(Relativistic corrections to) the static potential and ultrasoft effects

Mainly based on arXiv:1706.03971; Peset, Pineda and Stahlhofen

Antonio Pineda

Universitat Autònoma de Barcelona and IFAE-BIST

35th International Symposium on Lattice Field Theory (Granada), 18-24 June 2017

Heavy Quarkonium

Effective field theory description:
$m(hard) \gg mv(soft) \gg mv^2(ultrasoft)$
Potential Non-Relativistic QCD in the weak coupling regime is ideal for this.

$$\left.\begin{array}{l}\left(i\partial_0 - \frac{\mathbf{p}^2}{2m} - V_s(r)\right)\Phi(\mathbf{r}) = 0 \\ +\text{interaction with other low} \\ \quad\text{energy degrees of freedom}\end{array}\right\} \text{pNRQCD(Pineda, Soto)} \quad E \sim mv^2$$

$$V_s = V^{(0)} + \boxed{\frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2}} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots,$$

Heavy Quarkonium

Effective field theory description:
$m(hard) \gg mv(soft) \gg mv^2(ultrasoft)$
Potential Non-Relativistic QCD in the weak coupling regime is ideal for this.

$$\left.\begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m} - V_s(r)\right)\Phi(\mathbf{r}) = 0 \\ +\text{interaction with other low} \\ \quad\text{energy degrees of freedom} \end{array}\right\} \text{pNRQCD(Pineda, Soto)} \quad E \sim mv^2$$

$$V_s = V^{(0)} + \boxed{\frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2}} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots ,$$

Heavy Quarkonium

Effective field theory description:
$m(hard) \gg mv(soft) \gg mv^2(ultrasoft)$
Potential Non-Relativistic QCD in the weak coupling regime is ideal for this.

$$\left. \begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m} - V_s(r)\right)\Phi(\mathbf{r}) = 0 \\ +\text{interaction with other low} \\ \quad \text{energy degrees of freedom} \end{array} \right\} \text{pNRQCD(Pineda, Soto)} \quad E \sim mv^2$$

$$V_s = V^{(0)} + \boxed{\frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2}} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots,$$

Heavy Quarkonium

Effective field theory description:
$m(hard) \gg mv(soft) \gg mv^2(ultrasoft)$
Potential Non-Relativistic QCD in the weak coupling regime is ideal for this.

$$\left. \begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m} - V_s(r)\right)\Phi(\mathbf{r}) = 0 \\ +\text{interaction with other low} \\ \quad \text{energy degrees of freedom} \end{array} \right\} \text{pNRQCD(Pineda, Soto)} \quad E \sim mv^2$$

$$V_s = V^{(0)} + \boxed{\frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2}} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots ,$$

$$V^{(2,0)} = \boxed{\frac{1}{2}\left\{\mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r)\right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2}\mathbf{L}_1^2} + V_r^{(2,0)}(r) \quad \text{(SI)}$$
$$+ V_{LS}^{(2,0)}(r)\mathbf{L}_1 \cdot \mathbf{S}_1 \quad \text{(SD)}$$

$$V^{(1,1)} = \boxed{-\frac{1}{2}\left\{\mathbf{p}_1 \cdot \mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r)\right\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1 \cdot \mathbf{L}_2 + \mathbf{L}_2 \cdot \mathbf{L}_1)} + V_r^{(1,1)}(r)$$
$$+ V_{L_1 S_2}^{(1,1)}(r)\mathbf{L}_1 \cdot \mathbf{S}_2 - V_{L_2 S_1}^{(1,1)}(r)\mathbf{L}_2 \cdot \mathbf{S}_1 + V_{S^2}^{(1,1)}(r)\mathbf{S}_1 \cdot \mathbf{S}_2 + V_{\mathbf{S}_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}})$$

V=Wilson coefficient of the EFT (integrating out $\sim m, mv$)
Matching QCD & EFT $\rightarrow$ Wilson coefficients can be related with Wilson loops
$V^{(0)}$: Wilson, Susskind
$V^{(1,0)}$: Brambilla, Soto, Vairo, Pineda
$V_{\mathbf{p}^2}^{(2,0)}$, $V_{\mathbf{L}^2}^{(2,0)}$, $V_{\mathbf{p}^2}^{(1,1)}$, $V_{\mathbf{L}^2}^{(1,1)}$: Barchielli, Montaldi, Prosperi; Vairo, Pineda

...

