0) = \varphi_E(0)$
  - Using DFT to change between $\varphi^+(\omega)$ and $\varphi^+(\tau)$ incurrs finite volume artifacts

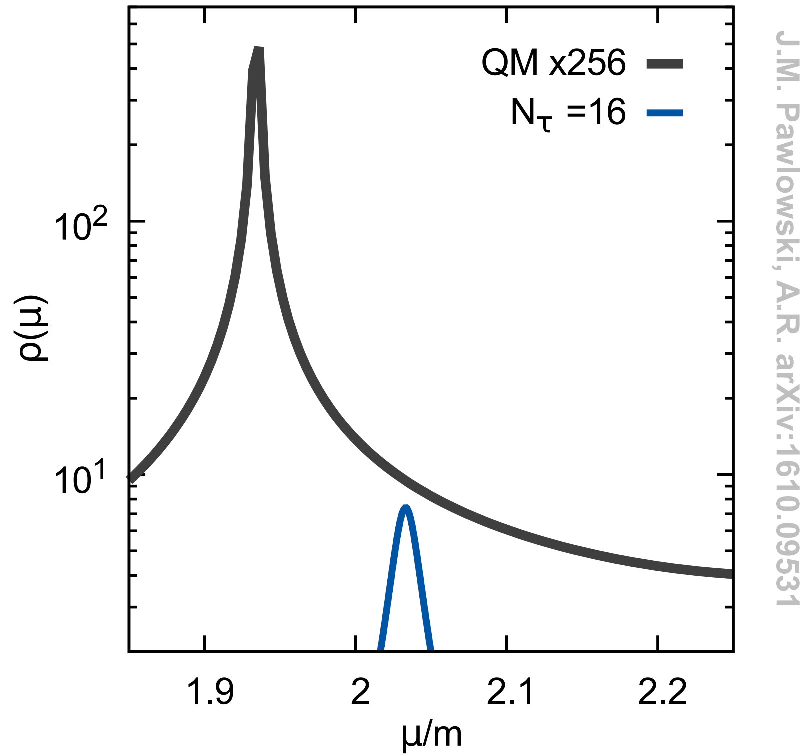


Spectra from Euclidean times

- Standard update for φ_E with $N_\tau=16$ $m=1$ $\lambda=24$ $N_\tau d\tau=1$ (compare to QM of A.H.O.)
- Spectrum from Euclidean time data with $\Delta G_E/G_E=10^{-3}$: peak position still off



Spectral reconstruction via Bayesian BR method. Y.Burnier, A.R. Phys.Rev.Lett. 111 (2013) 182003



