THE UNIVERSITY of ADELAIDE

SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

CoEPP
ARC Centre of Excellence for Particle Physics at the Terascale

Partonic structure from the Compton amplitude

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for QCDSF/UKQCD/CSSM
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Parton distributions from lattice QCD

Thanks Del Debbio & others for introduction

Lattice problem:
Historically limited to lowest moments in Bjorken x

Recent: Quasi PDFs

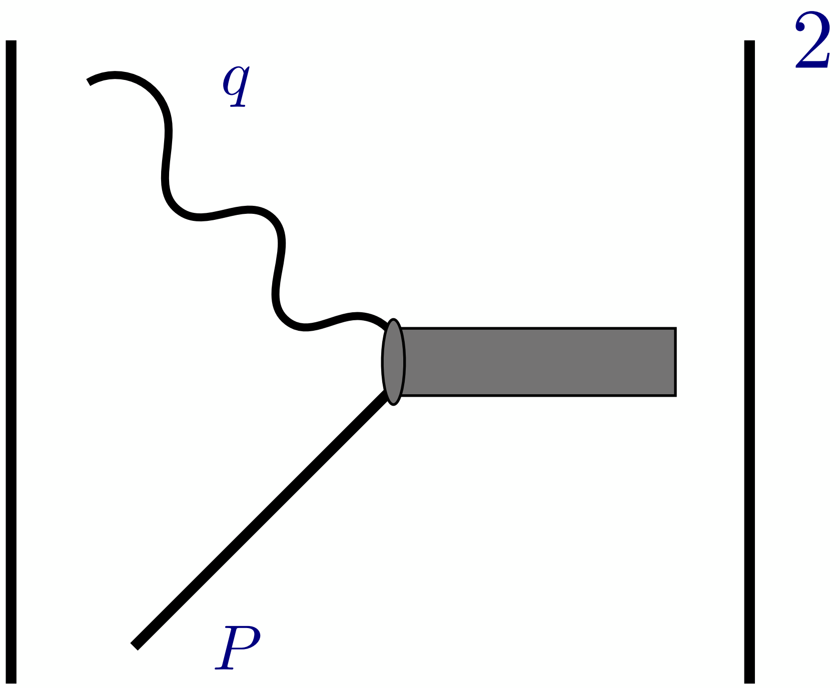
Here: Compton amplitude in unphysical region

Extracting the Compton amplitude on the lattice

Constraining the x dependence of PDFs

First: Hadron tensor and PDFs

Inelastic scattering

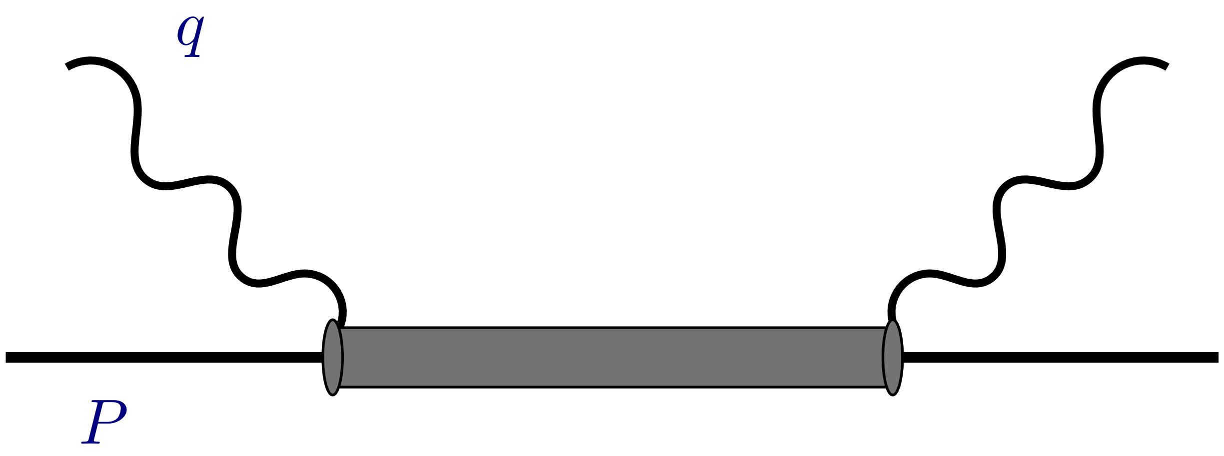


Cross section ∝ Hadron tensor

$$W_{\mu\nu} \sim \int d^4x \langle p|[J_\mu(x), J_\nu(0)]|p\rangle$$

Structure functions $F_{1,2}(P.q, Q^2)$

$$\boxed{F_i = \frac{1}{2\pi}\mathrm{Im}\, T_i}$$



Forward Compton amplitude

$$T_{\mu\nu} \sim \int d^4x \langle p|\mathrm{T} J_\mu(x) J_\nu(0)|p\rangle$$

Structure functions $T_{1,2}(P.q, Q^2)$

(Virtual) Compton amplitude

- Compton amplitude

$$T_{\mu\nu}(p,q) = \rho_{ss'} \int d^4x \langle p,s' | \mathrm{T} J_\mu(x) J_\nu(0) | p,s\rangle$$

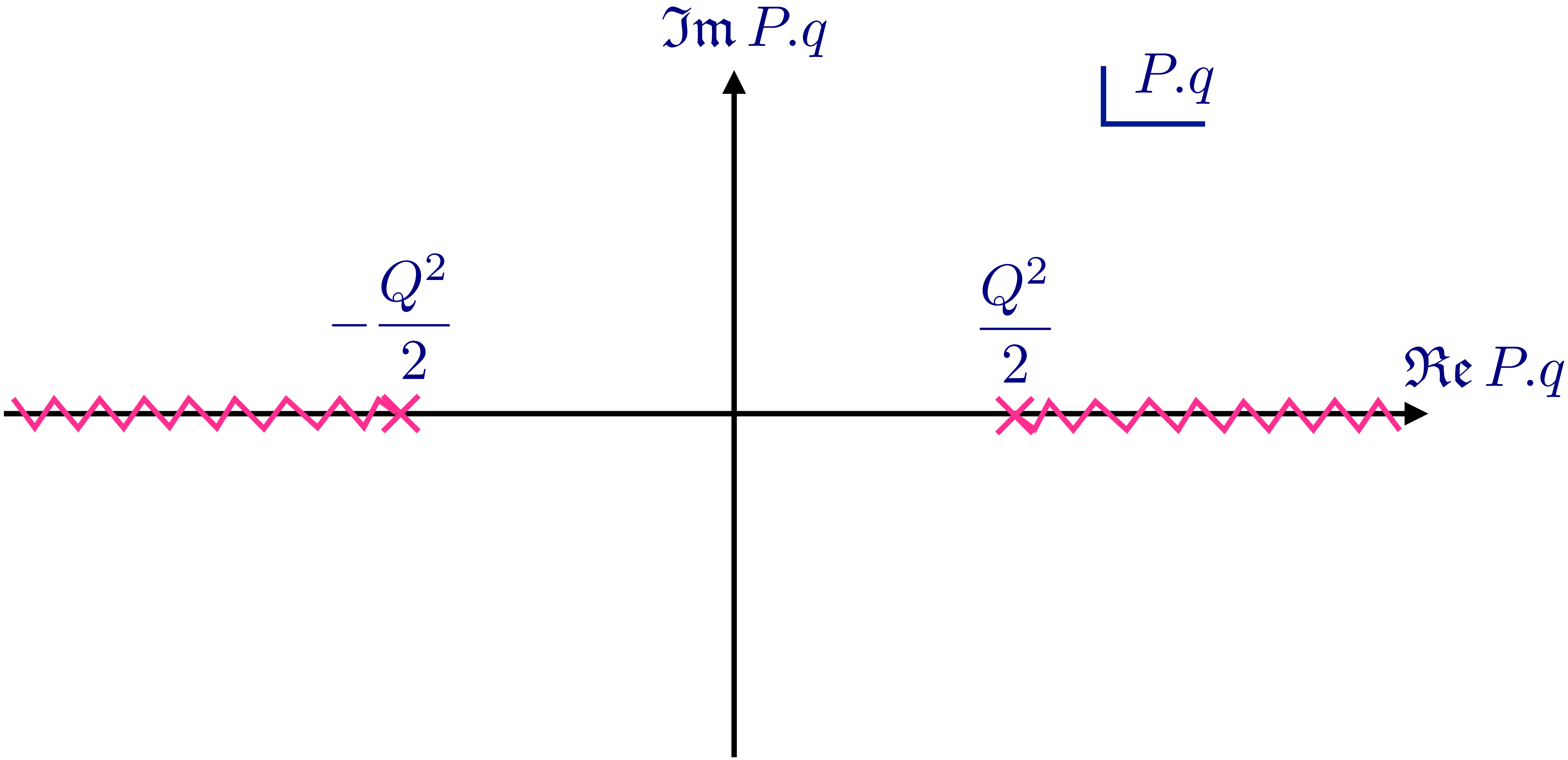
$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(P.q, Q^2) + \frac{1}{P.q}\left(p_\mu - \frac{P.q}{q^2} q_\mu\right)\left(p_\nu - \frac{P.q}{q^2} q_\nu\right) T_2(P.q, Q^2)$$