$$V^{(2,0)} = \boxed{\frac{1}{2}\left\{\mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r)\right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2}\mathbf{L}_1^2} + V_r^{(2,0)}(r) \quad \text{(SI)}$$

$$+ V_{LS}^{(2,0)}(r)\mathbf{L}_1 \cdot \mathbf{S}_1 \quad \text{(SD)}$$

$$V^{(1,1)} = \boxed{-\frac{1}{2}\left\{\mathbf{p}_1 \cdot \mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r)\right\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1 \cdot \mathbf{L}_2 + \mathbf{L}_2 \cdot \mathbf{L}_1)} + V_r^{(1,1)}(r)$$

$$+ V_{L_1 S_2}^{(1,1)}(r)\mathbf{L}_1 \cdot \mathbf{S}_2 - V_{L_2 S_1}^{(1,1)}(r)\mathbf{L}_2 \cdot \mathbf{S}_1 + V_{S^2}^{(1,1)}(r)\mathbf{S}_1 \cdot \mathbf{S}_2 + V_{\mathbf{S}_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}})$$

V=Wilson coefficient of the EFT (integrating out $\sim m$, mv)
Matching QCD & EFT $\rightarrow$ Wilson coefficients can be related with Wilson loops

$V^{(0)}$: Wilson, Susskind

$V^{(1,0)}$: Brambilla, Soto, Vairo, Pineda

$V_{\mathbf{p}^2}^{(2,0)}$, $V_{\mathbf{L}^2}^{(2,0)}$, $V_{\mathbf{p}^2}^{(1,1)}$, $V_{\mathbf{L}^2}^{(1,1)}$: Barchielli, Montaldi, Prosperi; Vairo, Pineda

...

$$V^{(2,0)} = \boxed{\frac{1}{2}\left\{\mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r)\right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2}\mathbf{L}_1^2} + V_r^{(2,0)}(r) \quad \text{(SI)}$$
$$+V_{LS}^{(2,0)}(r)\mathbf{L}_1\cdot\mathbf{S}_1 \quad \text{(SD)}$$

$$V^{(1,1)} = \boxed{-\frac{1}{2}\left\{\mathbf{p}_1\cdot\mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r)\right\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1\cdot\mathbf{L}_2+\mathbf{L}_2\cdot\mathbf{L}_1)} + V_r^{(1,1)}(r)$$
$$+V_{L_1S_2}^{(1,1)}(r)\mathbf{L}_1\cdot\mathbf{S}_2 - V_{L_2S_1}^{(1,1)}(r)\mathbf{L}_2\cdot\mathbf{S}_1 + V_{S^2}^{(1,1)}(r)\mathbf{S}_1\cdot\mathbf{S}_2 + V_{\mathbf{S}_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}})$$

V=Wilson coefficient of the EFT (integrating out $\sim m$, mv)
Matching QCD & EFT $\rightarrow$ Wilson coefficients can be related with Wilson loops
$V^{(0)}$: Wilson, Susskind
$V^{(1,0)}$: Brambilla, Soto, Vairo, Pineda
$V_{\mathbf{p}^2}^{(2,0)}$, $V_{\mathbf{L}^2}^{(2,0)}$, $V_{\mathbf{p}^2}^{(1,1)}$, $V_{\mathbf{L}^2}^{(1,1)}$: Barchielli, Montaldi, Prosperi; Vairo, Pineda

. . .

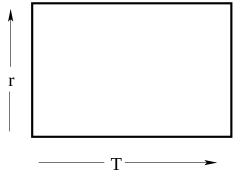
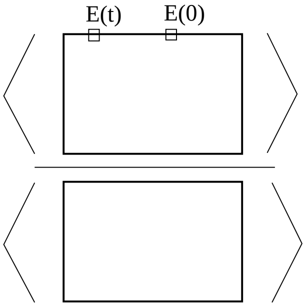


Figure: *Static Wilson loop. Time propagates from the left to the right. Horizontal lines correspond to the quark trajectories and the vertical lines to the Schwinger strings.*