Spectra from imaginary frequencies

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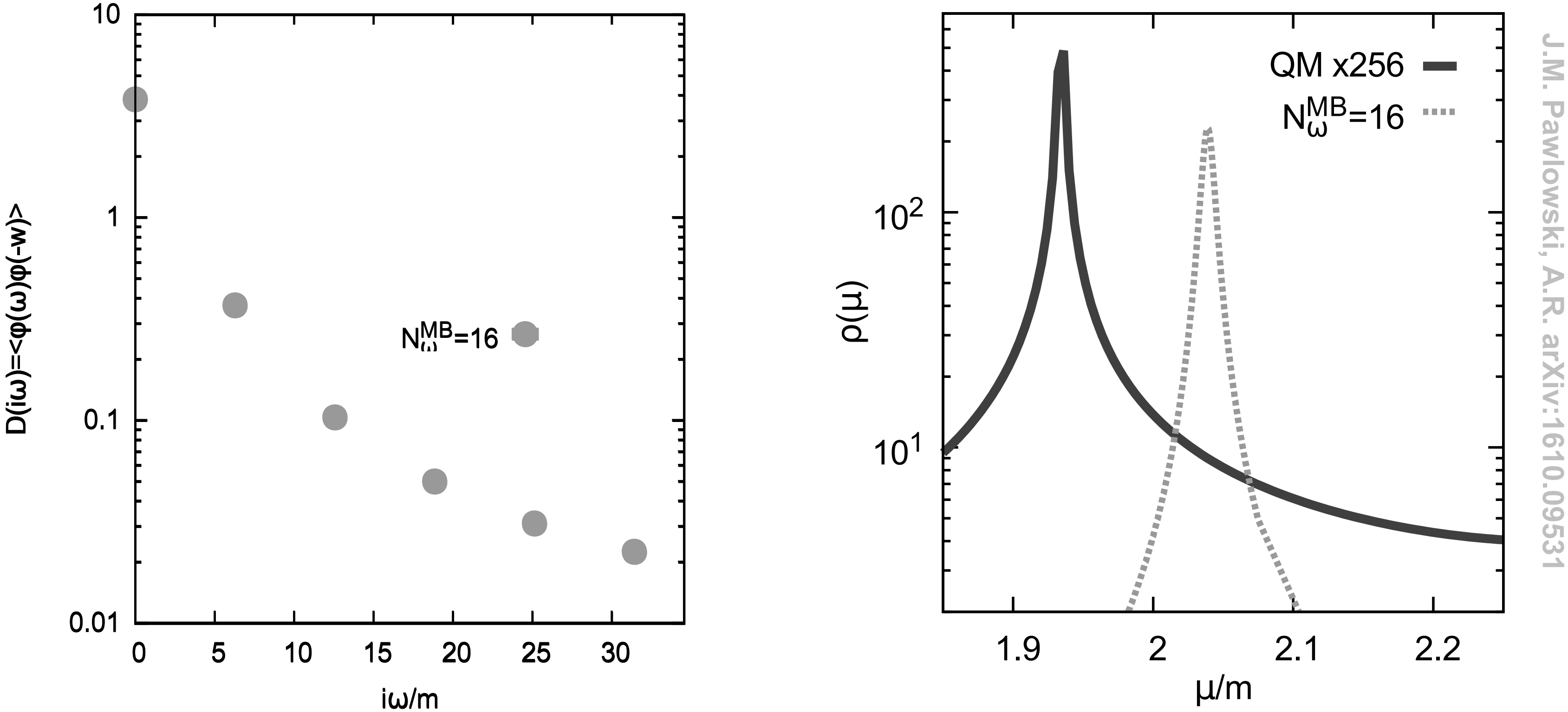

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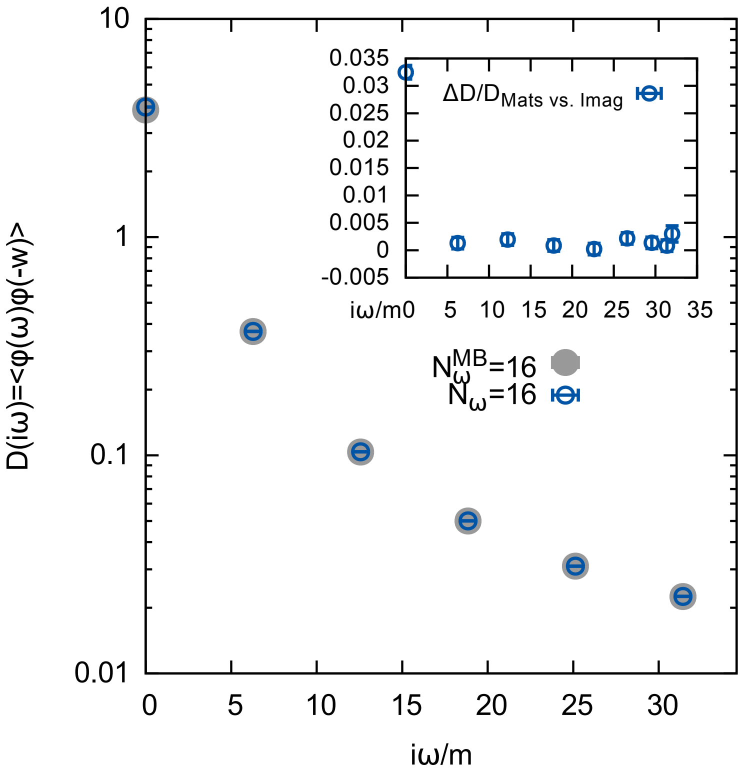

