Tensor Network study of the (1+1)-dimensional Thirring model

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Motivation

- Test the Tensor-Network methods in (1+1)-dimensional QFT with topology.

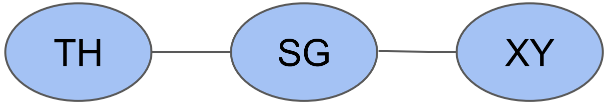
- The Thirring model is dual to the sine-Gordon theory, and the latter is also dual to the 2D classical XY model.

  → we can use the Thirring model to study the details of the topological phases in the latter two.

- Real-time dynamics of the topological phase transitions (Future work).

Plan of this talk

I. Models
   A. The massive Thirring model
   B. Bosonization -> The sine-Gordon model
   C. The 2D classical XY model

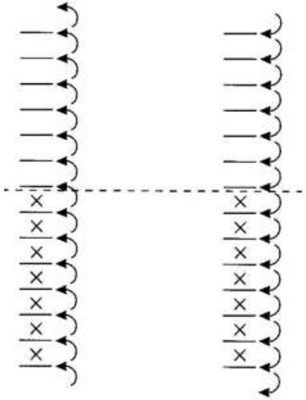


II. Preliminaries on the lattice calculation

III. Tensor Network methods
   A. Matrix Product State (MPS) & Density Matrix Renormalization Group (DMRG)

IV. Preliminary results

The massive Thirring model



- The action (in Euclidean space)

$$S_{Th}[\psi,\bar{\psi}] = -\int d^2x \left[\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + m_0\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma_{\mu}\psi)^2\right]$$

- Chiral anomaly could be presented because

$$exp\left(\int d^2x\left[-\frac{g}{2}(\bar{\psi}\gamma_{\mu}\psi)^2\right]\right) = \frac{1}{\sqrt{2g\pi}}\int \mathscr{D}A_{\mu}\ exp\left(\int d^2x\left[-\frac{1}{2g}A_{\mu}^2 - iA_{\mu}\bar{\psi}\gamma_{\mu}\psi\right]\right)$$

- Vector and axial currents in 2D

$$J_{\mu} = \bar{\psi}\gamma_{\mu}\psi\ ,\ J_{\mu}^{s} = \bar{\psi}\gamma_{\mu}\gamma_{s}\psi$$

are not independent because $\epsilon_{\mu\nu}\gamma^{\mu} = \gamma_{\nu}\gamma_{s}$

➡ Fermion number can change when the chiral symmetry is broken

Bosonization

$$\langle \prod_{i=1}^{n} \mu^2 e^{i\kappa(\phi(x_i)-\phi(x_i'))}\rangle\Big|_{ren.} = (2\pi)^{2n}\langle \prod_{i=1}^{n} \sigma_+(x_i)\sigma_-(x_i')\rangle$$

- The massive Thirring model & the sine-Gordon model

$$S_{Th}[\psi,\bar{\psi}] = -\int d^2x \left[\bar{\psi}\gamma^\mu\partial_\mu\psi + m_0\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2\right]$$

$$S_{SG}[\phi] = \frac{1}{t}\int d^2x \left[\frac{1}{2}(\partial_\mu\phi(x))^2 - \alpha_0 \cos\phi(x)\right]$$

- Kink number / Fermion number

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \epsilon^{0\nu}\partial_\nu\phi \leftrightarrow \int_{-\infty}^{\infty} dx\, \psi^+\psi$$

$$\sum_i \kappa_i = 0 \quad \text{(Neutral condition)}$$

$$\sigma_+(x) = \bar{\psi}_-(x)\psi_+(x)$$
$$\sigma_-(x) = \bar{\psi}_+(x)\psi_-(x)$$
$$\psi_\pm = \frac{1}{2}(1\pm\gamma_s)\psi$$

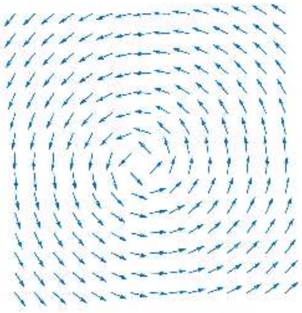
$$\bar{\psi}\gamma_\mu\psi \leftrightarrow \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi$$

$$\bar{\psi}\psi \leftrightarrow \frac{\Lambda}{\pi}cos\phi$$

$$\frac{4\pi}{t} = 1 + \frac{g}{\pi}$$

$$\frac{\alpha_0}{t} = \frac{m_0\Lambda}{\pi}$$

The 2D classical XY model



vortex

- Hamiltonian $$H_{XY} = -K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$
- Berezinskii-Kosterlitz-Thouless (BKT) phase transitions