Perturbative definition: only soft scale $\sim 1/r$

$$
\begin{aligned}
V^{(0)}(\mathbf{r}) &= \lim_{T\to\infty} \frac{i}{T} \log \langle W_\Box \rangle \Big|_{pert.} \\
V^{(1,0)} &= -\frac{1}{2} \lim_{T\to\infty} \int_0^T dt\, t\, \langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c \Big|_{pert.} \\
V^{(2,0)}_{\mathbf{p}^2}(r) &= \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \lim_{T\to\infty} \int_0^T dt\, t^2 \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_1(0)\rangle\rangle_c \Big|_{pert.}, \\
V^{(2,0)}_{\mathbf{L}^2}(r) &= \frac{i}{4}\left(\delta^{ij} - 3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right) \lim_{T\to\infty} \int_0^T dt\, t^2 \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_1(0)\rangle\rangle_c \Big|_{pert.}, \\
V^{(1,1)}_{\mathbf{p}^2}(r) &= i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \lim_{T\to\infty} \int_0^T dt\, t^2 \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_2(0)\rangle\rangle_c \Big|_{pert.}, \\
V^{(1,1)}_{\mathbf{L}^2}(r) &= i\frac{\delta^{ij} - 3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j}{2} \lim_{T\to\infty} \int_0^T dt\, t^2 \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_2(0)\rangle\rangle_c \Big|_{pert.},
\end{aligned}
$$

Perturbative definition: only soft scale $\sim 1/r$

$$V^{(0)} = \lim_{T\to\infty} \frac{i}{T} \log \langle W_\Box \rangle \Big|_{pert.} = -C_F \frac{\alpha}{r}\left(1 + \cdots + \# \frac{\alpha^3}{\pi} \ln(r\nu_{us}) + \cdots \right)$$

$$\begin{aligned} V^{(1,0)} &= -\frac{1}{2}\int_0^\infty dt\, t\, \langle\!\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c \Big|_{pert.} \\ &= -C_F C_A \frac{\alpha^2}{4\,r^2}\left(1 + \# \frac{\alpha}{\pi} \ln(r\nu_{us}) + \cdots \right) \end{aligned}$$

$$V^{(2,0)}_{\mathbf{p}^2} = \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \int_0^\infty dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0)\rangle\!\rangle_c \Big|_{pert.} = \mathcal{O}(\alpha^2 \ln(r\nu_{us})),$$

$$V^{(1,1)}_{\mathbf{p}^2} = i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \int_0^\infty dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0)\rangle\!\rangle_c \Big|_{pert.} = -C_F \frac{\alpha}{r}\left(1 + \# \frac{\alpha}{\pi} \ln(r\nu_{us}) + \cdots \right.$$

$$\cdots$$

Non-Perturbative ($\frac{1}{r} \gg \Delta V = V_o^{(0)} - V_s^{(0)} \sim \frac{\alpha}{r} \gg \Lambda_{\rm QCD}$)

$$E^{(0)} = \lim_{T\to\infty} \frac{i}{T} \log \langle W_\Box \rangle \Big|_{MC} = -C_F \frac{\alpha}{r} \left(1 + \cdots + \# \frac{\alpha^3}{\pi} \ln(\alpha) + \cdots \right)$$

$$\begin{aligned} E^{(1,0)} &= -\frac{1}{2} \int_0^\infty dt\, t \, \langle\langle g\mathbf{E}_1(t) \cdot g\mathbf{E}_1(0) \rangle\rangle_c \Big|_{MC} \\ &= -C_F C_A \frac{\alpha^2}{4\, r^2} \left(1 + \# \frac{\alpha}{\pi} \ln(\alpha) + \cdots \right) \end{aligned}$$

$$E^{(2,0)}_{\mathbf{p}^2} = \frac{i}{2} \hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \int_0^\infty dt\, t^2 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0) \rangle\rangle_c \Big|_{MC} = \mathcal{O}(\alpha^2 \ln \alpha),$$

$$E^{(1,1)}_{\mathbf{p}^2} = i \hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \int_0^\infty dt\, t^2 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0) \rangle\rangle_c \Big|_{MC} = -C_F \frac{\alpha}{r} \left(1 + \# \frac{\alpha}{\pi} \ln(\alpha) + \cdots \right),$$

$$\ldots$$

Motivation:

Ideal observables (clean) to study the $r \to 0, \infty$ limit and its interplay.
Light fermions play a minor role ($N_c \to \infty$).
Possible quantitative analysis of ultrasoft effects. Relevant for:
Determination of alpha strong from the static potential,
Validity of weak coupling analyses for heavy quarkonium.

