Dual Formulation of Lattice QCD in the Strong Coupling Regime

Wolfgang Unger, Bielefeld University
in collaboration with Giuseppe Gagliardi, Jangho Kim

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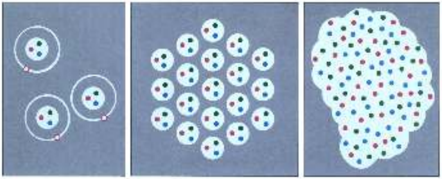

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Overview

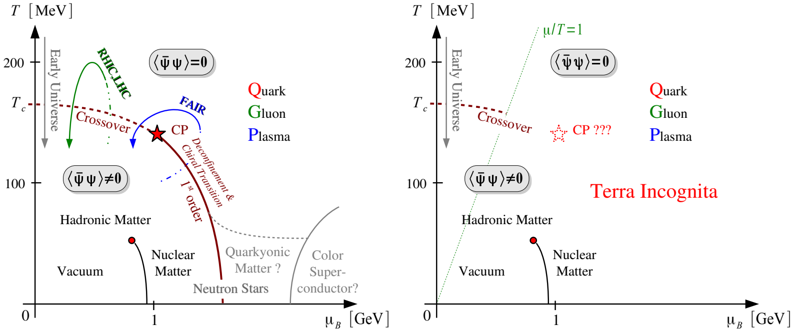
1. Why Lattice QCD in the Strong Coupling Regime?
2. Link Integration Revisited
3. Dual Representation in the Strong Coupling Regime
4. The Phase Diagram in the Strong Coupling Regime

   $\rightarrow$ see talk of Jangho Kim

Motivation

Why use strong coupling methods for finite density QCD?

QCD Phase Diagram and Finite Density Sign Problem



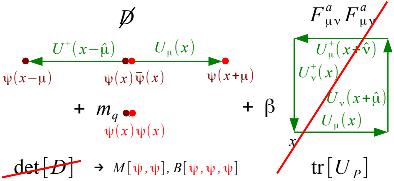
- **Sign problem:** no direct RHMC simulations at finite μ, fermion determinant-basesd algorithms cannot address large μ (yet)

Sign problem is **representation dependent!**

- **Change of Variables:** Hubbard Stratonovich trafos [Vairinhos & de Forcrand '14], Induced QCD [Brandt, Lohmayer & Wettig '16], Abelian Color Cycles [Marcha & Gattringer '17]

Why Lattice QCD at Strong Coupling?

Alternative approach:



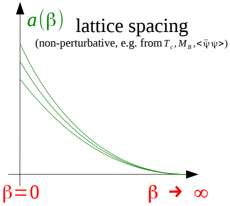
Limit of strong coupling: $\beta = \frac{6}{g^2} \to 0$

- gauge fields $U_\mu(x)$ can be integrated out
- **"dual" representation:** via color singlets on the links!
- at strong coupling: **mesons** and **baryons**

Advantage:

- has chiral symmetry breaking and confinement
- (almost) no sign problem
- fast simulations (no supercomputers necessary)

$\Rightarrow$ **complete phase diagram** can be calculated



Caveat:

- was limited to **infinitely strong coupling** $\to$ **coarse lattices**

Gauge corrections for $\beta > 0$ needed!

Strong Coupling Partition Function

Exact rewriting after Grassmann integration: Mapping onto **discrete system:**

[Rossi & Wolff '84], [Karsch & Mutter '89]

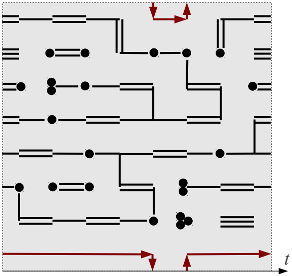
$$Z_F(m_q,\mu) = \sum_{\{k,n,\ell\}} \underbrace{\prod_{b=(x,\mu)} \frac{(N_c-k_b)!}{N_c! k_b!}}_{\text{meson hoppings } M_x M_y} \underbrace{\prod_x \frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral condensate } \bar\psi\psi} \underbrace{\prod_\ell w(\ell,\mu)}_{\text{baryon hoppings } \bar B_x B_y}$$