Mermin-Wagner Theorem -> No phase transition at finite T with short-range interactions in d ≤ 2

	order	$\left\langle \cos(\theta(r) - \theta(0)) \right\rangle$	phase
low T	quasi-long-range	$r^{-T/2\pi K}$	confined
high T	disorder	$\exp(-r/\xi)$	deconfined

- Relations between sine-Gordon and XY

$$S_{XY}[\theta_i] = \beta_T \left[-K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + \epsilon_c \sum_p n^2(p) \right]$$

ϵ_c: core energy of vortex;
$n(p)$: vortex density; p: plaquettes

Summarizing the relations

$$S_{Th}[\psi,\bar{\psi}] = -\int d^2x \left[\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + m_0\,\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma_{\mu}\psi)^2\right]$$

$$\bar{\psi}\gamma_{\mu}\psi \leftrightarrow \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_{\nu}\phi \quad , \quad \bar{\psi}\psi \leftrightarrow \frac{\Lambda}{\pi}cos\phi$$

, and in the zero-charge sector!

$$S_{SG}[\phi] = \frac{1}{t}\int d^2x \left[\frac{1}{2}(\partial_{\mu}\phi(x))^2 - \alpha_0\,\cos\phi(x)\right]$$

$$\partial_{\mu}j^{\mu} = 2\pi n(r) \quad , \quad j^{\mu} = \epsilon^{\mu\nu}\partial_{\nu}\phi$$

$$S_{XY}[\theta_i] = \beta_T\left[-K\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j) + \epsilon_c\sum_{p} n^2(p)\right]$$

Thirring	sine-Gordon	XY
g	$\frac{4\pi^2}{t} - \pi$	$\frac{T}{K} - \pi$
$\frac{m_0\Lambda}{\pi}$	$\frac{\alpha_0}{t}$	$2e^{-\beta_T\epsilon_c}$

Preliminaries on the lattice calculation

- Staggered fermions in the Hamiltonian formalism

$$\psi_u(x) \to \frac{1}{\sqrt{a}} c_{2n}$$

$$\psi_d(x) \to \frac{1}{\sqrt{a}} c_{2n+1}$$

$\Rightarrow$

$$H_{Th}^{(latt.)} = -\frac{i}{2a} \sum_{n=0}^{N-2} \left(c_n^\dagger c_{n+1} - c_{n+1}^\dagger c_n \right) + m_0 \sum_{n=0}^{N-1} (-1)^n c_n^\dagger c_n + \frac{2g}{a} \sum_{n=0}^{\frac{N}{2}-1} c_{2n}^\dagger c_{2n} c_{2n+1}^\dagger c_{2n+1}$$

- Jordan-Wigner transformation

$$c_n = exp\left(\pi i \sum_{j=1}^{n-1} S_j^z \right) S_n^-$$

$$c_n^\dagger = S_n^+ \, exp\left(-\pi i \sum_{j=1}^{n-1} S_j^z \right)$$

$\Rightarrow$

$$H_{spin} = -\frac{1}{2a} \sum_n \left(S_n^+ S_{n+1}^- + S_{n+1}^+ S_n^- \right) + m_0 \sum_n (-1)^n \left(S_n^z + \frac{1}{2} \right) + \frac{2g}{a} \sum_n \left(S_{2n}^z + \frac{1}{2} \right) \left(S_{2n+1}^z + \frac{1}{2} \right)$$

The spin model

- Hamiltonian

$$H_{spin} = -\frac{1}{2a}\sum_n \left(S_n^+ S_{n+1}^- + S_{n+1}^+ S_n^-\right) + m_0 \sum_n (-1)^n \left(S_n^z + \frac{1}{2}\right) + \frac{2g}{a}\sum_n \left(S_{2n}^z + \frac{1}{2}\right)\left(S_{2n+1}^z + \frac{1}{2}\right)$$

- The penalty term

$$H_{spin}^{(penalty)} = H_{spin} + \lambda\left(\sum_{n=0}^{N-1} S_n^z - S_{target}\right)^2$$

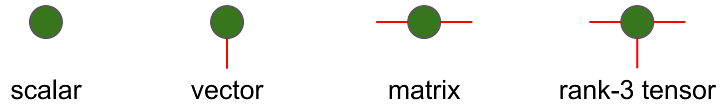
- kink number / fermion number / total Sz