- Looking ahead to lattice results shown at end, consider simple case

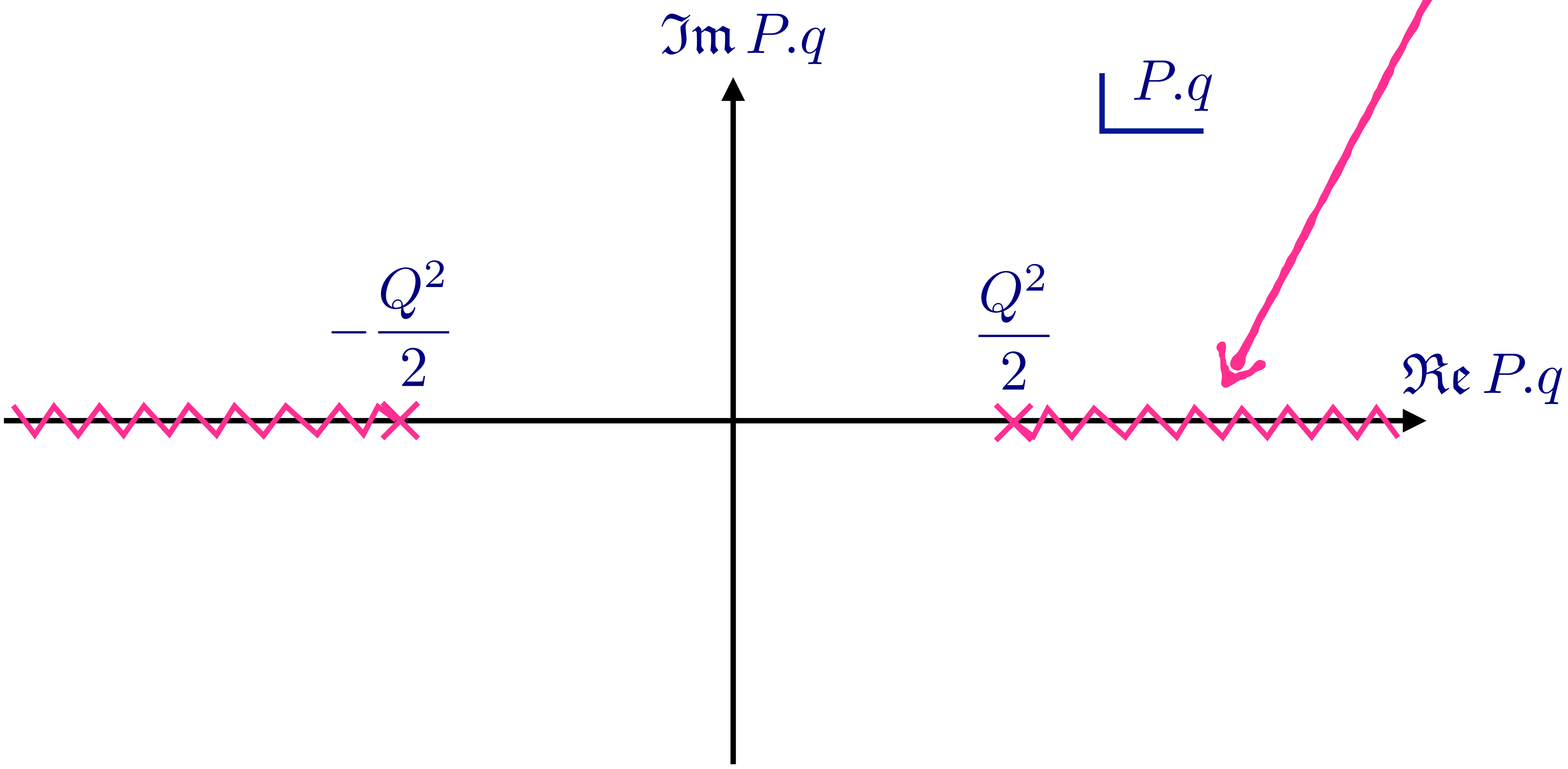
$$\mu = \nu = 3,\ \ q_3 = 0,\ \ P_3 = 0$$