$k_b \in \{0,\dots N_c\}$, $n_x \in \{0,\dots N_c\}$, $\ell_b \in \{0,\pm 1\}$, QCD: $N_c = 3$

- Grassmann constraint:

$$n_x + \sum_{\hat\mu=\pm\hat0,\dots\pm\hat d} \left(k_{\hat\mu}(x) + \frac{N_c}{2}|\ell_{\hat\mu}(x)|\right) = N_c$$

- weight $w(\ell,\mu)$ and sign $\sigma(\ell)\in\{-1,+1\}$ for oriented baryonic loop ℓ depends on loop geometry
- Sampled via Worm algorithm [Prokof'ev & Svistunov '01], [Adams & Chandrasekharan '03]



finite quark mass

Gauge Integrals

Revisiting Link Integration

In collaboration with Jangho Kim, Giuseppe Gagliardi

One-Link Integrals

- Link integration **no longer factorizes** when coupled by plaquettes from S_G:

$$U_P = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu(x+\hat{\nu})^\dagger U_\nu(x)^\dagger \equiv U_\mu(x) S_P, \qquad P = (x, \mu, \nu)$$

- Our strategy: **Brute force expansion of the full action**
on each link, the matrix $U \equiv U_\mu(x)$ is integrated out, with the quark matrices ${\mathcal{M}_i}^j = \bar{q}^j(y) q_i(x)$ and the staples S_P:

$$\mathcal{J}(\mathcal{M}, \mathcal{M}^\dagger, S_P, S_P^\dagger) = \int_G dU\, e^{\mathrm{tr}[\mathcal{M}^\dagger U] + \mathrm{tr}[\mathcal{M} U^\dagger]} e^{\frac{\beta}{2N_c} \sum_{P \supset U} (\mathrm{tr}[U S_P] + \mathrm{tr}[U^\dagger S_P^\dagger])}$$

$$= \int_G dU \sum_{\kappa_a, \kappa_b} \frac{1}{\kappa_a! \kappa_b!} \mathrm{tr}[\mathcal{M}^\dagger U]^{\kappa_a} \mathrm{tr}[\mathcal{M} U^\dagger])^{\kappa_b} \sum_{P \supset U} \sum_{n_P, \bar{n}_P} \frac{\left(\frac{\beta}{2N_c}\right)^{n_P + \bar{n}_P}}{n_P! \bar{n}_P!} \mathrm{tr}[U S_P]^{n_P} \mathrm{tr}[U^\dagger S^\dagger]^{\bar{n}_P}$$

$$= \sum_{\kappa_a, \kappa_b} \sum_{P \supset U} \sum_{n_P, \bar{n}_P} \frac{1}{\kappa_a! \kappa_b!} \frac{\left(\frac{\beta}{2N_c}\right)^{n_P + \bar{n}_P}}{n_P! \bar{n}_P!} \int_G dU \mathrm{tr}[\mathcal{M}^\dagger U]^{\kappa_a} \mathrm{tr}[\mathcal{M} U^\dagger])^{\kappa_b} \mathrm{tr}[U S_P]^{n_P} \mathrm{tr}[U^\dagger S_P^\dagger]^{\bar{n}_P}$$

$$= C(\beta, \{n_p, \bar{n}_p\})^{a,b}_{{i^j}_{,k}{}^l} \sum_{\{i_\alpha, j_\alpha, k_\beta, l_\beta\}} \prod_{\alpha=1}^{\kappa_a} \prod_{\beta=1}^{\kappa_b} {\mathcal{M}_{j_\alpha}}^{i_\alpha} {\mathcal{M}^\dagger{}_{l_\beta}}^{k_\beta} \mathcal{I}^{a+\kappa_a, b+\kappa_b}_{{i^j}_{,k}{}^l}, \quad a = \sum_{P \supset U} n_P, \quad b = \sum_{P \supset U} \bar{n}_P$$

