# Global Symmetries of Naive and Staggered Fermions in Arbitrary Dimensions

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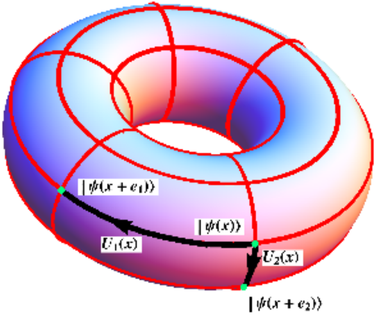
# $d$-dim Lattice

**on** $V = L_1 \times L_2 \times \ldots \times L_d$ **with gauge group**
$G = \mathrm{SU_f}(N_\mathrm{c} \geq 2), \mathrm{SU_a}(N_\mathrm{c} \geq 2)$




- wave function at a lattice site $x$: $|\psi(x)\rangle$
- gauge field at $x$ in direction $\mu$: $U_\mu(x)$
- translation operator in direction $\mu$:

$$T_\mu|\psi(x)\rangle = (-1)^{\delta_{\mu d}\delta_{x_d L_d}} U_\mu(x)|\psi(x+e_\mu)\rangle$$

- naive lattice Dirac operator:

$$D = \gamma_\mu(T_\mu - T_\mu^\dagger)$$

- generalized (Euclidean) Dirac matrices:

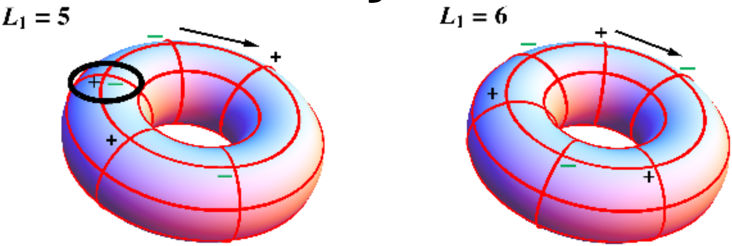
$$[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu}$$

# Symmetries

- **chiral symmetry** (only when $d$ is even):
  - $\gamma^{(5)} = \imath^{-d(d-1)/2}\gamma_1\gamma_2\cdots\gamma_d$
  - chirality: $[D, \gamma^{(5)}]_+ = 0$

- **charge conjugation** (only when group repr. real or quaternion):
  - complex conjugation: $K$
  - real repr.: $\mathrm{SU_a}(N_c \geq 2)$: $[K, U]_- = 0\ \forall U \in G$
  - quaternion repr.: $\mathrm{SU_f}(N_c = 2)$: $[K\tau_2, U]_- = 0\ \forall U \in G$
  - charge conjugation operator: $C = K\zeta$ or $C = K\tau_2\zeta$ with $\zeta$ product of $\gamma$-matrices
  - $[C, \imath^{d(d-1)/2}\gamma_\nu]_- = [C, \imath^{d(d-1)/2}\gamma^{(5)}]_- = [C, \imath^{d(d-1)/2}D]_- = 0$
  - $C^2 = (-1)^{(d+2)(d+1)d(d-1)/8}(K\zeta)^2$

# Artificial Symmetries




Definition of $\Gamma_1$ does not work !

Definition of $\Gamma_1$ does work !

- if $L_j$ even: $\Gamma_j|\psi(x)\rangle = (-1)^{x_j}|\psi(x)\rangle$
- $[\Gamma_\mu, T_\mu]_+ = 0$ and $[\Gamma_\mu, T_\nu]_- \overset{\mu\neq\nu}{=} 0$
- Define: $\Gamma_j^{(5)} = \Gamma_j\gamma_j$
- $[D, \Gamma_j^{(5)}]_+ = [\gamma^{(5)}, \Gamma_j^{(5)}]_+ = [C, \imath^{d(d-1)/2}\Gamma_j^{(5)}]_- = 0$
- $[\Gamma_i^{(5)}, \Gamma_j^{(5)}]_+ = 2\delta_{ij}$