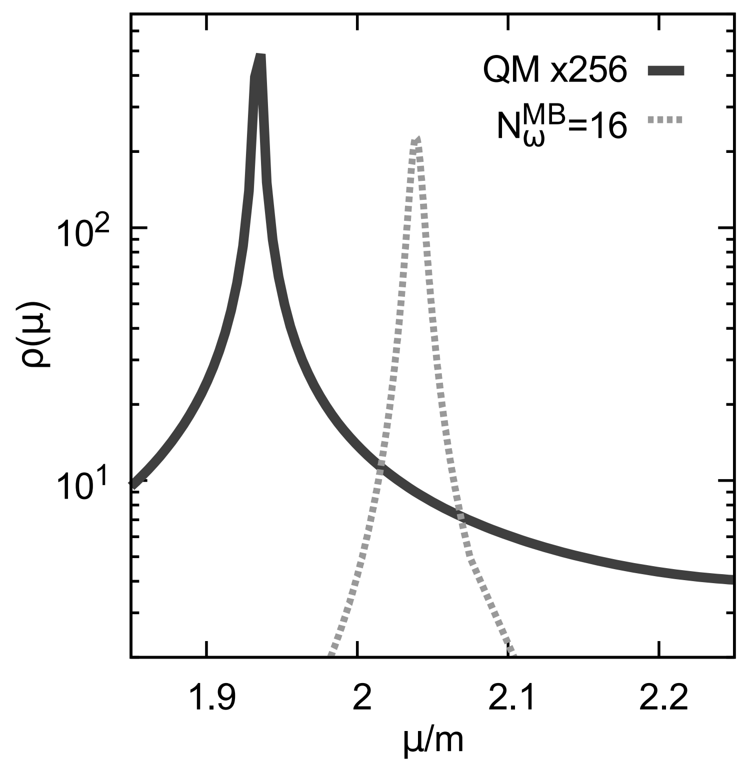

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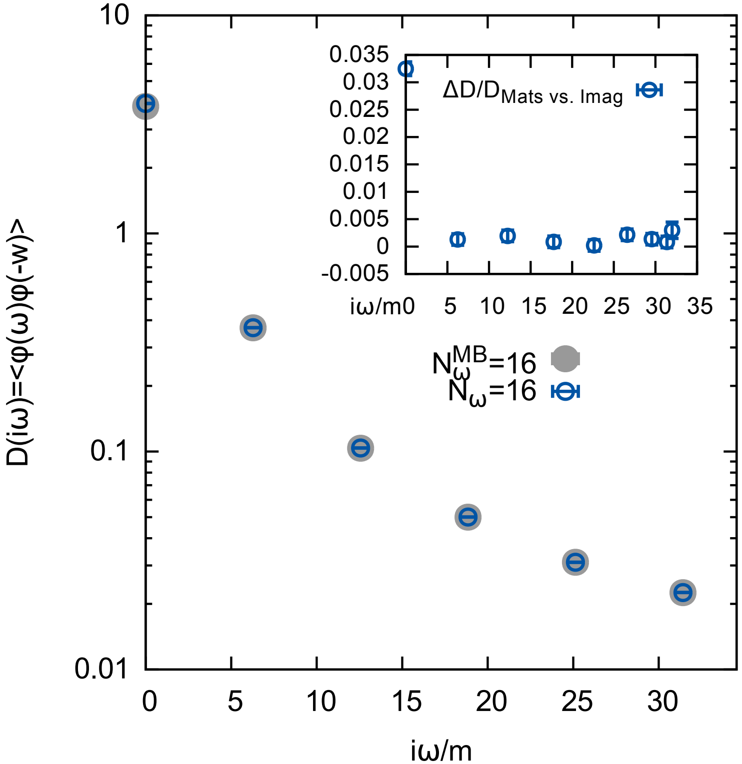