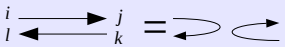
One-Link Integrals

- Integrals
$$\mathcal{I}^{a,b}_{i^{j},k^{l}} \equiv \int_G dU \prod_{\alpha=1}^{a} U_{i_\alpha}{}^{j_\alpha} \prod_{\beta=1}^{b} (U^\dagger)_{k_\beta}{}^{l_\beta}$$
are known, recursively [Creutz '80] or via **Weingarten functions:**
- Express everything in permutations σ, τ; for $\mathrm{U}(N_c)$ [Collins '03] $a = b \equiv n$:

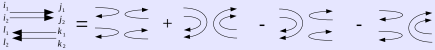
$$\mathcal{I}^{n,n}_{i^{j},k^{l}} = \sum_{\sigma,\tau \in S_n} \prod_{q=1}^{n} \left(\delta_{i_{a_q}}^{\ l_q} \delta_{k_q}^{\ j_{a_\pi(q)}} \right) \mathrm{Wg}^{n,N_c}([\sigma\tau^{-1}]), \quad \mathrm{Wg}^{n,N}(\rho) = \frac{1}{(n!)^2} \sum_{\substack{\lambda \vdash n \\ h(\lambda) \leq N_c}} \frac{(f^\lambda)^2}{D_\lambda} \chi^\rho_\lambda$$

- λ: irrep given by integer partition / Young diagram of bounded height
- f_λ: dimension of the irreducible representation of S_n
- D_λ: dimension of the irreducible representation of $U(N_c)$ and $SU(N_c)$
- ρ: the conjugacy class (given by the cycle structure of the permutation $\sigma\tau^{-1}$)
- χ^ρ_λ: the character of S_n example: $\mathrm{Wg}^{3,N_c}([21]) = \frac{-1}{(N_c{}^2-1)(N_c{}^2-4)}$

$$\int dU\, U_{ij} U^{+}_{kl} = \frac{1}{N_c} \delta_{il} \delta_{jk}:$$



$$\int dU\, U_{i_1 j_1} U_{i_2 j_2} U^{+}_{k_1 l_1} U^{+}_{k_2 l_2} = \frac{1}{N_c^2-1} \left(\delta_{i_1 l_1} \delta_{i_2 l_2} \delta_{j_1 k_1} \delta_{j_2 k_2} + \delta_{i_1 l_2} \delta_{i_2 l_1} \delta_{j_1 k_2} \delta_{j_2 k_1} \right) - \frac{1}{N_c (N_c^2-1)} \left(\delta_{i_1 l_2} \delta_{i_2 l_1} \delta_{j_1 k_1} \delta_{j_2 k_2} + \delta_{i_1 l_1} \delta_{i_2 l_2} \delta_{j_1 k_2} \delta_{j_2 k_1} \right)$$



One-Link Integrals

- Integrals

$$\mathcal{I}^{a,b}_{i}{}^{j}{}_{,k}{}^{l} \equiv \int_G dU \prod_{\alpha=1}^{a} U_{i_\alpha}{}^{j_\alpha} \prod_{\beta=1}^{b} (U^\dagger)_{k_\beta}{}^{l_\beta}$$

are known, recursive [M. Creutz '80] or via **Weingarten functions:**

- We extended and **showed for** $\mathrm{SU}(N_c)$ $(q = (a-b)/N_c)$:

$$\mathcal{I}^{a,b}_{i}{}^{j}{}_{,k}{}^{l} = B_{N_c}(q) \sum_{\mathcal{D}_b=1}^{a} \sum_{\mathcal{E}_{a-b}=1}^{a} \mathcal{I}^{a-b,0}_{i_{\{e\}}}{}^{j_{\{e\}}} \sum_{\sigma,\tau\in S_b} \prod_{r=1}^{b} \left(\delta_{i_{d_{\sigma(r)}}}{}^{l_r} \delta_{k_r}{}^{j_{d_{\tau(r)}}} \right) \mathrm{Wg}^{b,q+N_c}([\sigma\tau^{-1}])$$

$$\mathcal{I}^{qNc,0}_{i_{\{e\}}}{}^{j_{\{e\}}} = [\epsilon_{i_{e_1},\ldots i_{e_{N_c}}} \ldots \epsilon_{i_{e_{(q-1)N_c+1}},\ldots i_{e_{qN_c}}}][\epsilon^{j_{e_1},\ldots j_{e_{N_c}}} \ldots \epsilon^{j_{e_{(q-1)N_c+1}},\ldots j_{e_{qN_c}}}]$$

$$B_{N_c}(q) = (qN_c)! \prod_{s=0}^{N_c-1} \frac{s!}{(q+s)!}$$

- $\mathcal{D}_b = \{d_1, \ldots d_b\}$: subindex set that enters Kronecker deltas
- $\mathcal{E}_{a-b} = \{e_1, \ldots e_{a-b}\}$: subindex set that enters epsilon tensors

