QCD at finite isospin chemical potential: Reweighting

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with Bastian Brandt & Gergely Endrődi

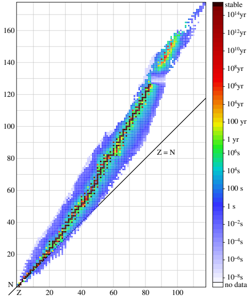
Outline

- QCD with isospin: motivation
- introduction of a symmetry breaking parameter λ
- $\lambda \to 0$ limit
- explore phase diagram for $\mu_B > 0$ via reweighting
- summary & outlook

Motivation

- two quark flavors u, d
- isospin density $n_I = n_u - n_d$
- $n_I \neq 0$:
  - systems with charged pions
  - neutron stars (udd)
  - heavy-ion collisions (N>Z)

(taken from thescienceexplorer)



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- $n_I \neq 0$:
  - systems with charged pions
  - neutron stars (udd)
  - heavy-ion collisions (N>Z)
- no sign problem $\Rightarrow$ lattice simulations
- analogies to baryon density
  - Silver Blaze
  - hadron condensation (small eigenvalues)
  - saturation

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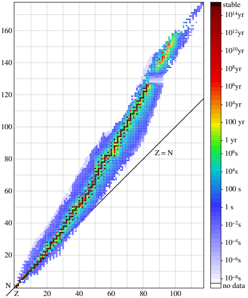

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Symmetry breaking

- QCD with light quarks

$$\mathcal{M} = \not{D} + m_{ud}\mathbb{1}$$

- corresponding symmetry (without considering trivial $U(1)_V$)

$$\mathrm{SU}(2)_V$$

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$$\mathrm{SU}(2)_V \rightarrow \mathrm{U}(1)_{\tau_3}$$

- problem: cannot directly observe spontaneous symmetry breaking
  - pion condensate $\langle \bar{\psi}\gamma_5\tau_{1,2}\psi \rangle = 0$ (Goldstone mode)
  - accumulation of zero eigenvalues

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  - no zero eigenvalues
- extrapolate results $\lambda \rightarrow 0$

Simulation Details

- QCD partition function

$$\mathcal{Z} = \int \mathcal{D}[U] \; (\det \mathcal{M})^{\frac{1}{4}} \, e^{-S_G}$$

- $N_f = 2$ rooted staggered fermions, $\eta_5 = (-1)^{n_t+n_x+n_y+n_z}$

$$\mathcal{M} = \begin{pmatrix} \not{D}_{\mu_I} + m & \lambda\eta_5 \\ -\lambda\eta_5 & \not{D}_{-\mu_I} + m \end{pmatrix}$$

- plaquette gauge action S_G
- positivity of determinant:

$$\det \mathcal{M} = \det \left(M^\dagger M + \lambda^2 \right) \in \mathbb{R}_{>0} \qquad M = \not{D}_{\mu_I} + m$$

  ensured by $\quad \eta_5 \tau_1 \mathcal{M} \tau_1 \eta_5 = \mathcal{M}^\dagger$
- first studies [Kogut, Sinclair '02, de Forcrand, Stephanov, Wenger '07]
- 8^4 lattice, $T = 0$

Observables

- condensates, isospin density

$$\left\langle \bar{\psi}\psi \right\rangle = \frac{T}{V}\frac{\partial \ln \mathcal{Z}}{\partial m} \qquad \langle \pi \rangle = \frac{T}{V}\frac{\partial \ln \mathcal{Z}}{\partial \lambda} \qquad \langle n_I \rangle = \frac{T}{V}\frac{\partial \ln \mathcal{Z}}{\partial \mu_I}$$