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \epsilon^{0\nu}\partial_\nu \phi \leftrightarrow \int_{-\infty}^{\infty} dx\, \psi^\dagger \psi \to \sum_n c_n^\dagger c_n = \sum_n \left(S_n^z + \frac{1}{2}\right)$$

- cosine operator / chiral condensate / staggered total Sz

$$\frac{\Lambda}{\pi} cos\phi \leftrightarrow \bar{\psi}\psi \to \sum_n (-1)^n c_n^\dagger c_n = \sum_n (-1)^n \left(S_n^z + \frac{1}{2}\right)$$

Tensor Network 101

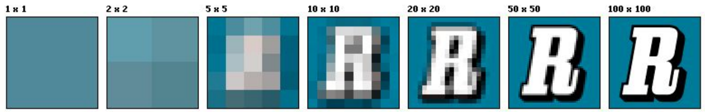


In short: make a low rank approximation on the many-body Hilbert space

- Singular Value Decomposition (SVD)

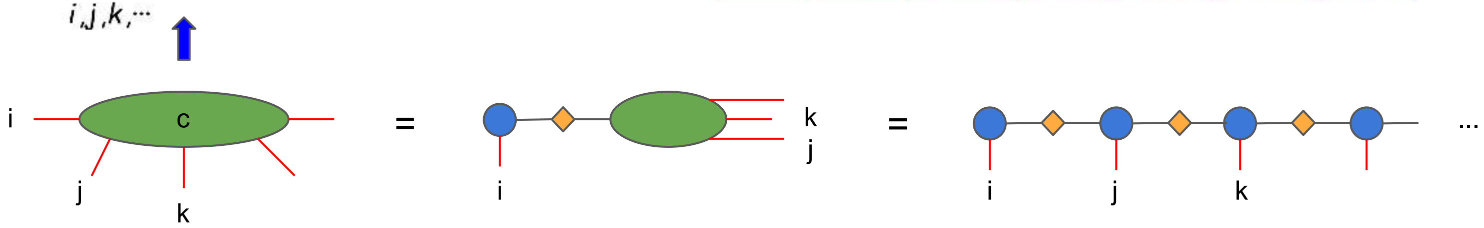
$$M_{m\times n} = U_{m\times r} S_{r\times r} V^{\dagger}_{r\times n}, r = min(m, n)$$

$$S = \begin{pmatrix} 0.909 & 0 & 0 & 0 \\ 0 & 0.326 & 0 & 0 \\ 0 & 0 & 0.256 & 0 \\ 0 & 0 & 0 & 0.026 \end{pmatrix}$$



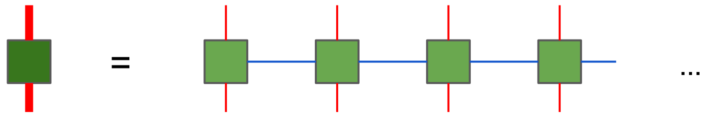
- Matrix Product State (MPS)

$$|\psi\rangle = \sum_{i,j,k,\cdots} \underline{c_{ijk\cdots}} |ijk\cdots\rangle$$



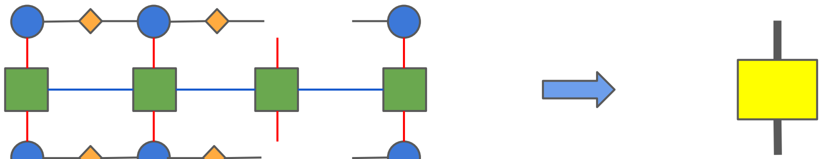
Tensor Network 101

- Matrix Product Operator (MPO)



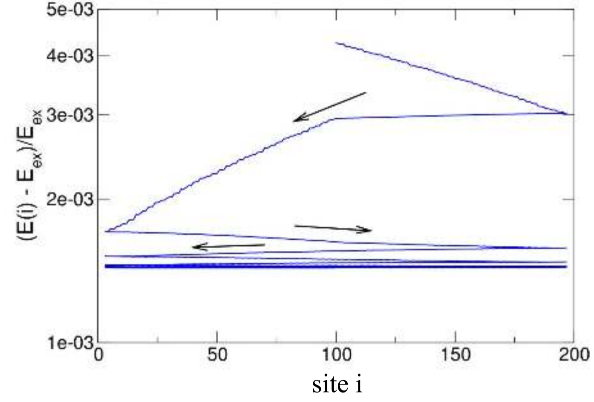
$$M_n = \begin{pmatrix} \mathbb{1} & -\frac{1}{2a}S^+ & -\frac{1}{2a}S- & 2\lambda S^z & \frac{2g}{a}S^z\,\delta_{0,n\%2} & \beta_n S^z + \gamma\mathbb{1} \\ 0 & 0 & 0 & 0 & 0 & S^- \\ 0 & 0 & 0 & 0 & 0 & S^+ \\ 0 & 0 & 0 & \mathbb{1} & 0 & S^z \\ 0 & 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