One-Link Integrals

- Depending on how many fermions hoppings contribute to the link integral: restrict the contributions to the Weingarten function:

$$\mathrm{Wg}^{n,N}{}_{\lambda}(\rho) = \frac{1}{(n!)^2}\frac{(f^{\lambda})^2}{D_\lambda}\chi^{\rho}_{\lambda}, \quad \mathrm{Wg}^{n,N}{}_{\Lambda}(\rho) = \sum_{\substack{\lambda\vdash n\\ \lambda\in\Lambda}} \mathrm{Wg}^{n,N}{}_{\lambda}(\rho)$$

- At strong coupling: fermions dictate anti-symmetric irrep $\Lambda = \{[1^n]\}, \quad n \leq N_\mathrm{c}$

$$\mathcal{I}^{n,n}_{i^j,k^l}(\Lambda) = \sum_{\mathcal{D}_n=1}^{n} \sum_{\sigma,\tau\in S_n} \prod_{r=1}^{n} \left(\delta_{i_{d_{\sigma(r)}}}^{\;l_r}\, \delta_{k_r}^{\;j_{d_\tau(r)}}\right) \mathrm{Wg}^{n,N_\mathrm{c}}_{\lambda=[1^n]}([\sigma\tau^{-1}])$$

$$= \sum_{\mathcal{D}_n=1}^{n} \sum_{\rho\vdash n} \prod_{r=1}^{n} \mathrm{tr}_\rho[\ldots] \frac{(N_\mathrm{c}-n)!}{N_\mathrm{c}!\,n!}\mathrm{sgn}(\rho), \qquad \rho = [\sigma\tau^{-1}]$$

$$\mathcal{I}^{N_\mathrm{c},0}_{i^j,k^l}(\Lambda) = \sum_{\mathcal{E}_{N_\mathrm{c}}=1}^{N_\mathrm{c}} [\epsilon_{i_{e_1},\ldots i_{e_{N_\mathrm{c}}}}][\epsilon^{j_{e_1},\ldots j_{e_{N_\mathrm{c}}}}] = \det[\ldots]$$

- reproduces well known result!
- *sgn*(ρ) is cancelled by ordering fermions in Grassmann integration

One-Link Integrals

- Depending on how many fermions hoppings contribute to the link integral: restrict the contributions to the Weingarten function:

$$\mathrm{Wg}^{n,N}{}_{\lambda}(\rho) = \frac{1}{(n!)^2}\frac{(f^{\lambda})^2}{D_\lambda}\chi^{\rho}_{\lambda}, \quad \mathrm{Wg}^{n,N}{}_{\Lambda}(\rho) = \sum_{\substack{\lambda \vdash n \\ \lambda \in \Lambda}} \mathrm{Wg}^{n,N}{}_{\lambda}(\rho)$$

- For the leading order gauge corrections, the additional gauge link from the plaquette allows partial symmetrization: $\Lambda = \{[1^n], [21^{n-2}]\}$

$$\mathcal{I}^{n,n}_{i^j,k^l}(\Lambda) = \sum_{\mathcal{D}_n=1}^{n} \sum_{\sigma,\tau \in S_n} \prod_{r=1}^{n} \left(\delta_{i_{d_{\sigma(r)}}}^{l_r} \delta_{k_r}^{j_{d_\tau(r)}} \right) \mathrm{Wg}^{n,N_c}_{\lambda=[1^n]}([\sigma\circ\tau^{-1}]) + \mathrm{Wg}^{n,N_c}_{\lambda=[21^{n-2}]}([\sigma\circ\tau^{-1}])$$