- $E^{(0)} D = 3 + 1 \to$ NNNLO (ultrasoft effects)
- $E^{(0)}$ D=2+1 $\to$ NNLO (ultrasoft effects)
- $E^{(0)}$ $\mathcal{N}$=4 SUSY $\to$ NLO (ultrasoft effects)
- Relativistic corrections: $E^{(1,0)}$, $E^{(2,0)}$, $\cdots$ D=3+1 $\to$ (N)LO (ultrasoft effects)

$$E(r) = V_W(r) + \delta E_{US}$$

V_W: Soft $\to$ Peset, Pineda, Stahlhofen

ULTRASOFT

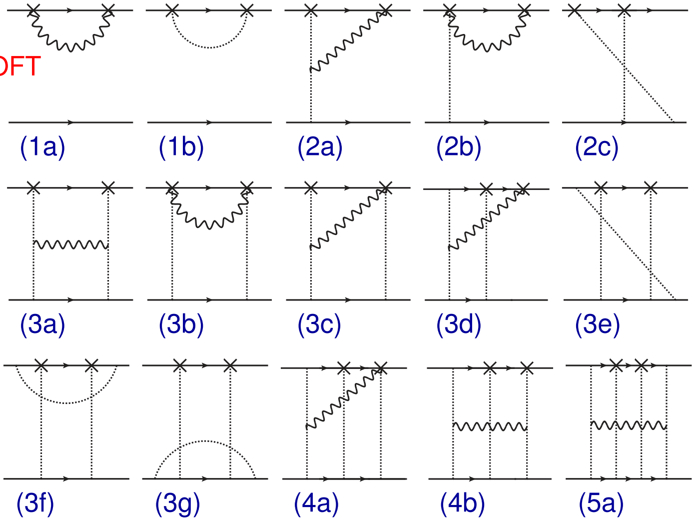


Figure: Dotted lines represent the A^0, wiggled lines the **A** components of a gluon. Vertical lines are associated with potential gluons. The ultrasoft gluon is depicted as diagonal, horizontal or curvy line.

ULTRASOFT

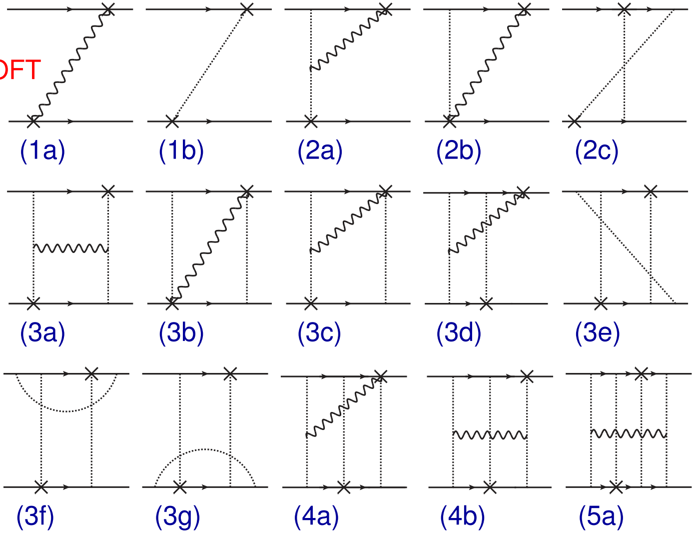


Figure: Dotted lines represent the A^0, wiggled lines the **A** components of a gluon. Vertical lines are associated with potential gluons. The ultrasoft gluon is depicted as diagonal, horizontal or curvy line.

Relativistic corrections to the static energy $D = 4$

$$E^{(1,0)}(r) = V^{(1,0)}(r) + \delta E^{(1,0)}_{US}(r) =$$

$$-\frac{\alpha^2 C_A C_F}{4r^2}\left\{1+\frac{\alpha}{2\pi}\left[\frac{8}{3}C_A\ln(C_A\alpha e^{\gamma_E})+\frac{47}{18}C_A-\frac{49}{18}T_f n_f+2\beta_0\ln(e^{\gamma_E}\nu r)\right]\right\},$$

$$E^{(2,0)}_{\mathbf{p}^2}(r) = V^{(2,0)}_{\mathbf{p}^2}(r) + \delta E^{(2,0)}_{\mathbf{p}^2,US}(r) = -\frac{\alpha^2 C_A C_F}{3\pi r}\left\{\ln(C_A\alpha e^{\gamma_E})+\frac{1}{4}\right\},$$

$$E^{(2,0)}_{L^2}(r) = V^{(2,0)}_{L^2}(r) + \delta E^{(2,0)}_{L^2,US}(r) = -\frac{\alpha^2 C_A C_F}{3\pi r}\left\{2\ln(C_A\alpha e^{\gamma_E})-1\right\},$$