$$\Rightarrow T_{33}(P,q) = T_1(P.q, Q^2)$$

Analytic structure of Compton amplitude



Analytic structure of Compton amplitude

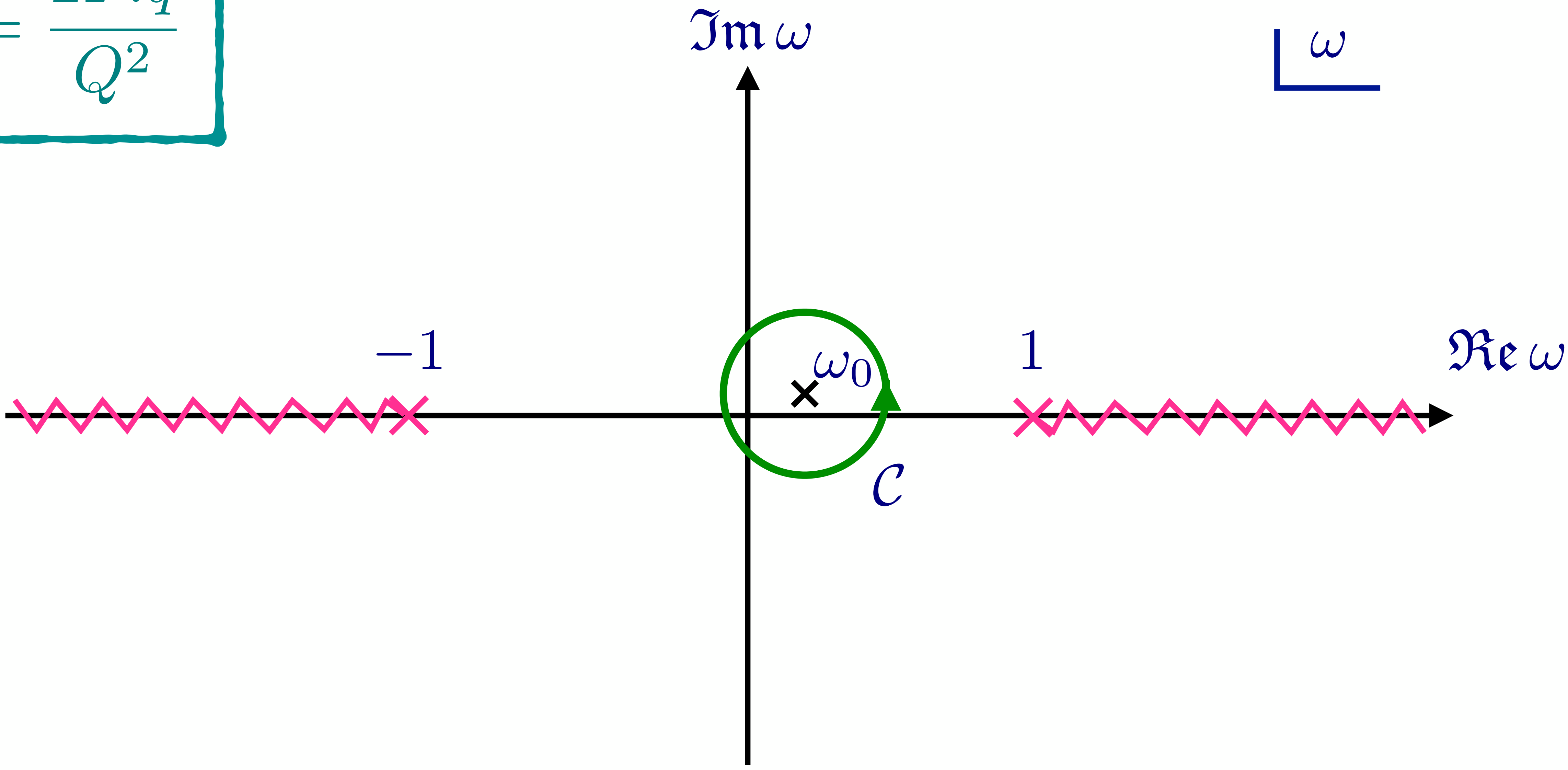


Physical scattering region

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$

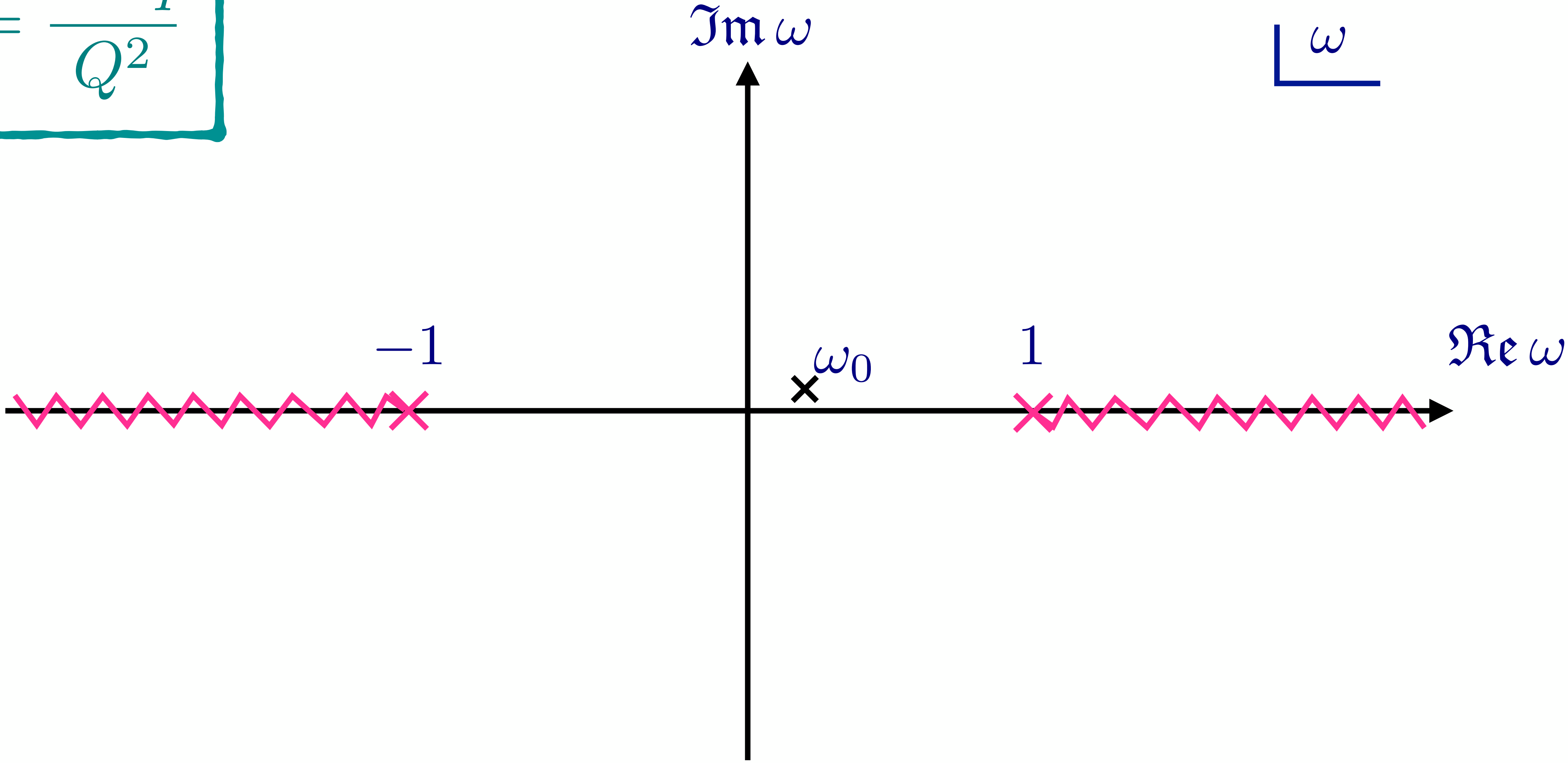


$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i} \oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$

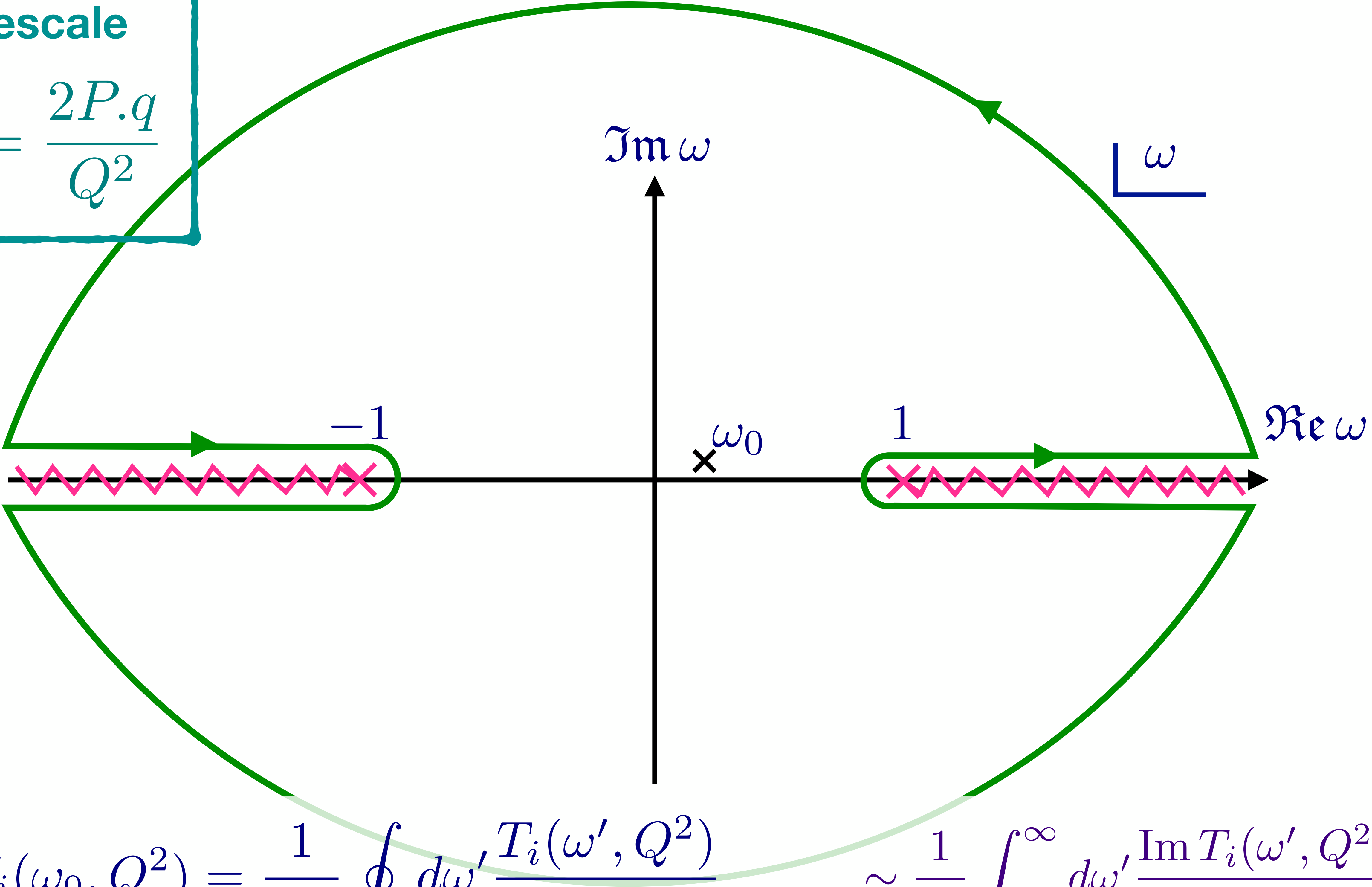


$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0} \qquad \sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\, T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$