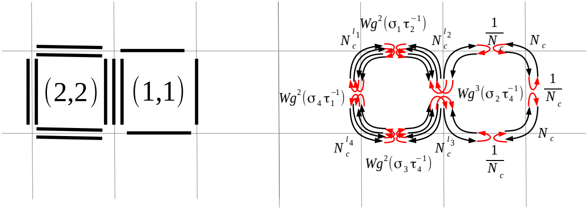
$$= \sum_{\mathcal{D}_n=1}^{n} \sum_{\sigma,\tau \in S_n} \prod_{r=1}^{n} \left(\delta_{i_{d_{\sigma(r)}}}^{l_r} \delta_{k_r}^{j_{d_\tau(r)}} \right) \frac{1}{(n!)^2} \left(\frac{(N_c-n)!}{N_c!}\mathrm{sgn}([\sigma\circ\tau^{-1}]) + \frac{(N_c+1-n)!}{(N_c+1)!}\chi^{[\sigma\circ\tau^{-1}]}_{[21^{n-2}]} \right)$$

$$= \sum_{\mathcal{D}_n=1}^{n} \sum_{\rho \vdash n} \mathrm{tr}_\rho[\ldots] \frac{(N_c-n)!}{N_c!n!} \left(\mathrm{sgn}\rho + \frac{N_c+1-n}{N_c+1}\chi^{\rho}_{[21^{n-2}]} \right)$$

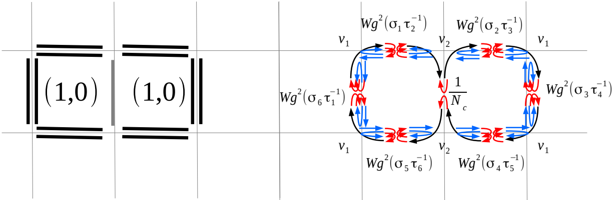
- reproduces the $\mathcal{O}(\beta)$ correction
- $\mathrm{tr}_\rho[\ldots] = \prod_i \mathrm{tr}[(\ldots)^i]^{\rho}_i, \qquad \sum_i i\rho_i = n$

Putting together the link integrals

- "Dimer Approach" to Pure Gauge Theory: all irreps λ contribute, color contraction along plaquettes. **"k-dimers" carry two permutations**

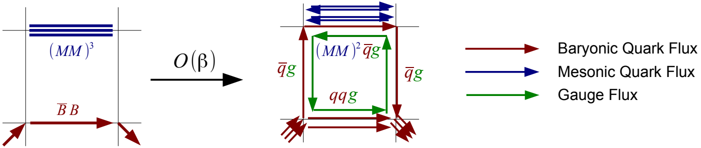


- Together with Staggered Fermions: only some part of the Weingarten functions enter, additional site weights $v_i(N_c)$ from Grassmann integration



Lattice QCD in the Strong Coupling Regime

Dual Variable (Color Singlet) Representation



Plaquette Occupation Numbers and Flux Variables

New interpretation of **dual representation:**

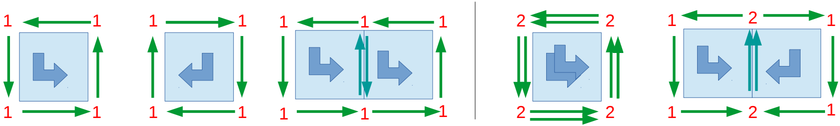
- at strong coupling limit: dimers=meson hoppings, 3-fluxes=baryons
- away from strong coupling limit: **dimers = color singlets** (=U(3) sector),
  3-fluxes = color triplets
- in principle: also 6-flux, 9-flux,... sectors from pure gauge sector, but neglected here

Plaquette occupation numbers at plaquette coordinate P:

- **equivalence classes** of difference of fundamental plaquettes $\mathrm{Tr}[U_P]$ and anti-fundamental plaquettes $\mathrm{Tr}[U_P^\dagger]$ from gauge action:

$$n_P = n_f(P) - n_{\bar{f}}(P) \mod 3 \qquad \Rightarrow \qquad \beta \mapsto u(\beta)$$