$$E^{(1,1)}_{\mathbf{p}^2}(r) = V^{(1,1)}_{\mathbf{p}^2}(r) + \delta E^{(1,1)}_{\mathbf{p}^2,US}(r) =$$

$$-\frac{\alpha C_F}{r}\left\{1+\frac{\alpha}{3\pi}\left[C_A\left(2\ln(C_A\alpha e^{\gamma_E})-\frac{3}{2}\right)+\beta_0\left(\frac{3}{2}\ln(e^{\gamma_E}\nu r)+\frac{1}{2}\right)\right]\right\},$$

$$E^{(1,1)}_{L^2}(r) = V^{(1,1)}_{L^2}(r) + \delta E^{(1,1)}_{L^2,US}(r) =$$

$$\frac{\alpha C_F}{2r}\left\{1-\frac{\alpha}{3\pi}\left[2C_A\left(4\ln(C_A\alpha e^{\gamma_E})-1\right)+\beta_0\left(\frac{1}{4}-\frac{3}{2}\ln(e^{\gamma_E}\nu r)\right)\right]\right\},$$

Comparison with lattice

$$E_b(r) = -\frac{2}{3}E_{\mathbf{L}^2}^{(1,1)}(r) - E_{\mathbf{p}^2}^{(1,1)}(r),$$

$$E_c(r) = -E_{\mathbf{L}^2}^{(1,1)}(r),$$

$$E_d(r) = \frac{2}{3}E_{\mathbf{L}^2}^{(2,0)}(r) + E_{\mathbf{p}^2}^{(2,0)}(r),$$

$$E_e(r) = E_{\mathbf{L}^2}^{(2,0)}(r).$$

Lattice data: Koma, Koma & Wittig

Short distances

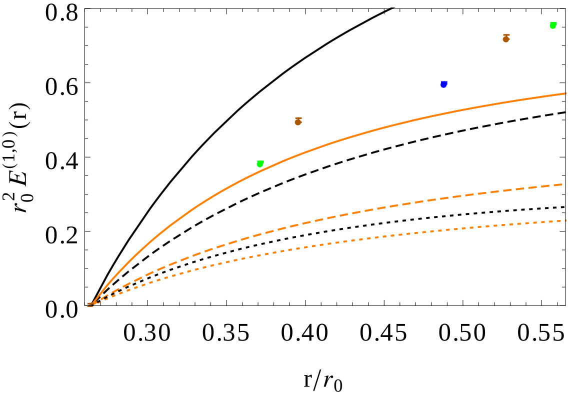


Figure: Comparison of the lattice data and $E^{(1,0)}$. The continuous lines correspond to the LO (orange) and NLO (black) at fixed scale $\nu = 2 \times 1/0.263821\ r_0^{-1}$. The dashed lines correspond to the LO (orange) and NLO (black) setting $\nu = 3/r$ and the dotted lines correspond to the LL (orange) and NLL (black) setting $\nu = 3/r$ and $\nu_{us} = 2C_A\alpha(3/r)/r$.

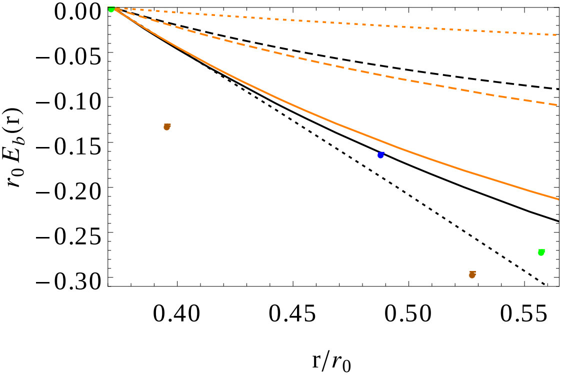


Figure: The solid lines correspond to the LO (orange) and NLO results (black) at fixed scale $\nu = 2 \times 1/0.371627\ r_0^{-1}$. The dashed lines correspond to the LO (orange) and NLO results (black) setting $\nu = 4/r$. The dotted lines correspond to the LL (orange) and NLL results (black) setting $\nu_{us} = C_A \alpha(\nu)/r$ and $\nu = 4/r$.