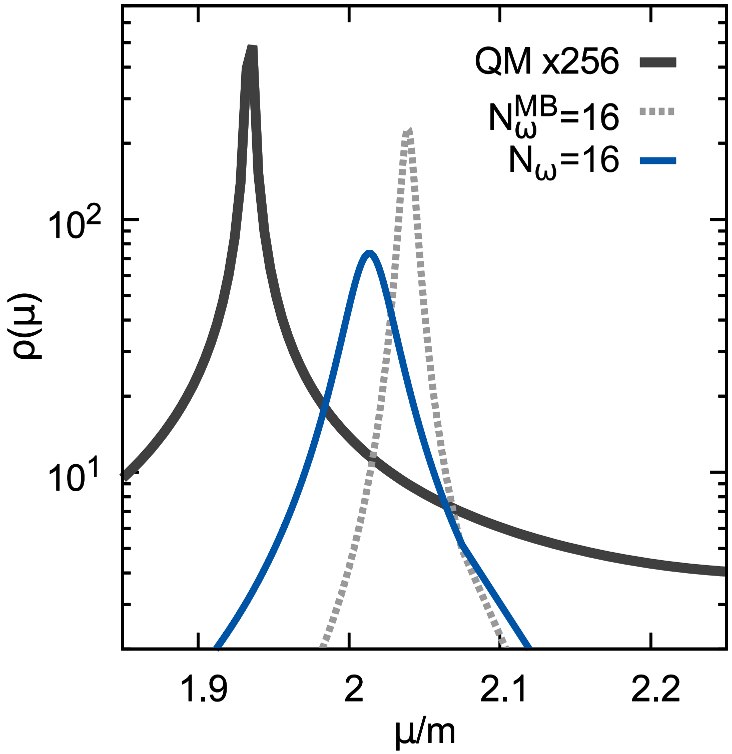

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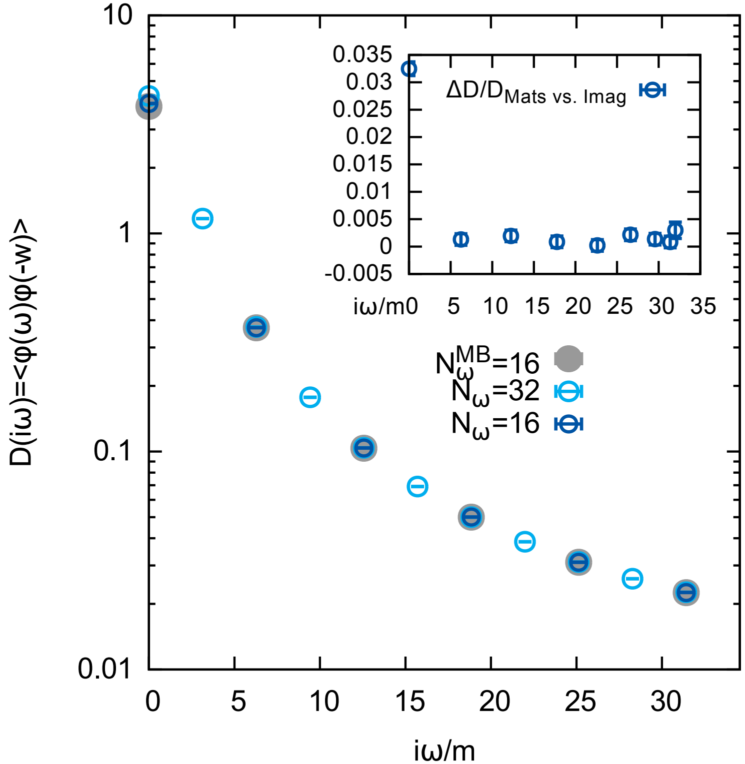