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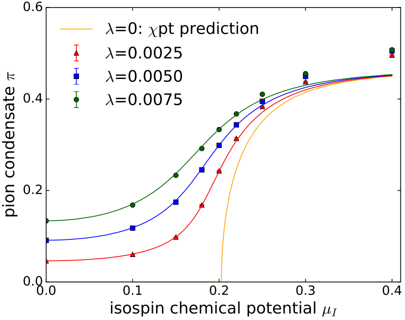
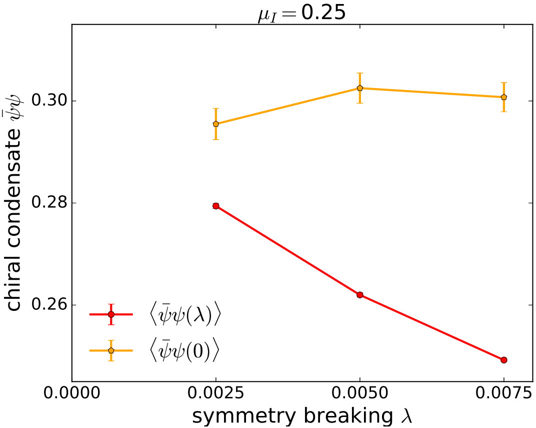
- $O(\lambda)$ strongly depend on λ
- for $\lambda \to 0$:

$$\langle \pi \rangle \begin{cases} = 0 & : \mu_I \leq m_\pi/2 \\ > 0 & : \mu_I > m_\pi/2 \end{cases}$$



λ-extrapolation

- naive limit (steep): $\langle O \rangle = \lim_{\lambda \to 0} \int \mathcal{D}[U]\, e^{-S_G} \det [\mathcal{M}(\lambda)]^{\frac{1}{4}}\, O(\lambda)$
- better (almost flat): $\langle O \rangle = \lim_{\lambda \to 0} \int \mathcal{D}[U]\, e^{-S_G} \det [\mathcal{M}(\lambda)]^{\frac{1}{4}}\, O(0)$



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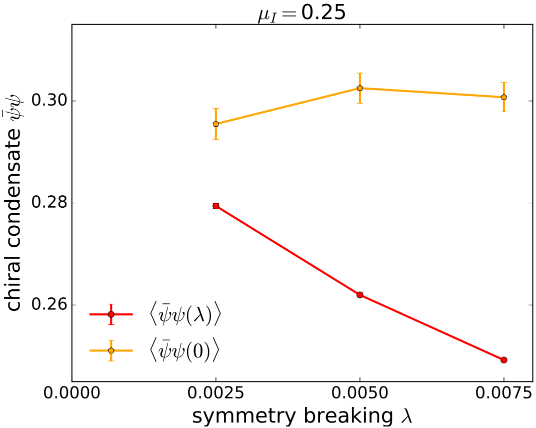

- instead of taking lim: reweight $\lambda \to 0$

Reweighting in λ

- reweighting factor

$$R_\lambda = \left[\frac{\det(M^\dagger M)}{\det(M^\dagger M + \lambda^2)}\right]^{\frac{1}{4}} \in \mathbb{R}$$

- leading order expansion in λ

$$R_\lambda \approx R_{LO} = \exp\left(-\frac{\lambda T}{2V}\pi\right)$$

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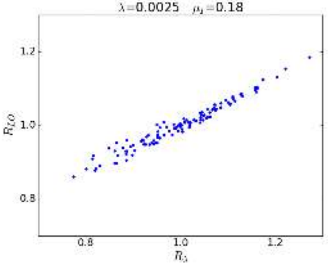
$$R_\lambda \approx R_{LO} = \exp\left(-\frac{\lambda T}{2V} \pi \right)$$