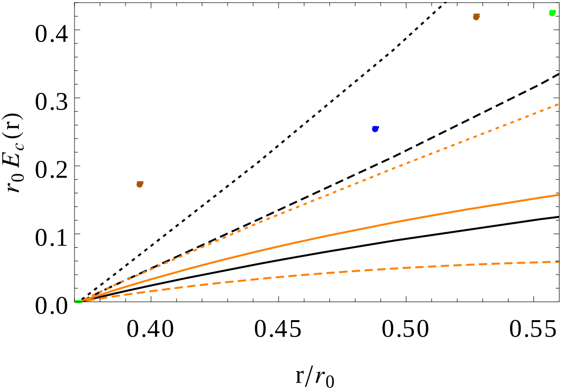


Figure: Comparison to the lattice data for E_c. The solid lines correspond to the LO (orange) and NLO results (black) at fixed scale $\nu = 2 \times 1/0.371627 r_0^{-1}$. The dashed lines correspond to the LO (orange) and NLO (black) results setting $\nu = 1/r$. The dotted lines correspond to the LL (orange) and NLL (black) results setting $\nu_{us} = C_A \alpha(\nu)/r$ and $\nu = 1/r$

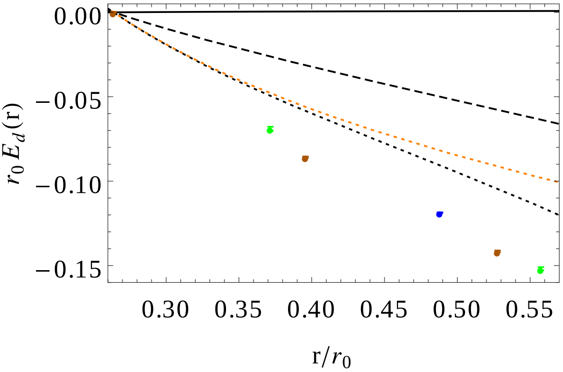


Figure: Comparison of the lattice data and E_d. The continuous black line corresponds to the NLO at fixed scale $\nu = 2/0.263821 r_0^{-1}$. The dashed black line corresponds to the NLO setting $\nu = 2/r$ and the dotted lines correspond to the LL (orange) and NLL (black) setting $\nu = 2/r$ and $\nu_{us} = C_A\alpha(2/r)/r$.

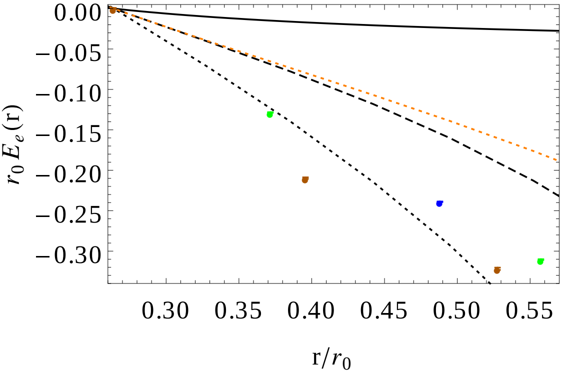


Figure: Comparison of the lattice data and E_e. The continuous black line corresponds to the NLO at fixed scale $\nu = 2/0.263821 r_0^{-1}$. The dashed black line corresponds to the NLO setting $\nu = 1/r$ and the dotted lines correspond to the LL (orange) and NLL (black) setting $\nu = 1/r$ and $\nu_{us} = C_A \alpha(1/r)/r$.

Long distances

$$E_b(r) = -\frac{2}{3}E^{(1,1)}_{\mathbf{L}^2}(r) - E^{(1,1)}_{\mathbf{p}^2}(r),$$
$$E_c(r) = -E^{(1,1)}_{\mathbf{L}^2}(r),$$
$$E_d(r) = \frac{2}{3}E^{(2,0)}_{\mathbf{L}^2}(r) + E^{(2,0)}_{\mathbf{p}^2}(r),$$
$$E_e(r) = E^{(2,0)}_{\mathbf{L}^2}(r).$$

Lattice data: Koma, Koma & Wittig

Effective string theory:
Barchielli et al.; Perez-Nadal & Soto

$$E^{(1,0)} = \frac{\sigma}{2\pi}\ln(\sigma r^2) + \mu_1,$$
$$E_b(r) = E_e(r) = -\frac{r\sigma}{9},$$
$$E_c(r) = E_d(r) = -\frac{r\sigma}{6}.$$