# Reduced Dirac Operator

- **Schur's lemma:** There is a basis in which $D$ has the form:

$$D = \begin{cases} D_{\mathrm{red}} \otimes \mathrm{diag}\,(\mathbb{1}, -\mathbb{1}), & d - N_{\mathrm{ev}} \text{ odd}, \\ D_{\mathrm{red}} \otimes \mathbb{1}, & d - N_{\mathrm{ev}} \text{ even} \end{cases}$$

- Reduced Dirac Operator:

$$D_{\mathrm{red}} = \sum_{\mu=1}^{N_{\mathrm{ev}}} D_\mu^{(\mathrm{red})} + \sum_{\mu=N_{\mathrm{ev}}+1}^{d} D_\mu^{(\mathrm{red})} \gamma_\mu$$

- $D_{\mathrm{red}}$ is maximally Kramer's degenerate
- $D_{\mathrm{red}}$ is only chiral if $d - N_{\mathrm{ev}}$ is even
- $D_{\mathrm{red}} = D_{\mathrm{staggered}}$ for $d = N_{\mathrm{ev}}$

# Cartan Classification Scheme

| | RMT | Matrix Form |
|---|---|---|
| Hermitian | GUE($n$) | $H = H^\dagger$ |
| real symmetric | GOE($n$) | $H = H^T = H^*$ |
| real self-dual | GSE($n$) | $H = \theta_2 H^T \theta_2 = \theta_2 H^* \theta_2$ |
| im. antisymmetric | GAOE$_\nu$($n$) | $H = -H^T = -H^*$ |
| im. anti-self-dual | GASE($n$) | $H = -\theta_2 H^T \theta_2 = -\theta_2 H^* \theta_2$ |
| chiral Herm. | $\chi$GUE$_\nu$($n$) | $H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}$ |
| chiral real | $\chi$GOE$_\nu$($n$) | $H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix},\ W = W^*$ |
| chiral quaternion | $\chi$GSE$_\nu$($n$) | $H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix},\ W = \theta_2 W^* \theta_2$ |
| chiral symmetric | GBOE($n$) | $H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix},\ W = W^T$ |
| chiral antisymmetric | GBSE$_\nu$($n$) | $H = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix},\ W = -W^T$ |

# Symmetry Breaking Patterns

| $d-N_{\rm ev}$ | real repr. | complex repr. | quaternion repr. |
|---|---|---|---|
| $8m$ | $\mathrm{U}(2N_{\rm eff}) \downarrow \mathrm{USp}(2N_{\rm eff})$ | $\mathrm{U}(N_{\rm eff})\times\mathrm{U}(N_{\rm eff}) \downarrow \mathrm{U}(N_{\rm eff})$ | $\mathrm{U}(2N_{\rm eff}) \downarrow \mathrm{O}(2N_{\rm eff})$ |
| $8m+1$ | $\mathrm{O}(2N_{\rm eff}) \downarrow \mathrm{U}(N_{\rm eff})$ | $\mathrm{U}(2N_{\rm eff}) \downarrow \mathrm{U}(N_{\rm eff})\times\mathrm{U}(N_{\rm eff})$ | $\mathrm{USp}(2N_{\rm eff}) \downarrow \mathrm{U}(N_{\rm eff})$ |
| $8m+2$ | $\mathrm{O}(2N_{\rm eff})\times\mathrm{O}(2N_{\rm eff}) \downarrow \mathrm{O}(2N_{\rm eff})$ | $\mathrm{U}(N_{\rm eff})\times\mathrm{U}(N_{\rm eff}) \downarrow \mathrm{U}(N_{\rm eff})$ | $\mathrm{USp}(2N_{\rm eff})\times\mathrm{USp}(2N_{\rm eff}) \downarrow \mathrm{USp}(2N_{\rm eff})$ |
| $8m+3$ | $\mathrm{O}(2N_{\rm eff}) \downarrow \mathrm{O}(N_{\rm eff})\times\mathrm{O}(N_{\rm eff})$ | $\mathrm{U}(2N_{\rm eff}) \downarrow \mathrm{U}(N_{\rm eff})\times\mathrm{U}(N_{\rm eff})$ | $\mathrm{USp}(4N_{\rm eff}) \downarrow \mathrm{USp}(2N_{\rm eff})\times\mathrm{USp}(2N_{\rm eff})$ |
| $8m+4+l$ | quat. repr. for $8m+l$ | see $8m+l$ | real repr. for $8m+l$ |