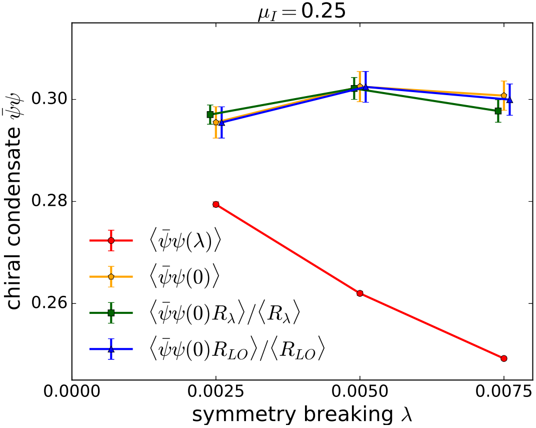
- any observable measured at $\lambda = 0$ can then be expressed as:

$$\langle O \rangle = \frac{\langle O R_\lambda \rangle}{\langle R_\lambda \rangle} \approx \frac{\langle O R_{LO} \rangle}{\langle R_{LO} \rangle}$$



- R_λ expensive
- R_{LO} cheap
- for $\lambda \to 0$: $R_\lambda = R_{LO}$

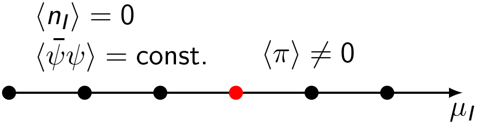
Reweighting in λ: comparison



- $O(\lambda)$: steep
- $O(0)$: flat
- LO $\widehat{=}$ full reweighting

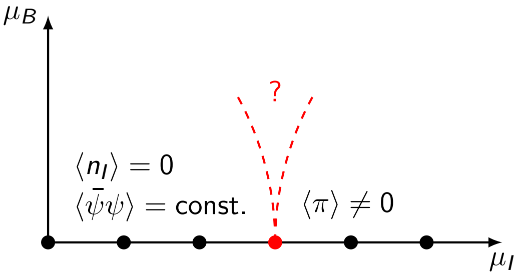
Reweighting in μ

Reweighting in μ: motivation



- until now: $\mu_B = 0$

Reweighting in μ: motivation



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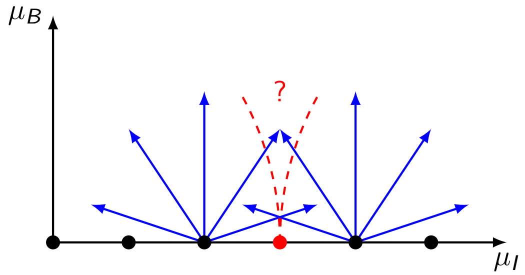
Reweighting in μ: motivation



- until now: $\mu_B = 0$
- want to explore the $\mu_B > 0$ direction: phase boundary
- reweight from pure μ_I into $\mu_I - \mu_B$ plane
- look for:
  - lines of constant observables
  - small error on reweighted observables (big overlap, small sign problem)

Reweighting in μ: how-to

ensemble: $(\mu_I, \lambda, 0) \quad \rightarrow \quad (\mu_I, 0, 0) \quad \rightarrow \quad (\mu_I', 0, \mu_B)$

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$$\mathcal{M} = \begin{pmatrix} M_{\mu_I} & \eta_5\lambda \\ -\eta_5\lambda & M_{-\mu_I} \end{pmatrix} \rightarrow \begin{pmatrix} M_{\mu_I} & 0 \\ 0 & M_{-\mu_I} \end{pmatrix} \rightarrow \begin{pmatrix} M_{\mu_u} & 0 \\ 0 & M_{\mu_d} \end{pmatrix}$$

with $\mu_u = \mu_B + \mu_I' \qquad \mu_d = \mu_B - \mu_I'$

Reweighting in μ: how-to

$$\text{ensemble:} \qquad (\mu_I, \lambda, 0) \quad \to \quad (\mu_I, 0, 0) \quad \to \quad (\mu_I', 0, \mu_B)$$

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with $\qquad \mu_u = \mu_B + \mu_I' \qquad \mu_d = \mu_B - \mu_I'$

- can now factorize

$$\det \mathcal{M} = \det M(\mu_u) \det M(\mu_d)$$

- reweighting factor for $(\mu_I, 0, 0) \to (\mu_I', 0, \mu_B)$:

$$R_\mu = \left[\frac{\det M(\mu_u) \det M(\mu_d)}{\det M(\mu_I) \det M(-\mu_I)} \right]^{\frac{1}{4}} \in \mathbb{C}$$