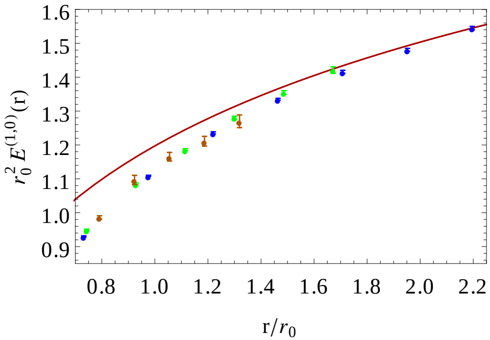
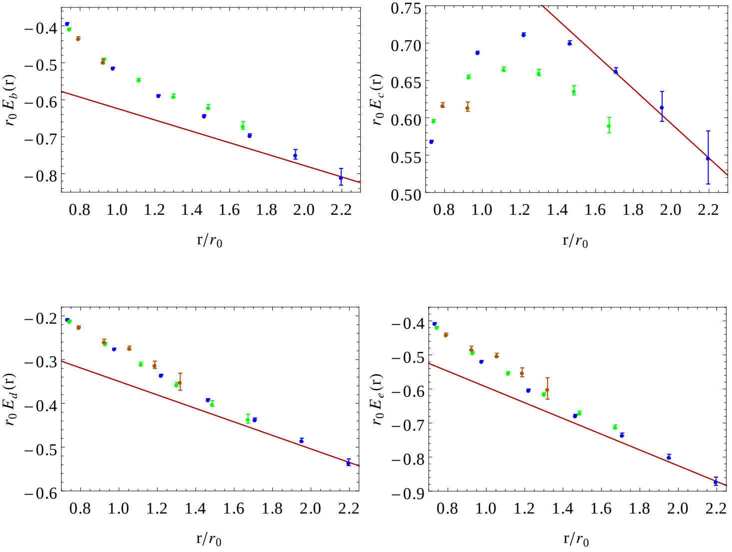


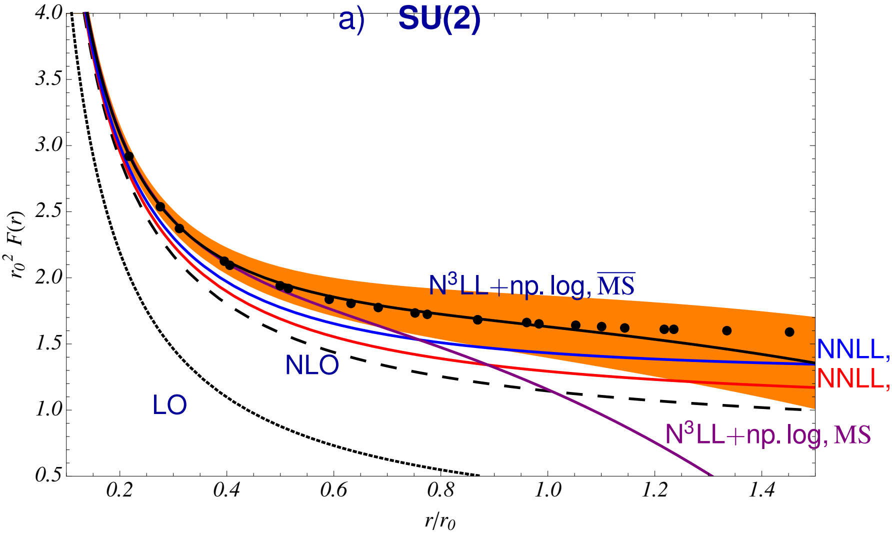
Figure: Comparison of the lattice predictions and the predictions from string effective theory at long distances for $E_1^{(1,0)}$, E_b and E_c.