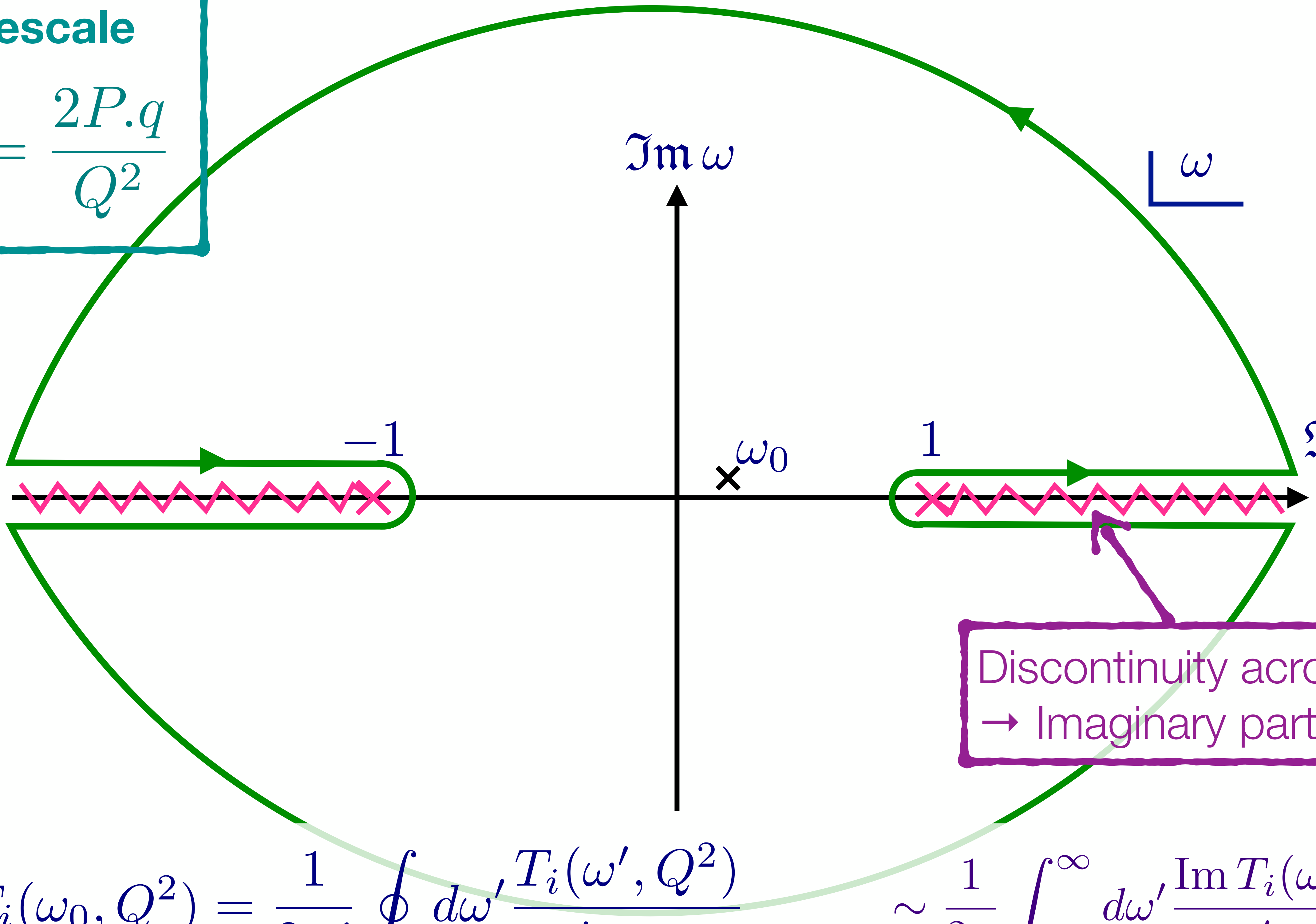


$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0} \sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\, T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$



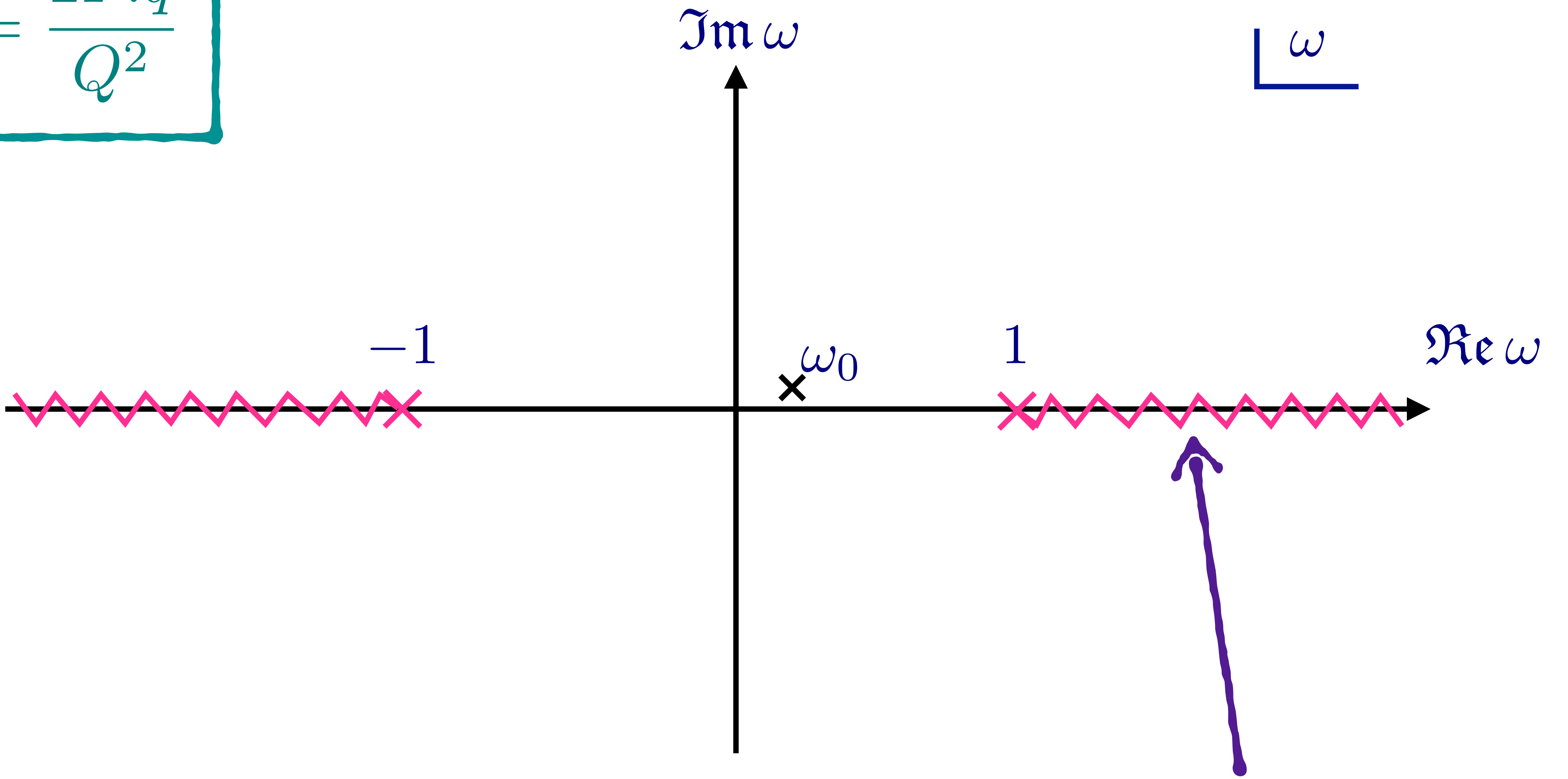
Discontinuity across cut
→ Imaginary part

$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0} \sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\,T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$