Reweighting in μ: determinant reduction

- need to compute $\det M(\mu)$ for many different μ
- determinant reduction [Toussaint '90 & Fodor, Katz '02]

$$\det M(\mu) = e^{-3V_s L_t \mu} \det\left(P - e^{L_t \mu}\right)$$
$$= e^{-3V_s L_t \mu} \prod_{i=1}^{6V_s} \left(p_i - e^{L_t \mu}\right)$$

- analytic μ-dependence: calculate eigenvalues of P just once
- construction of P is well defined

Reweighting in μ: observables

- combined reweighting step

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- have to measure observables in target ensemble ($\mu_B \neq 0$):

$$\bar{\psi}\psi = \frac{T}{4V}\frac{\partial}{\partial m}\left[\ln\det M(\mu_u) + \ln\det M(\mu_d)\right]$$

  via numerical derivative

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via numerical derivative,

$$n_I = \frac{T}{4V} \frac{\partial}{\partial \mu_I} \left[\ln \det M(\mu_u) + \ln \det M(\mu_d) \right]$$

via closed formula (follows from determinant reduction):

$$\frac{\partial \ln \det M(\mu)}{\partial \mu} = -3 L_t V_s - L_t \sum_{i=1}^{6 V_s} \frac{e^{L_t \mu}}{p_i - e^{L_t \mu}}$$

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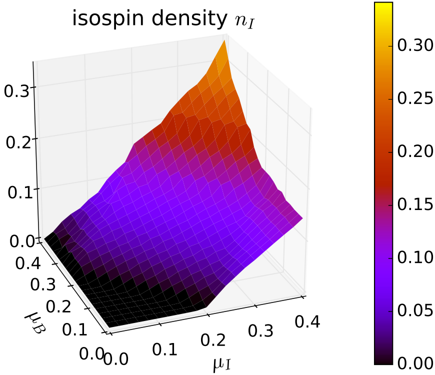
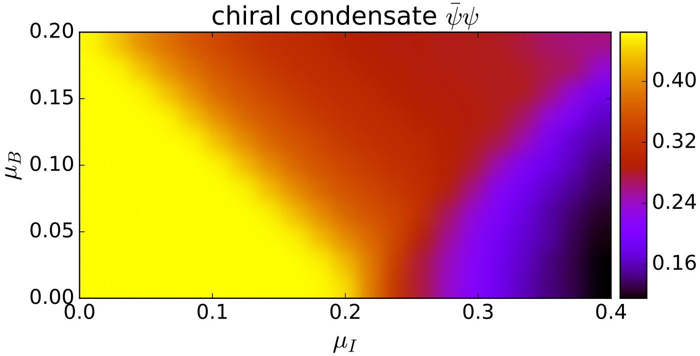
- measure $\pi(\mu_B \neq 0)$?

Reweighting in μ: results ($T = 0$)

- take auxiliary ensemble with biggest overlap
- unphysical surfaces
  - severe sign problem
  - underestimated error
- similar for chiral condensate



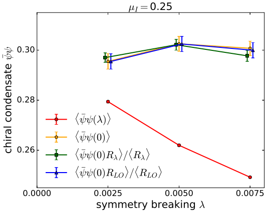
Reweighting in μ: results ($T = 0$)



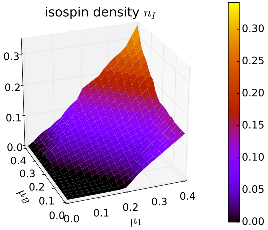
- phase boundary bends to the right for increasing μ_B

Summary, outlook & transition

- lattice QCD with isospin chemical potential



- observe pion condensation with $\lambda \neq 0$
- reweight $\lambda \to 0$



- determinant reduction + reweighting to explore the $\mu_I - \mu_B$ phase-space
- finite T: next talk by B. Brandt