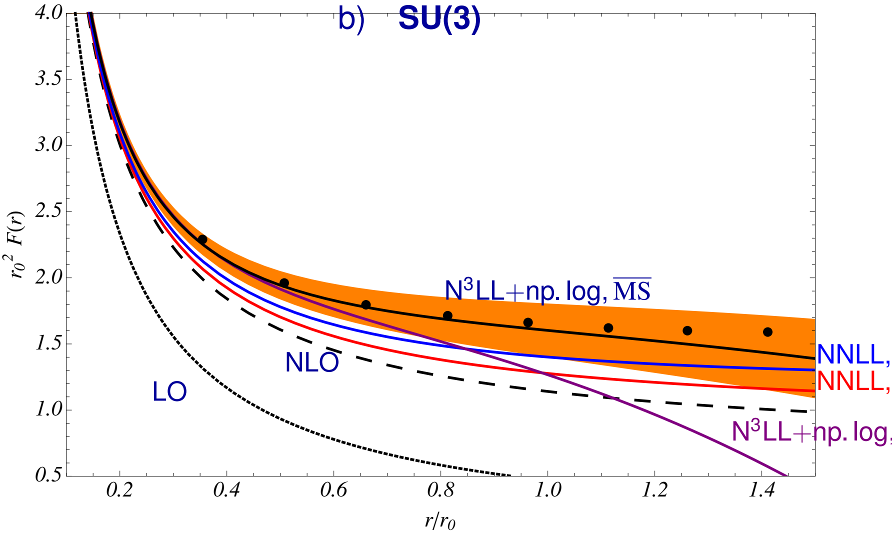


$r_0 E_b(r)$
−0.4
−0.5
−0.6
−0.7
−0.8
0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
r/r_0
$r_0 E_c(r)$
0.75
0.70
0.65
0.60
0.55
0.50
0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
r/r_0
$r_0 E_d(r)$
−0.2
−0.3
−0.4
−0.5
−0.6
0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
r/r_0
$r_0 E_e(r)$
−0.4
−0.5
−0.6
−0.7
−0.8
−0.9
0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
r/r_0

Static potential $D=3$ ($\Delta V \sim C_A\alpha\ln(r\nu)$); Pineda & Stahlhofen

$$
\begin{aligned}
E_s(r) &= C_F\alpha\ln(r^2\nu_s^2\pi e^{\gamma_E}) + \frac{7\pi}{4}C_F C_A\alpha^2 r + C_A^2 C_F\alpha^3\, r^2\Bigg\{\frac{1}{6}\ln^3(r\Delta V_{\rm MS}) \\
&+\ln^2(r\Delta V_{\rm MS})\frac{1}{4}(2\gamma_E-1-2\ln 2) \\
&+\ln(r\Delta V_{\rm MS})\left[\frac{13\pi^2}{384}+\frac{1}{8}\left(4\gamma_E^2+4\ln^2 2-2\gamma_E(1+4\ln 2)-2(11+\ln\pi)\right)\right. \\
&+\left[c_{2,0}^{\rm MS}+\frac{\pi^2}{2304}(39\gamma_E-715+564\ln 2+39\ln\pi)+\frac{209}{24}-\frac{43}{24}\zeta(3)\right. \\
&+\frac{1}{48}(\gamma_E+\ln\pi)\big(\ln^2\pi+3(2\ln 2-1)\ln\pi-\gamma_E(3+18\ln 2+4\ln\pi) \\
&\quad\left.+7\gamma_E^2+12\ln^2 2-66\big)\right] \\
&+\frac{C_A\alpha}{\Delta V_{\rm MS}}\left[\left(\frac{43}{6}-\frac{157}{384}\pi^2\right)\left(\ln\left[\frac{\Delta V_{\rm MS}}{C_A\alpha}\right]-\frac{1}{2}(\ln(16\pi)+\gamma_E)-\frac{1}{8}\right)\right. \\
&\quad\left.\left.+2B_G+c_{3,0}^{\rm MS}\right]\Bigg\}\right.,
\end{aligned}
$$


a) SU(2)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.2
0.4
0.6
0.8
1.0
1.2
1.4
$r_0^2\ F(r)$
r/r_0
N^{3}LL+np. log, $\overline{\mathrm{MS}}$
NNLL,
NNLL,
NLO
LO
N^{3}LL+np. log, MS


b) SU(3)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
$r_0^2\ F(r)$
0.2
0.4
0.6
0.8
1.0
1.2
1.4
r/r_0
$N^3LL+np.\ log,\ \overline{MS}$
NNLL,
NNLL,
$N^3LL+np.\ log,$
NLO
LO

CONCLUSIONS

- Obtained the short-distance behavior of the $1/m$ potential and of the $1/m^2$ spin-independent momentum-dependent heavy quarkonium potentials, with $\mathcal{O}(\alpha^3)$ and $\mathcal{O}(\alpha^2)$ respectively.
- Our (N)LO computation is sensitive to ultrasoft effects. Therefore, obtaining these results requires a selective resummation of an infinite number of diagrams.
- Preliminary comparison with lattice simulations.
- Long distance requires corrections to leading order predictions by string theory.
- More quantitative analysis for the $D = 2 + 1$ static potential and comparison with lattice.