- plaquette orientations induce **link fluxes** f_b and define **flux sites** f_x:



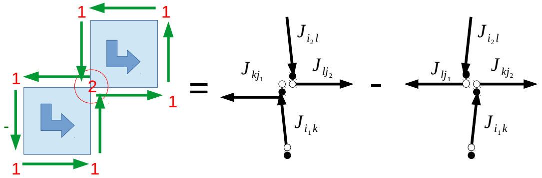
Grassmann integration

Free color indices need to be **contracted** at each site (for given ensemble of plaquettes)

- in general, gives rise to tensor networks/vertex model → (too) difficult!

Simplifications in the U(3) sector:

- plaquette occupations $n_p, n_{p'} \in \mathbb{Z}$ can **only differ by** ± 1 if p, p' adjacent
- plaquette occupations $n_p, n_{p'} \in \mathbb{Z}$ cannot share a site if they do not share a link:



⇒ Gauge fluxes f_b form self-avoiding loops: $\ell_f = \{b | f_b = \pm 1\}$

- Ensemble of plaquette world sheets bounded by quarks
- Does not apply to the additional SU(3) contributions: so far restricted to first non-trivial contribution (3-flux sector)

MDP+P partition sum: monomers+dimers+worldlines+worldsheets

$$Z(m_q,\mu)=\sum_{\{k,n,\ell,n_p\}}\underbrace{\prod_{b=(x,\mu)}\frac{(N_c-k_b)!}{N_c!(k_b-|f_b|)!}}_{\text{singlet hoppings } M_xM_y}\underbrace{\prod_x\frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral condensate } \bar\psi\psi}\underbrace{\prod_{\ell_3}w(\ell_3,\mu)}_{\text{triplet hoppings } \bar B_x B_y}\underbrace{\prod_{\ell_f}\tilde w(\ell_f,\mu)}_{\text{weight modifications}}\underbrace{\prod_P\frac{\left(\frac{\beta}{2N_c}\right)^{n_P+\bar n_P}}{n_P!\bar n_P!}}_{\text{gluon propagation}}$$

$k_b\in\{0,\dots N_c\}$, $n_x\in\{0,\dots N_c\}$, $\ell_b\in\{0,\pm1\}$, $f_b=\partial n_p$, $f_x=\frac{1}{2}\sum_b|f_b|$

- color constraint:

$$n_x+\sum_{\hat\nu=\pm\hat 0,\dots\pm\hat d}\left(k_{\hat\nu}(x)+\frac{N_c}{2}|\ell_{\hat\nu}(x)|\right)=N_c+f_x$$

- 3-flux weight $\tilde w(\ell_f)$ involves additional **site weights** v_x and **link weights** $w_b(B)$:

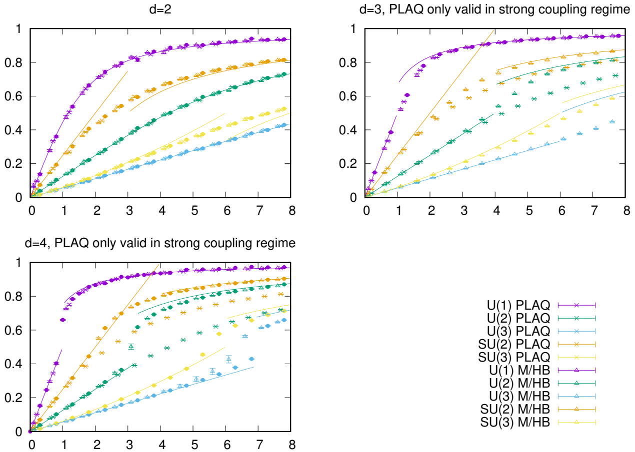
at $\mathcal{O}(\beta)$: $\quad w(B_1)=\frac{1}{N_c!(N_c-1)!},\quad w(B_2)=\frac{(N_c-1)!}{N_c!},$

new at $\mathcal{O}(\beta^2)$: $\quad w(B_3)=\frac{1}{N_c!(N_c-1)!(N_c-2)!},\quad w(B_4)=\frac{(N_c-1)!(N_c-2)!}{N_c!}$

Sign of a configuration factorizes in **3-flux sign** and **gauge flux sign**!