- Density Matrix Renormalization Group (DMRG)

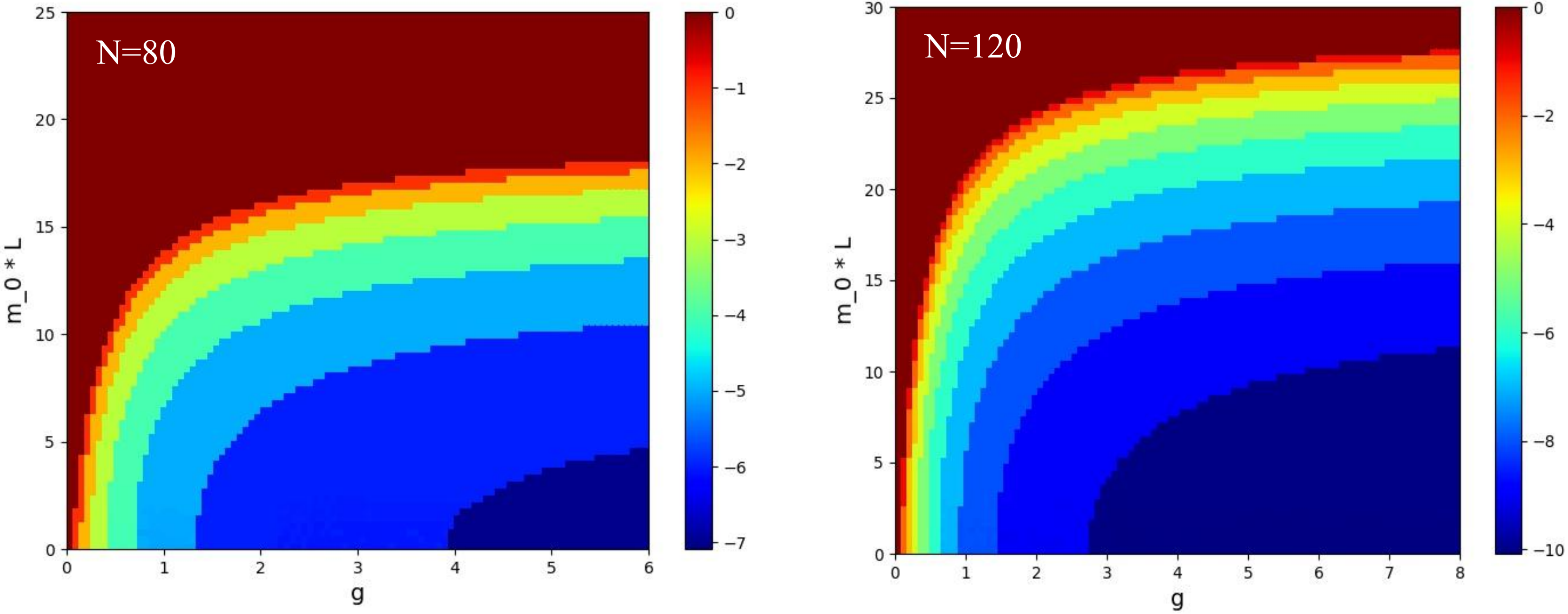


Variationally optimize each site by minimizing the energy

Michael Sekania (2011)



Preliminary results for $\langle 0|\sum_n S_n^z|0\rangle$ (~ Fermion number)



Summary & Future works

- The change of fermion number (total Sz) was observed without the penalty term.
- The investigation towards BKT phase transitions requires the penalty term to constrain the fermion number being within zero-charge sector (under progress).