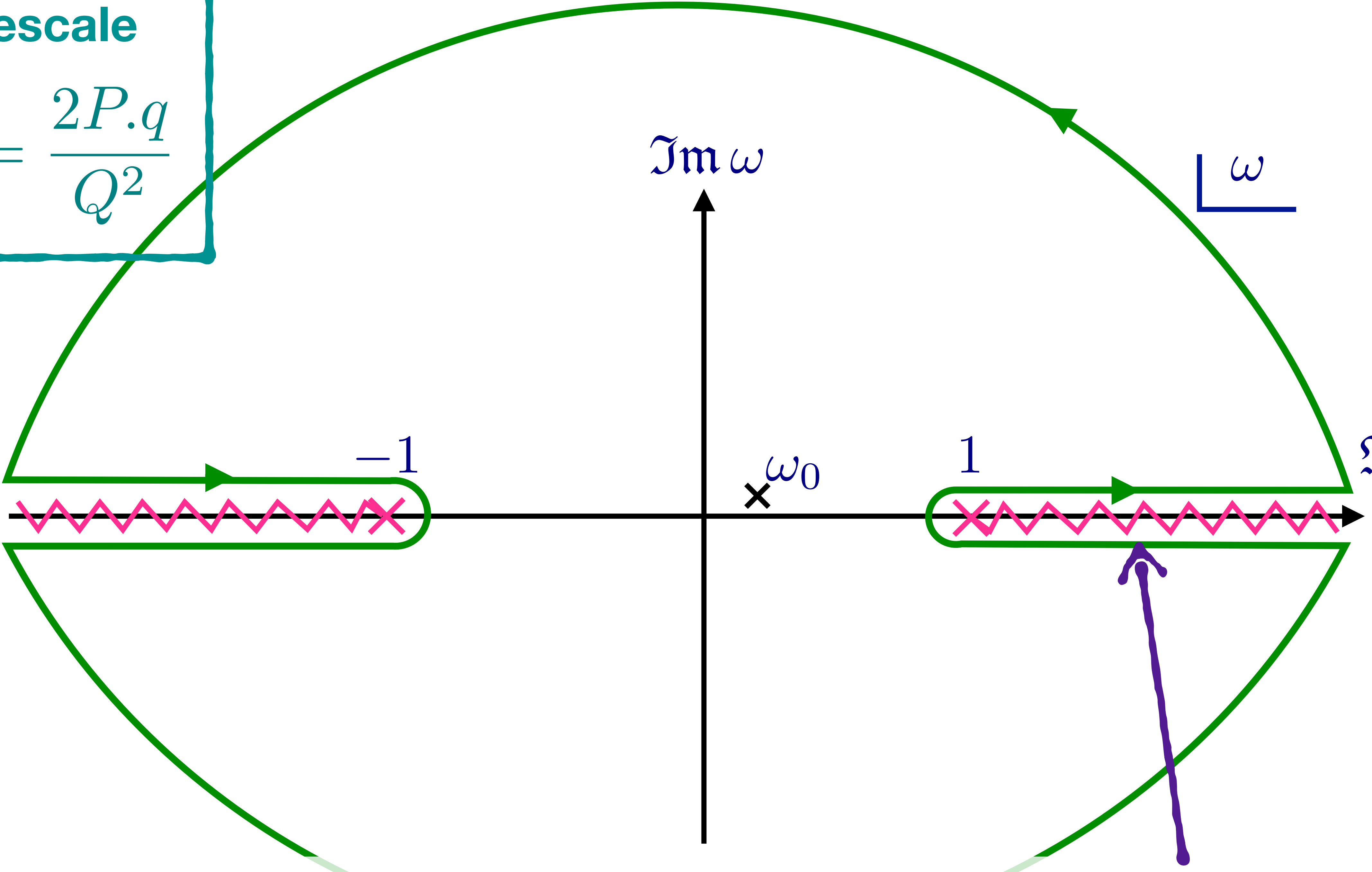
$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0}$$

$$\sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\, T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$



$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0} \sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\,T_i(\omega', Q^2)}{\omega' - \omega_0}$$

PDF moments & dispersion integrals

- Re-express integral over familiar Bjorken x, e.g.

“Actual PDF”

$$T_1(\omega,Q^2)-T_1(\omega,0)=\frac{4\omega^2}{2\pi}\int_1^\infty d\omega'\,\frac{\mathrm{Im}\,T_1(\omega',Q^2)}{\omega'(\omega^2-\omega'^2)}=4\omega^2\int_0^1 dx\,x\,\frac{F_1(x,Q^2)}{1-(\omega x)^2}$$

$$\boxed{x=1/\omega'}$$

- PDF moments from small-ω behaviour of Compton amplitude

$$T_1(\omega,Q^2)-T_1(\omega,0)=\sum_{n=2,4,\ldots}4\omega^2\int_0^1 dx\,x^{n-1}F_1(x,Q^2)$$

Feynman–Hellmann at second order: Virtual Compton amplitude from an energy shift

Matrix elements from “Feynman–Hellmann”

- Feynman–Hellmann in quantum mechanics:

$$\frac{dE_n}{d\lambda} = \langle n|\frac{\partial H}{\partial\lambda}|n\rangle$$

  - matrix elements of the derivative of the Hamiltonian determined by derivative of corresponding energy eigenstates

- Lattice QCD: evaluate energy shifts with respect to weak external fields

- Analogous to considering the energy of a fermion in a weak uniform magnetic field:

$$E(\mathbf{B}) = m - \boldsymbol{\mu}.\mathbf{B} + \frac{|e\mathbf{B}|}{2m} - 2\pi\beta_M|\mathbf{B}|^2 + \mathcal{O}(\mathbf{B}^3)$$

Feynman–Hellmann (1st order)

- Suppose we want $\langle H|\mathcal{O}|H\rangle$
- Proceed by $S \to S + \lambda \int \mathrm{d}^4 x \, \mathcal{O}(x)$

real parameter

local operator, e.g. $\bar{q}(x)\gamma_5\gamma_3 q(x)$

- FH tells us

$$\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)} \left\langle H \middle| \frac{\partial S(\lambda)}{\partial \lambda} \middle| H \right\rangle$$

- Calculation of matrix element ≡ hadron spectroscopy [2-pt functions only]

➡ $$\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)} \langle H|\mathcal{O}|H\rangle$$

External momentum current

- Modify Lagrangian with external field containing a spatial Fourier transform [constant in time]

$$\mathcal{L}(y) \to \mathcal{L}_0(y) + \lambda 2 \cos(\vec{q}.\vec{y})\overline{q}(y)\gamma_\mu q(y)$$

- Action perturbation: $$\frac{\partial S}{\partial \lambda} = \int d^4y\, 2\cos(\mathbf{q}.\mathbf{y})\overline{q}(y)\gamma_\mu q(y)$$

- To access form factors, key physical difference

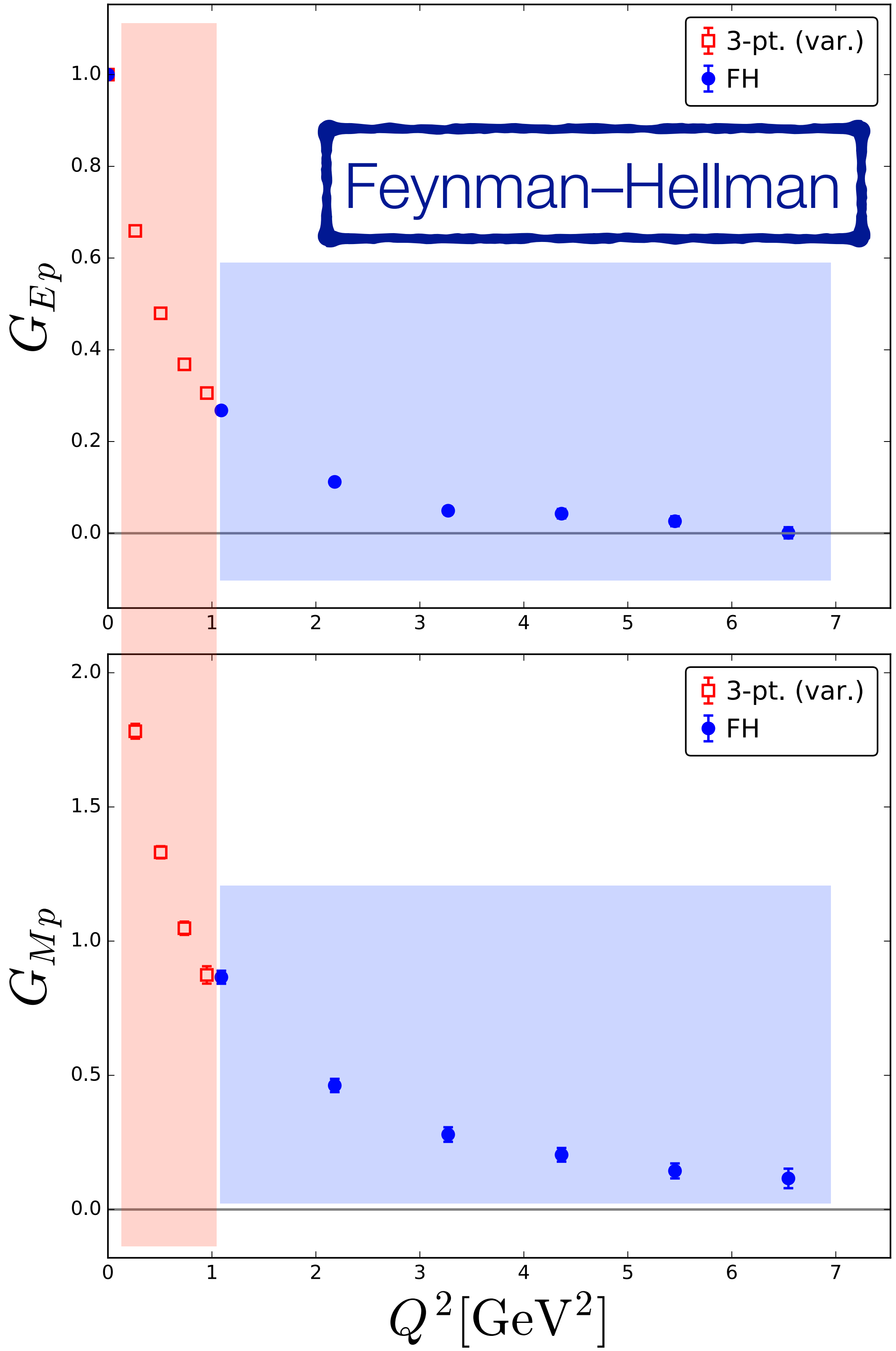
$$\langle \mathbf{p}|J(\mathbf{q})|\mathbf{p}\rangle = 0 \quad \text{for } |\mathbf{q}| > 0$$

  - Hence no first-order energy shift (we will use this in a moment)

- **If** $E(\mathbf{p}) = E(\mathbf{p} \pm \mathbf{q})$ **“Breit frame” kinematics**
  - must use degenerate perturbation theory ⇒ recover linear energy shift!