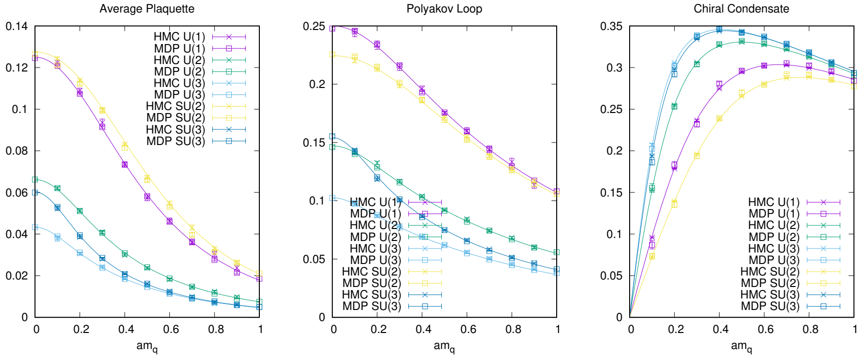
$$\sigma(C)=\prod_{\ell_f}\sigma(\ell_f)\prod_{\ell_3}\sigma(\ell_3),\quad \sigma(\ell)=(-1)^{1+w(\ell)+N_{(\ell)}}\prod_{\tilde\ell}\eta_\mu(x)$$

Crosschecks for Pure Gauge Theory



- Plaquette algorithm for $\mathrm{U}(N_c)$ and $\mathrm{SU}(N_c)$ Pure Gauge Theory: extensions of the U(1) random surface ensemble [Korzec & Wolff '13]
- for $d > 2$ and $N_c > 1$: validity confined to the strong coupling regime

Crosschecks with staggered quarks for U(N_c), SU(N_c)



- Plaquette and Polykov Loop agree with analytic prediction and HMC ($\mu = 0$)

Conclusions

Results:

- Integrals and resulting invariants (dimers, fluxes) related to Weingarten functions
- Grassmann integration simplifies in U(3) sector
  $\rightarrow$ gauge fluxes form self-avoiding loops
- **Direct Simulations with $\mathcal{O}(\beta)^2$ corrections** possible,
  extending the reweighting result [de Forcrand, Langelage, Philipsen, U. '14]
  $\rightarrow$ see talk of Jangho Kim

Prospects for the future:

- In practice restricted to rather small β due to the sign problem
- Improve plaquette algorithm $\rightarrow$ incorporate **character expansion**
- Surface worm [Delgado, Gattriger & Schmidt '12] instead of local plaquette update?

Backup: Strong Coupling LQCD at Finite Temperature

How to vary the temperature?

- $aT = 1/N_\tau$ is discrete with N_τ even
- $aT_c \simeq 1.5 \qquad \Rightarrow \qquad$ we cannot address the phase transition!

Solution: introduce an **anisotropy** γ in the Dirac couplings:

$$\mathcal{L}_{\mathrm{F}} = \sum_\mu \frac{\gamma^{\delta_{\mu 0}}}{2} \eta_\nu(x) \left(e^{\mu \delta_{\mu 0}} \bar{\chi}(x) U_\nu(x) \chi(x+\hat{\mu}) - e^{-\mu \delta_{\mu 0}} \bar{\chi}(x+\hat{\mu}) U_\mu^\dagger(x) \chi(x) \right)$$

$$Z_F(m_q, \mu, \gamma) = \sum_{\{k,n,\ell\}} \prod_{b=(x,\mu)} \frac{(N_c - k_b)!}{N_c! k_b!} \gamma^{2 k_b \delta_{\mu 0}} \prod_x \frac{N_c!}{n_x!} (2am_q)^{n_x} \prod_\ell w(\ell, \mu)$$

- Mean-field at strong coupling: $\frac{a}{a_t} \equiv \xi(\gamma) = \gamma^2$

$\Rightarrow$ mean-field defintition of the temperature: $aT \simeq \frac{\gamma^2}{N_\tau}$