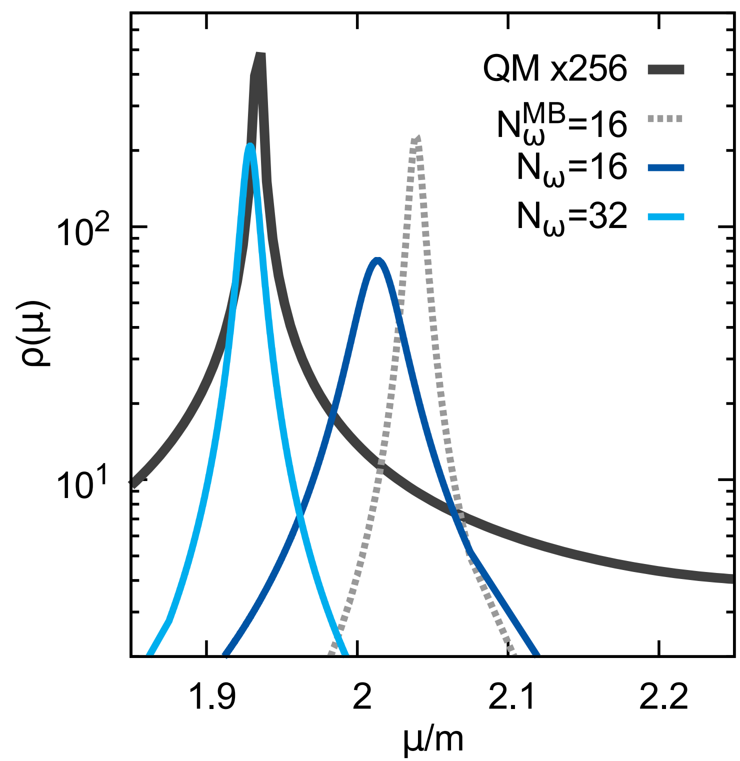

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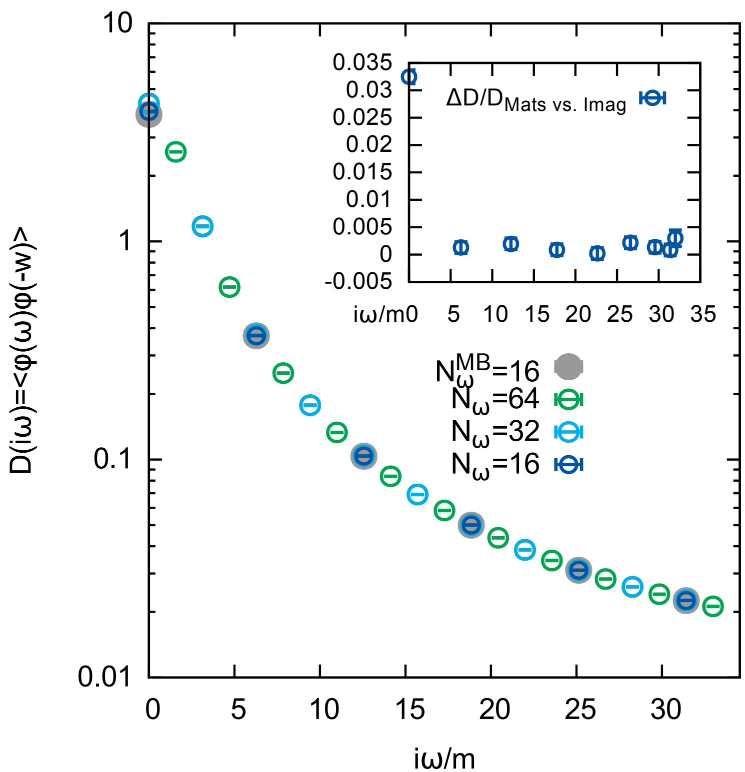