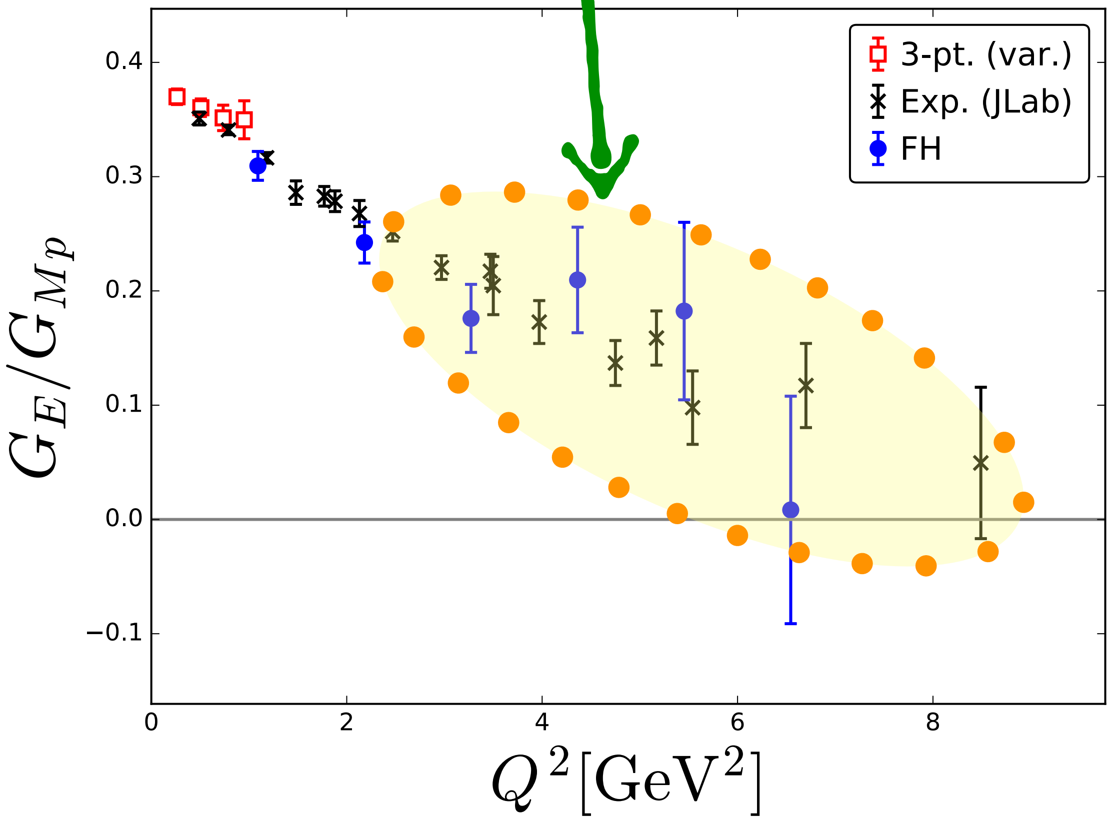
Proton Form Factors

3-pt functions



Feynman–Hellman

Phenomenologically-interesting region.
Domain dominated by model calculations…
previously prohibitive to lattice study.



Second-order “Feynman-Hellmann”
(with external momentum)

Feynman–Hellmann (2nd order)

- Two-point correlator

$$\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}} \frac{1}{\mathcal{Z}(\lambda)} \int \mathcal{D}\phi\, \chi(x)\chi^\dagger(0)e^{-S(\lambda)} = \sum_N \frac{\left|{}_\lambda\langle\Omega|\chi|N,\mathbf{p}\rangle_\lambda\right|^2}{2E_{N,\mathbf{p}}(\lambda)} e^{-E_{N,\mathbf{p}}(\lambda)x_0}$$

Integral over all fields

only interested in perturbative shift of ground-state energy

$$\simeq A_{\mathbf{p}}(\lambda)e^{-E_{\mathbf{p}}(\lambda)x_0}$$

“Momentum” quantum# at finite field

$$|N,\mathbf{p}\rangle_\lambda$$

$$\mathbf{p}\equiv\mathbf{p}+n\mathbf{q},\ \ n\in\mathbb{Z}$$

Feynman–Hellmann (2nd order)

- Differentiate spectral sum

$$\frac{\partial}{\partial\lambda}\sum_N \frac{|{}_\lambda\langle\Omega|\chi|N,\mathbf{p}\rangle_\lambda|^2}{2E_N(\mathbf{p},\lambda)}e^{-E_{N,\mathbf{p}}(\lambda)x_4} = \sum_N\left[\frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial\lambda} - A_{N,\mathbf{p}}(\lambda)x_4\frac{\partial E_{N,\mathbf{p}}}{\partial\lambda}\right]e^{-E_{N,\mathbf{p}}(\lambda)x_4}$$

$$\to\left[\frac{\partial A_{\mathbf{p}}(\lambda)}{\partial\lambda} - A_{\mathbf{p}}(\lambda)x_4\cancel{\frac{\partial E_{\mathbf{p}}}{\partial\lambda}}\right]e^{-E_{\mathbf{p}}(\lambda)x_4}$$

- And again

Not Breit frame, ω<1 ⇒ 0

$$\frac{\partial^2}{\partial\lambda^2}\left[\cdots\right] = \sum_N\left[\frac{\partial^2 A_{N,\mathbf{p}}(\lambda)}{\partial\lambda^2} - 2\frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial\lambda}x_4\cancel{\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial\lambda}} - A_{N,\mathbf{p}}(\lambda)x_4\frac{\partial^2 E_{N,\mathbf{p}}(\lambda)}{\partial\lambda^2} + A_{N,\mathbf{p}}(\lambda)x_4^2\cancel{\left(\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial\lambda}\right)^2}\right]$$

$$\to\left[\frac{\partial^2 A_{\mathbf{p}}(\lambda)}{\partial\lambda^2} - A_{\mathbf{p}}(\lambda)x_4\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2}\right]e^{-E_{\mathbf{p}}(\lambda)x_4}$$

Quadratic energy shift

Watch for temporal enhancement $\sim x_4\, e^{-E_{\mathbf{p}}x_4}$

Feynman–Hellmann (2nd order)

- **Differentiate path integral**

$$\frac{\partial}{\partial\lambda}\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}(\lambda)}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)e^{-S(\lambda)}$$

$$=\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}(\lambda)}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)\left[-\frac{\partial S}{\partial\lambda}-\frac{1}{\mathcal{Z}(\lambda)}\frac{\partial\mathcal{Z}}{\partial\lambda}\right]e^{-S(\lambda)},$$

"Disconnected" operator insertions;
drop for simplicity

- Differentiate again, take zero-field limit and note: $\frac{\partial^2 S}{\partial\lambda^2}=0$

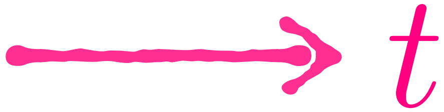
$$\frac{\partial^2}{\partial\lambda^2}[\cdots]\Big|_{\lambda=0}=\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}_0}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)\left(\frac{\partial S}{\partial\lambda}\right)^2 e^{-S_0}$$

Current insertions integrated
over 4-volume

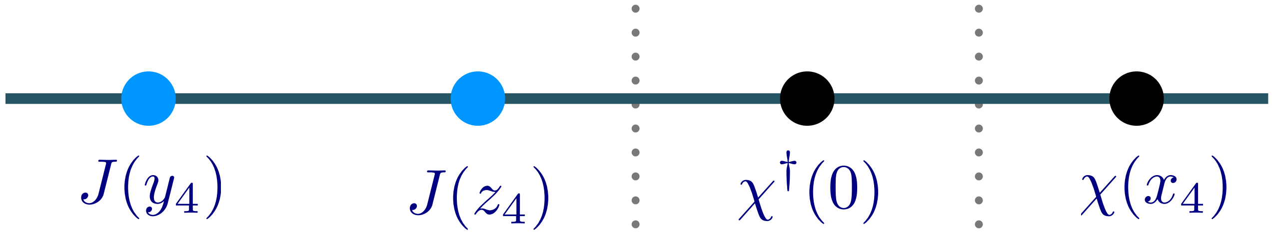
$$\frac{\partial S}{\partial\lambda}=\int d^4y\, 2\cos(\mathbf{q}.\mathbf{y})\overline{q}(y)\gamma_\mu q(y)$$