$N_{\rm eff}=d_{\rm deg}N_{\rm f}$ with $d_{\rm deg}$ degeneracy (without Kramers) of $D$

# Comparison with RMT

**Two Observables**

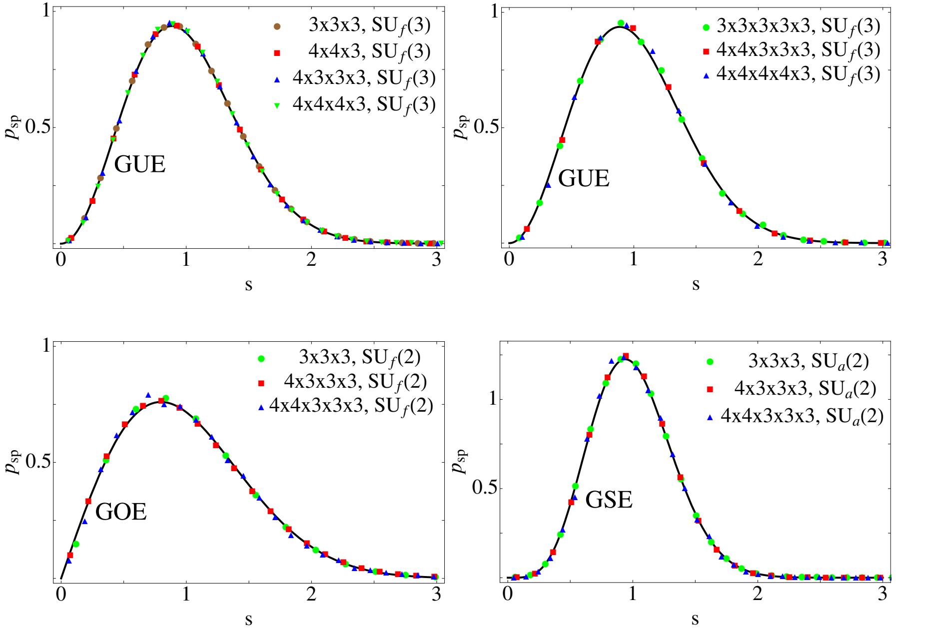
- Dyson index $\beta_{\mathrm{D}} = 1, 2, 4$
- Microscopic level density: e.g. for $\beta_{\mathrm{D}} = 2$

$$\rho_\nu(x) = \frac{|x|}{2} \left( J_\nu^2(x) - J_{\nu+1}(x) J_{\nu-1}(x) \right)$$

- Level spacing distribution (Wigner's surmise)

$$p_{\mathrm{sp}}(s) \propto s^{\beta_{\mathrm{D}}} \exp\left[ -\gamma s^2 \right]$$