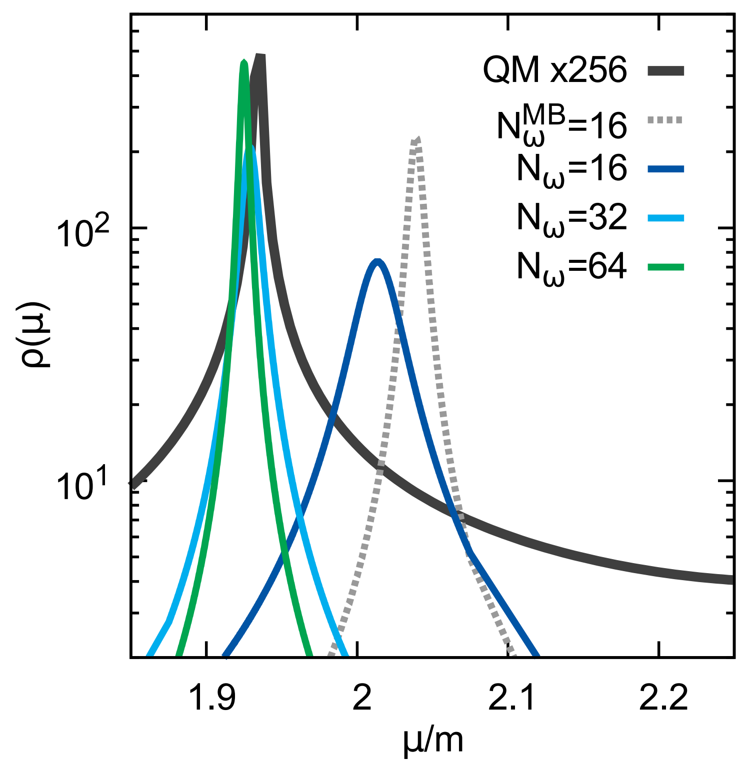

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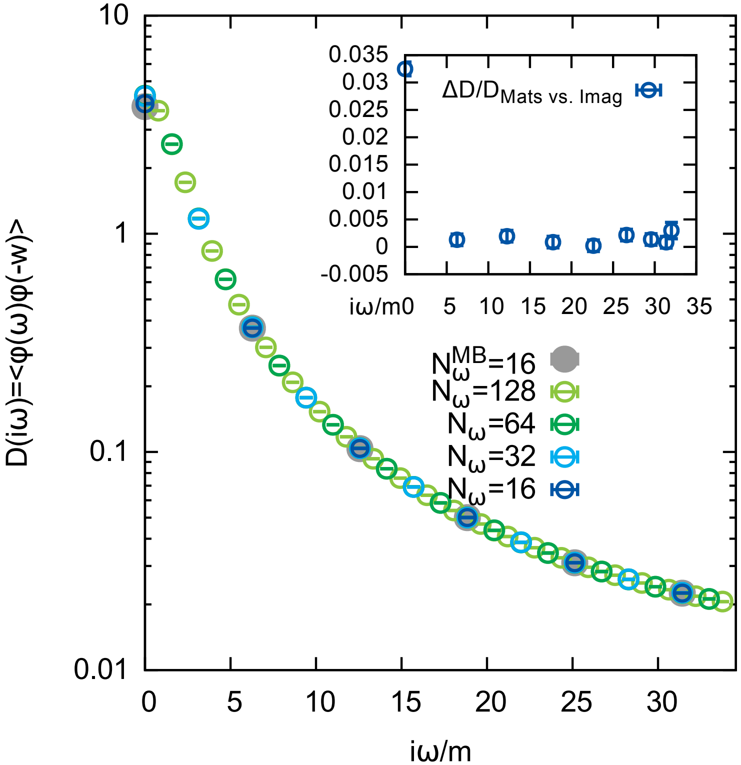