Field time orderings

ignore finite T

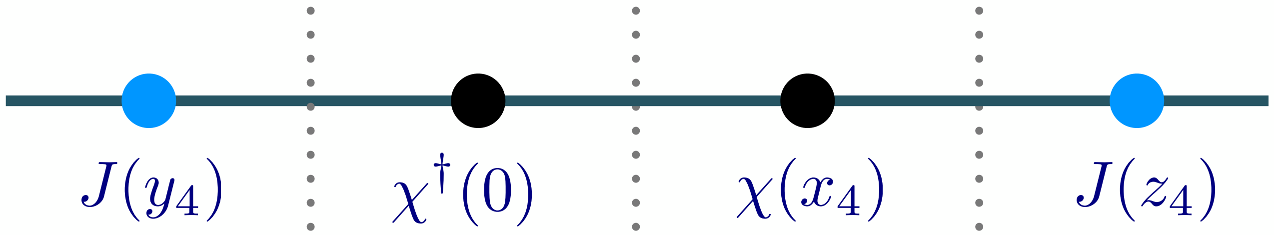


- Current insertion possibilities
  - Both currents “outside” (together)



$$\langle \chi(x)\chi^\dagger(0)\mathrm{T}(J(y)J(z))\rangle, \quad y_4, z_4 < 0 < x_4$$
$$\sim e^{-E_X x_4}, \quad E_X \gtrsim E_{\mathbf{p}}$$

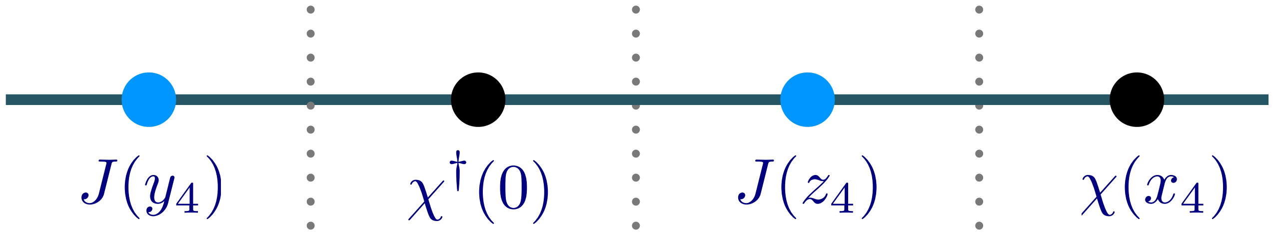
  - Both currents “outside” (opposite)



$$\langle J(z)\chi(x)\chi^\dagger(0)J(y)\rangle, \quad y_4 < 0 < x_4 < z_4$$
$$\sim e^{-E_X x_4}, \quad E_X \gtrsim E_{\mathbf{p}}$$

$E_X = E_{\mathbf{p}}$ ⇒ changes amplitudes

  - One current “inside”

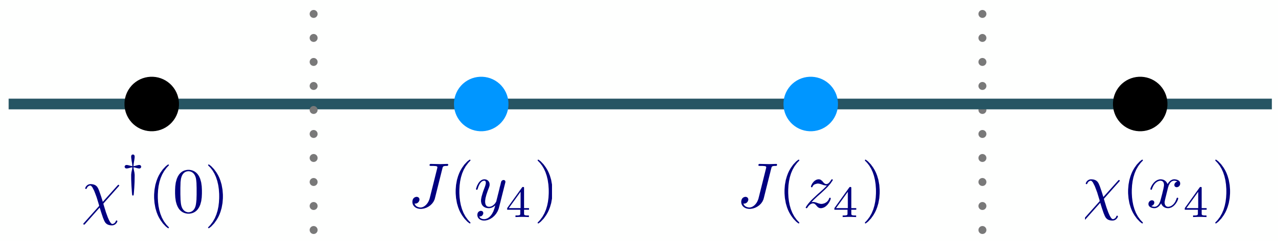


$$\langle \chi(x)J(z)\chi^\dagger(0)J(y)\rangle, \quad y_4 < 0 < z_4 < x_4$$
$$\sim \frac{\partial E_{\mathbf{p}}}{\partial \lambda} x_4\, e^{-E_{\mathbf{p}} x_4} \to 0$$

linear energy shift
(and changed amplitude)

Field time orderings

- Both currents between creation/annihilation



$$\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}} \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi\, \chi(x)\chi^\dagger(0) \left(\frac{\partial S}{\partial \lambda}\right)^2 e^{-S_0}$$

$$= \sum_{N,N'} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{N,\mathbf{k}}} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2E_{N',\mathbf{k}'}} \int d^3x \int d^4z \int d^4y\, e^{-i\mathbf{p}.\mathbf{x}} \left(e^{i\mathbf{q}.\mathbf{z}} + e^{-i\mathbf{q}.\mathbf{z}}\right) \left(e^{i\mathbf{q}.\mathbf{y}} + e^{-i\mathbf{q}.\mathbf{y}}\right)$$

$$\times \langle\Omega|\chi(x)|N,\mathbf{k}\rangle \langle\mathbf{k}|\mathrm{T}J(z)J(y)|\mathbf{k}'\rangle \langle N',\mathbf{k}'|\chi^\dagger(0)|\Omega\rangle,$$

$$\vdots$$

$$\to \frac{A_\mathbf{p}}{2E_\mathbf{p}} x_4 e^{-E_\mathbf{p} x_4} \int d^4\xi \left(e^{iq.\xi} + e^{-iq.\xi}\right) \langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$

Note $q_4 = 0 \quad \Rightarrow \mathbf{q}.\boldsymbol{\xi} = q.\xi$

Final steps

- Equate spectral sum and path integral representation
  - Asymptotically, we have

$$-A_{\mathbf{p}}\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2}x_4 e^{-E_{\mathbf{p}}x_4} = \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}}x_4 e^{-E_{\mathbf{p}}x_4}\int d^4\xi\left(e^{iq.\xi}+e^{-iq.\xi}\right)\langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$

$$\boxed{\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2} = -\frac{1}{2E_{\mathbf{p}}}\int d^4\xi\left(e^{iq.\xi}+e^{-iq.\xi}\right)\langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle}$$

Renormalisation is trivial, not worth creating a slide

$$\overline{q}\gamma_\mu q \to Z_V \overline{q}\gamma_\mu q$$

Compton amplitude → PDFs

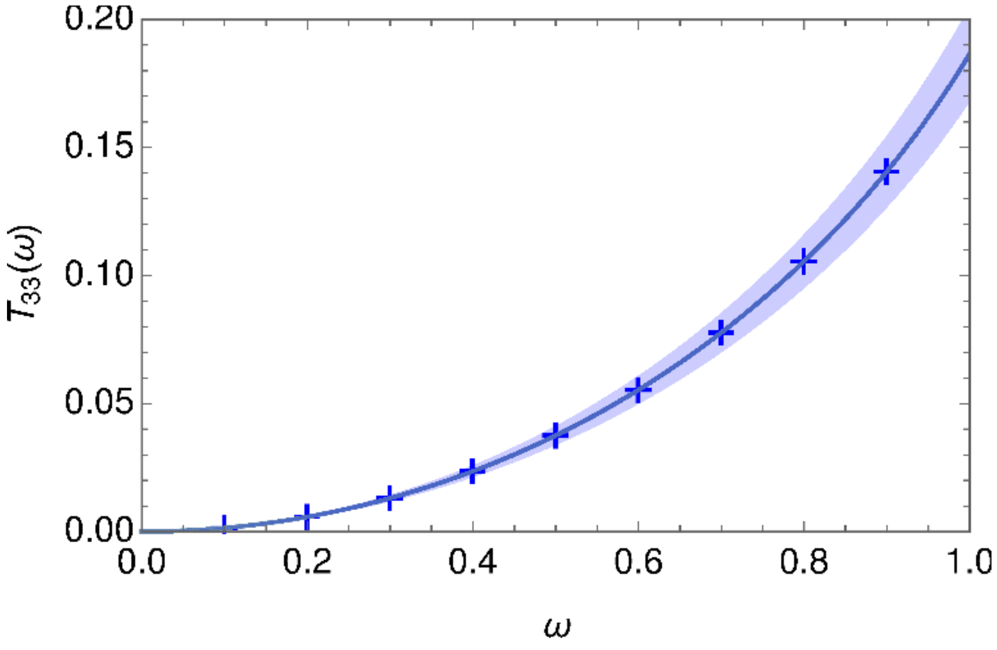
Taylor expansion

- Consider moments of structure function

$$\mu_{2m-1} = \int_0^1 dx\, x^{2m-1} F_1(x)$$

- Series expansion of Compton amplitude

$$T_{33}(\omega)/4 = \omega^2 \mu_1 + \omega^4 \mu_3 + \omega^6 \mu_5 + \ldots$$



input PDFs: MSTW(LO)