# Comparison: MC vs. RMT



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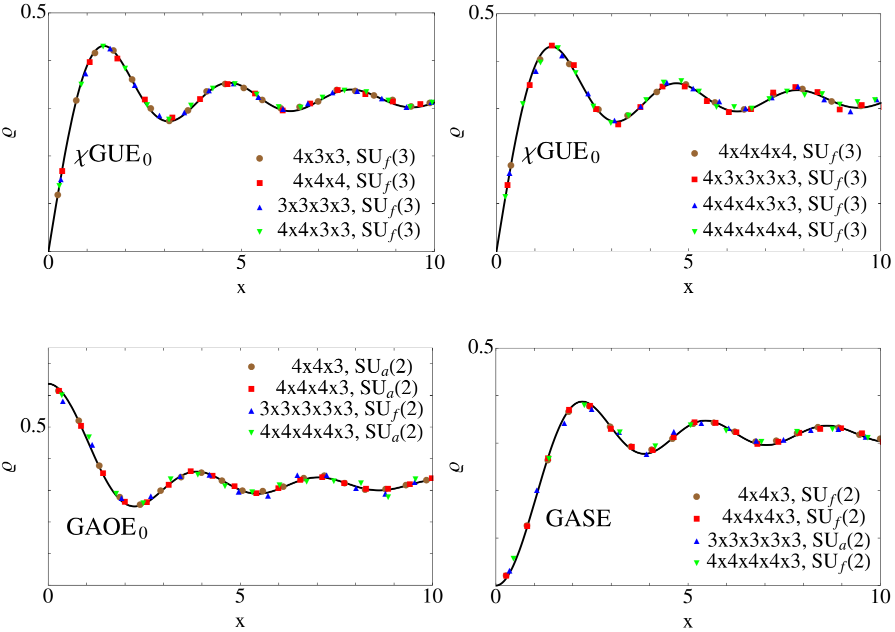

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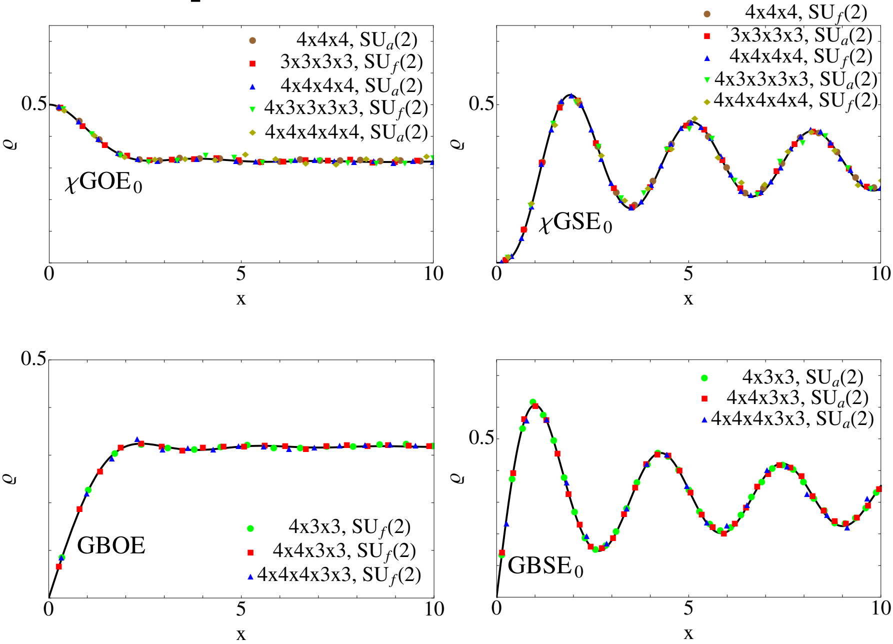

# Summary & Outlook

- **No exact zero modes** in naive and staggered discretization for any dimension **d>2**!
- Global symmetries are **Bott periodic** in the dimension $d$ and the number $N_{\rm ev}$ of directions with even partition
- Global symmetries are those of **($d$-$N_{ev}$)-dim continuum theory**
- Odd partition in **all** directions have always the correct symmetry!
- Staggered fermions have **always** the global symmetries of the continuum theory at **d=8**!

<u>Open Problems:</u>

- Do the global symmetries change when taking the continuum limit?
- **If yes:** How is this happening?

# Thank You for Your Attention!

M. K., J. J. M. Verbaarschot, and S. Zafeiropoulos

*Dirac Spectra of 2-dimensional QCD-like theories*

Phys. Rev. D **90**, 085013 (2014)
arXiv:1405.0433 [hep-lat]

M.K. and T. R. Würfel

*Shift of Symmetries of Naive Fermions in QCD-like Lattice Theories*

accepted for publication in Phys. Rev. D
arXiv:1703.08083 [hep-lat]