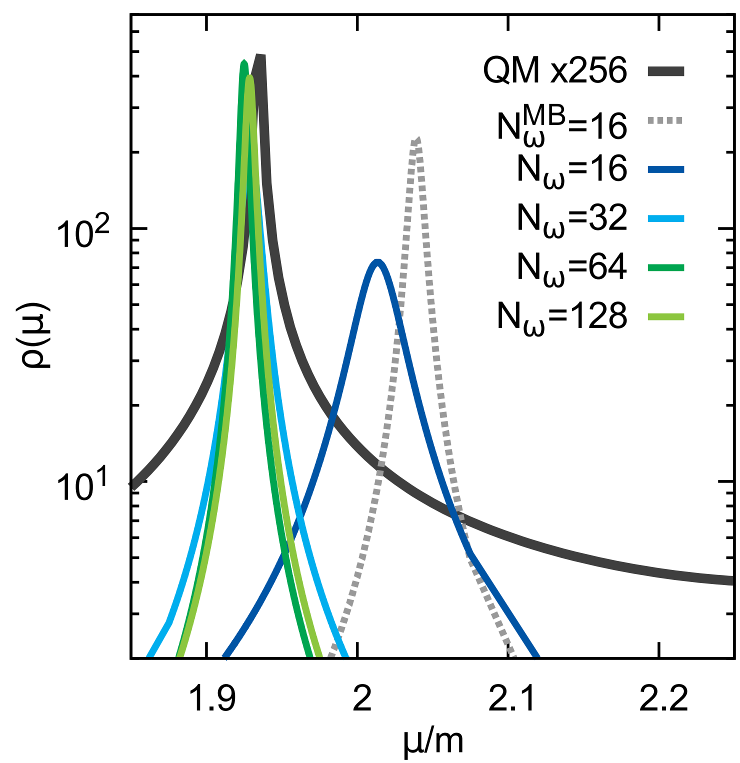

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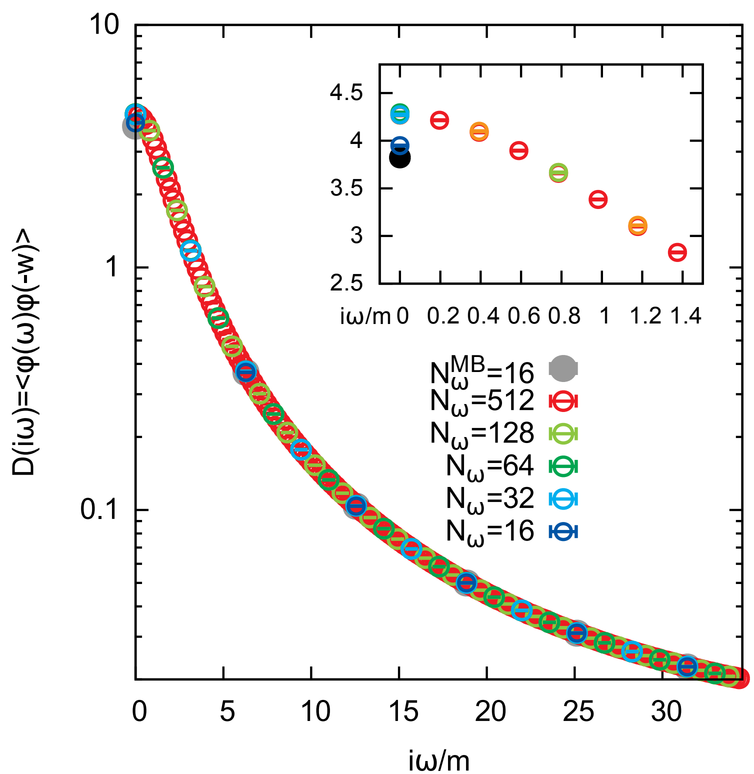