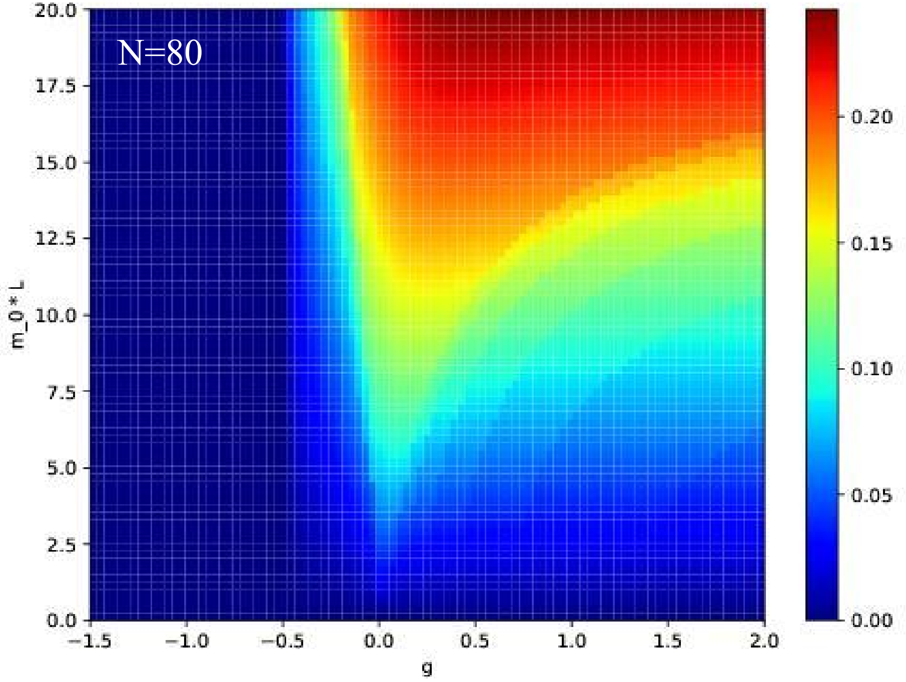
$$\langle \prod_{i=1}^{n} \mu^2 e^{i\kappa(\phi(x_i)-\phi(x_i'))}\rangle\Big|_{ren.} = (2\pi)^{2n}\langle \prod_{i=1}^{n} \sigma_+(x_i)\sigma_-(x_i')\rangle \quad , \quad \sum_i \kappa_i = 0 \quad \text{(Neutral condition)}$$

- Extrapolate the results to the full Hilbert space and the thermodynamic limit.
- Investigate what would happen if we start a quantum quench from one topological sector to another (future work).

Thank you for your attention!

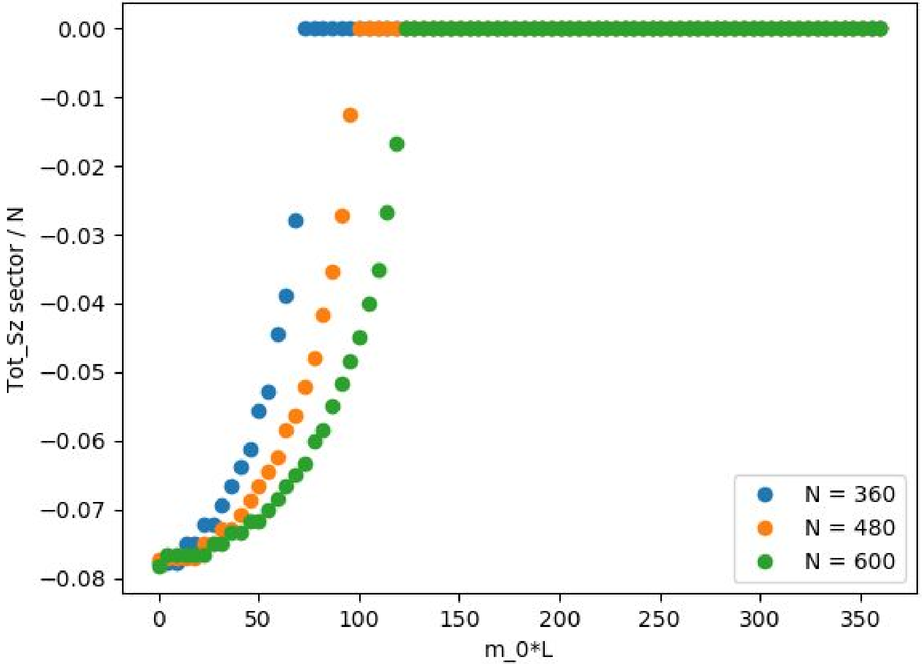
Backup

Backup - Chiral condensate



(Preliminary)

Backup - Finite size scaling to the total Sz sector



(Preliminary)