“Inversion”

- Discrete approximation to parton distribution $F_1(x)$
- Consider discretised integral

$$T_{33}(\omega_n) = \sum_{m=1}^{M} K_{nm} F_1(x_m), \quad x_m = \frac{m}{M} \qquad K_{nm} = \frac{4\omega_n^2 x_m}{1-(\omega_n x_n)^2}$$

$$N < M$$

- Use singular value decomposition to invert $N\times M$ matrix

$$K = U\,[\mathrm{diag}(w_1,\ldots,w_{N'},\underline{w_{N'+1}\ldots,w_N})]\,V^\top$$

$N\times M$ “diag”

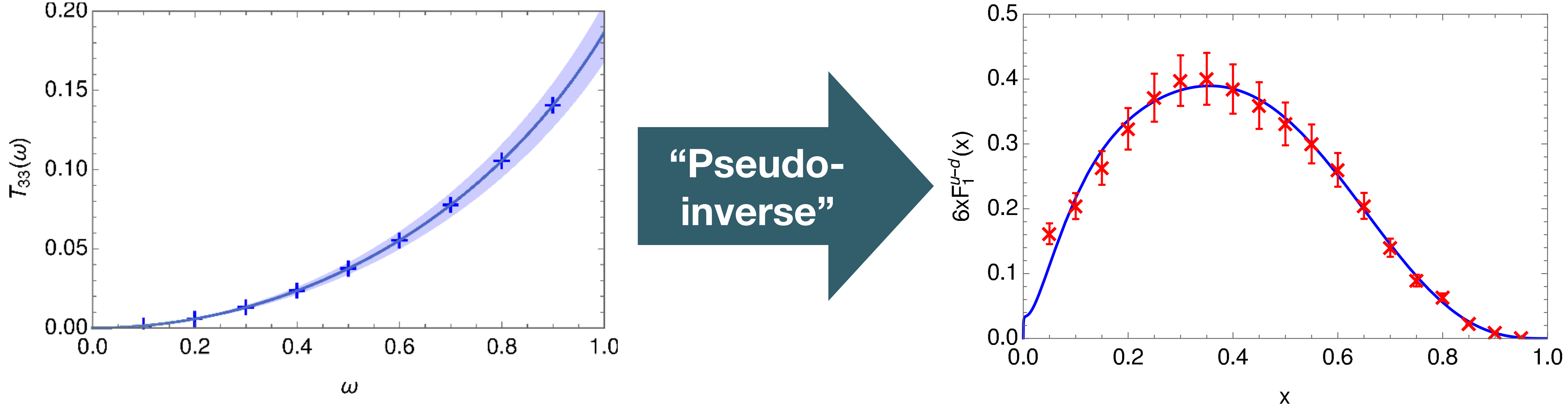
$$w_{N'+1}\ldots,w_N \simeq 0,\ \ N' \leq N$$

  - Pseudoinverse

$$K^{-1} = V\,[\mathrm{diag}(1/\omega_1,\ldots,1/\omega_{N'},0,\ldots,0)]\,U^\top$$

Input

Output



“Pseudo-inverse”

$$T_{33} = 4\omega \int_0^1 dx \frac{\omega x}{1-(\omega x)^2} F_1^{u-d}(x)$$

$$2xF_1^{u-d}(x) = \frac{1}{3} x \left[u(x) - d(x)\right]$$

input PDFs: MSTW(LO)

Chambers et al., PRL(2017)

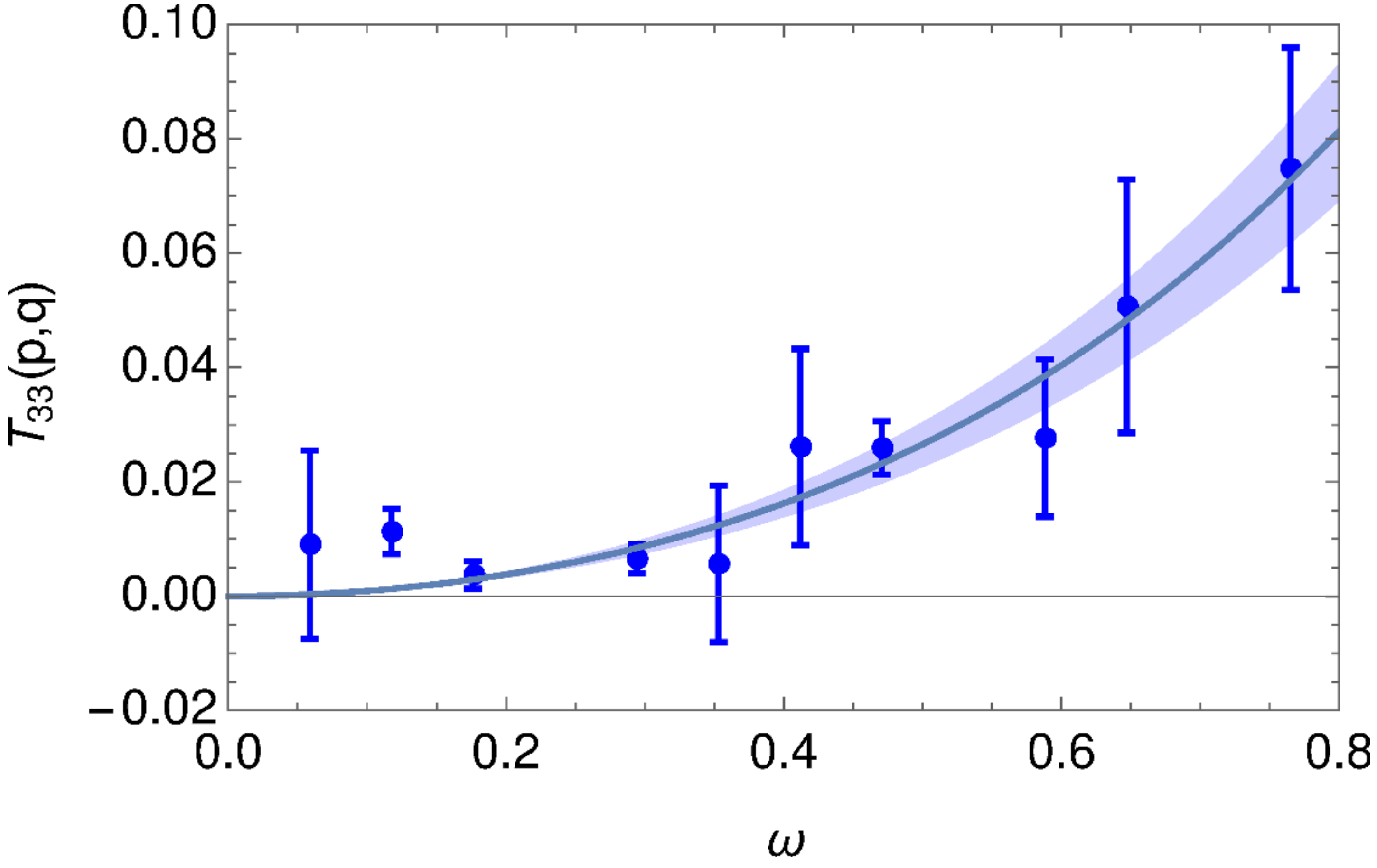
Toy model test

Numerical test: "Lattice results"

$$\vec{q} = (3,5,0)\,\frac{2\pi}{L}$$

$pL/(2\pi)$	ω
(-2,1,0)	2/34
(-1,1,0)	4/34
(1,0,0)	6/34
(0,1,0)	10/34
(2,0,0)	12/34
(-1,2,0)	14/34
(1,1,0)	16/34
(0,2,0)	20/34
(2,1,0)	22/34
(1,2,0)	26/34

Compton amplitude from quadratic energy shift
(subtraction term removed)



Chambers et al., PRL(2017)

Lattice specs

SU(3) symmetric point $m_\pi \simeq 400\,\mathrm{MeV}$
32^3x64, a≈0.074 fm
O(900) configs

(Virtual) Compton amplitude
accessible on the lattice

Nonperturbative constraint on
hadronic structure functions
→ PDFs + higher twist