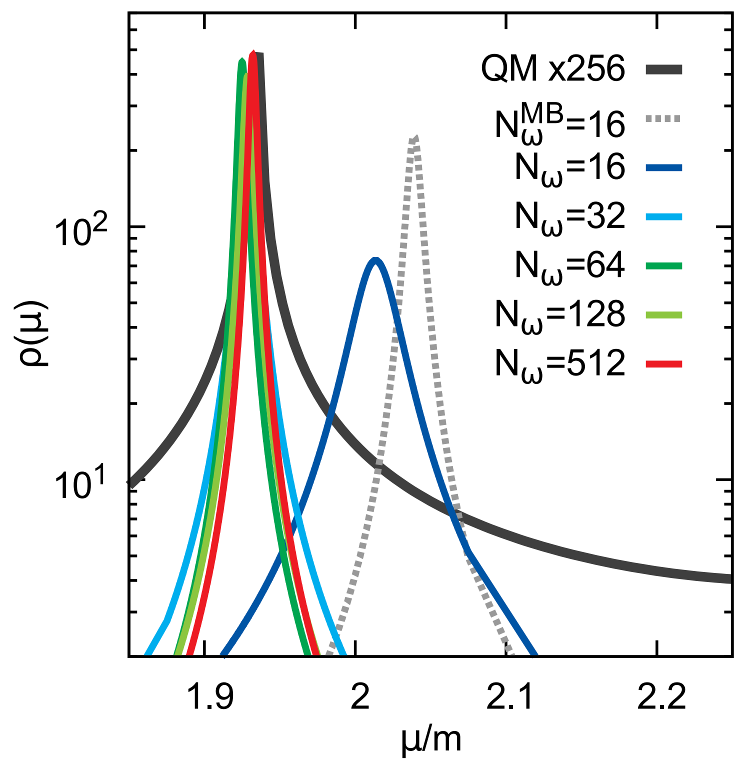

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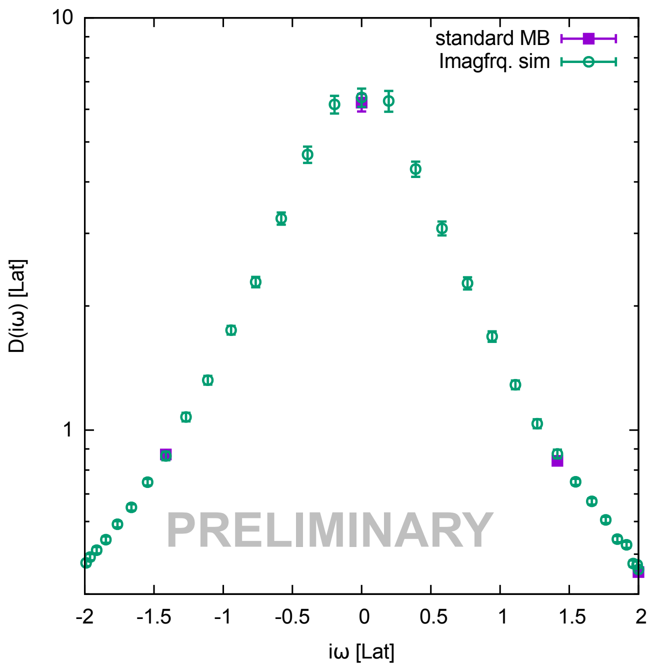
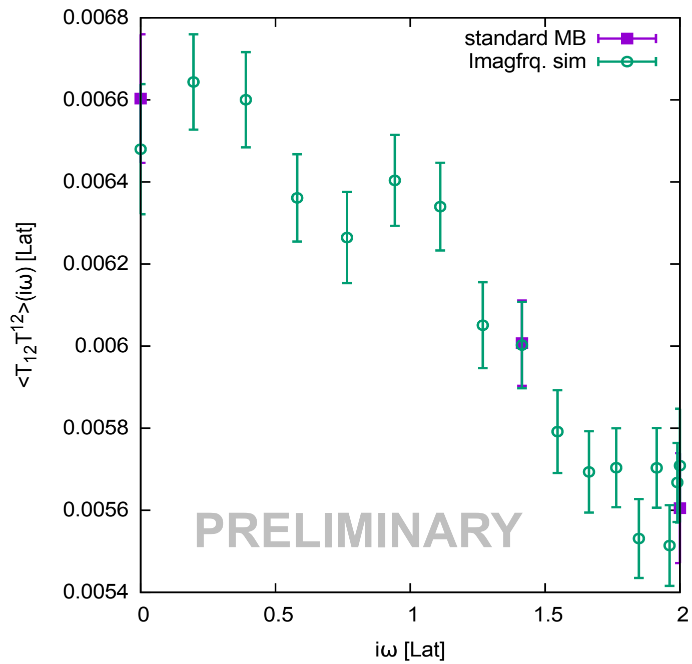
Towards genuine field theories

- Complex scalar field in 3+1d $m^2=0.2$ $\lambda=1$ $d\tau=dx=1$ $N=8$ $N_\tau=4$ $N_\omega=32$





J.M. Pawlowski, A.R. in preparation

- Observables $<\varphi(x)\,\varphi^*(y)>$ and $<T_{12}T^{12}> \approx <(T_{11}-T_{22})\,(T^{11}-T^{22})>$

$$\left(T_{11}-T_{22}\right)(x)=\partial_1\varphi(x)\partial_1\varphi^*(x)-\partial_2\varphi(x)\partial_2\varphi^*(x)$$

- First low statistics runs promising but for gauge theory still conceptual challenges

Summary

- Spectral reconstructions from standard Euclidean simulations exponentially hard
- Relevant physics encoded in correlators away from Matsubara frequencies
- Proposal for a different treatment of thermal fields
  - Consider initial-value problem on the Schwinger-Keldysh contour
  - Rotate the real-time branches into a non-compact Euclidean time
  - Setup mixed representation stochastic quantization for the forward branch since $\rho(\omega)$ can be related to $G^{++}(i\omega)$ via rational integral kernel
- First numerical test in a (0+1)d scalar toy model
  - Agreement of correlators on Matsubara frequencies (except at $\omega=0$ due to finite volume)
  - Reconstructed spectra improve significantly if imaginary frequencies between Matsubara are resolved.
- ***First test*** for complex scalar in 3+1 dim. and ***ongoing efforts*** for gauge fields