Non-perturbative determination of the anisotropy $\xi \equiv a/a_\tau(\gamma)$ [arXiv:1701.08324]

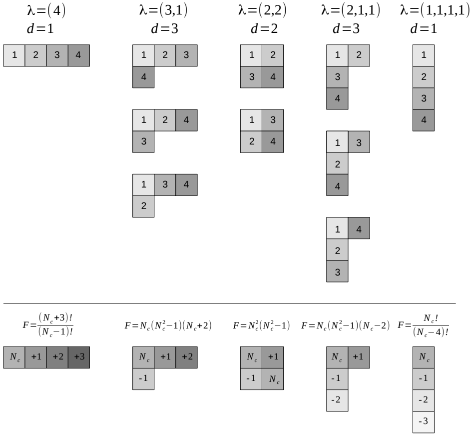
Backup: Young Tableaux - Color Structure

So, what are the C_λ^τ (and for $\mathcal{O}(\beta)$, $\mathcal{O}(\beta^2)$: C_λ^{τ,ρ_1}, $C_\lambda^{\tau,\rho_1,\rho_2}$, etc) ?

- answer: related to **irreducible representations** of the **symmetric group** S_n with $n = \kappa_1 + r = \kappa_2 + s$

Young Tableaux $\lambda = (\lambda_1, \ldots \lambda_k)$:

- Standard Young tableaux correspond to **irreps of** S_n with dimension $d_\lambda = \frac{n!}{H_\lambda}$ with Hook lengths H_λ
- used to determine dimension $D_\lambda = \frac{F_\lambda}{H_\lambda}$ of irreps of SU(N_c)



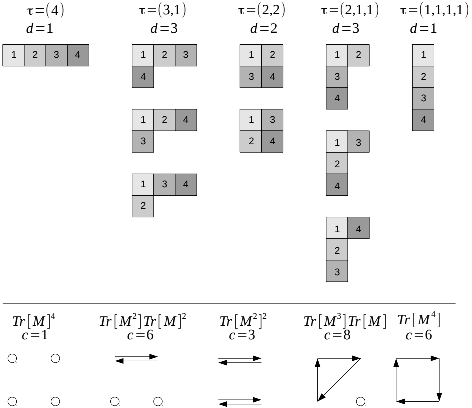
Backup: Young Tableaux - Flavor Structure

So, what are the C_λ^τ (and for $\mathcal{O}(\beta)$, $\mathcal{O}(\beta^2)$: C_λ^{τ,ρ_1}, $C_\lambda^{\tau,\rho_1,\rho_2}$, etc) ?

- answer: related to **irreducible representations** of the **symmetric group S_n** with $n = \kappa_1 + r = \kappa_2 + s$

Young Tableaux $\tau = (t_1, \dots t_k)$:

- conjugacy classes
  $\leftrightarrow$ cycle structure of S_n
  $\leftrightarrow$ flavor permutations
- at a given order n, the **trace structure** $\tau = (t_1, \dots t_k)$ with $T = \prod \mathrm{Tr}[M^i_{xy}]^{t_i}$ is equivalent to a partition of n: $\sum_i i t_i = n$
- example: $\pi = (136)(24)(58)(7)$
  $\rightarrow \quad t_1 = 1,\ t_2 = 2,\ t_3 = 1$



Backup: Table of Characters and Invariants

τ λ :	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)	Sum
(4,0,0,0)	1	3	2	3	1	10
(2,1,0,0)	1	1	0	-1	-1	0
(0,2,0,0)	1	-1	2	-1	1	2
(1,0,1,0)	1	0	-1	0	1	1
(0,0,0,1)	1	-1	0	1	-1	0
Sum	5	2	3	2	1	

Table: Characters χ_λ^τ for $n = 4$

τ λ :	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)	Sum
(4,0,0,0)	1	9	4	9	1	24
(2,1,0,0)	6	18	0	-18	-6	0
(0,2,0,0)	3	-9	12	-9	3	0
(1,0,1,0)	8	0	-16	0	8	0
(0,0,0,1)	6	-18	0	18	-6	0
Sum	24	0	0	0	0	

Table: Invariants C_λ^τ for $n = 4$