Abelian Color Cycles approach for the dualization of non-abelian lattice gauge theories

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Dual approach

- It consists in exactly mapping the system to new degrees of freedom, called **dual variables**.
- Different representations of a theory better describe different aspects of the system.
  [Savit, Rev. Mod. Phys. **D38** (1988)]
- Method developed in order to overcome the sign problem.
  [Gattringer, PoS(LATTICE 2013)002]
- Abelian color cycles: new approach for the dualization of non-abelian field theories, resulting in a strong coupling expansion.
  [Gattringer and CM, Nucl. Phys. B **916** (2017), [1609.00124]]
  [Gattringer and CM, PoS(LATTICE2016)034, [1611.01022]]

Dual approach

General strategy:

- The Boltzmann factor is decomposed into local factors which are then expanded.
- The conventional d.o.f. are integrated out, leading to **constraints for the dual variables**.
- These constraints allow for geometrical interpretation of the system:

$$\text{dual variables for matter} \longleftrightarrow \text{loops}$$
$$\text{dual variables for gauge fields} \longleftrightarrow \text{surfaces}$$

- Dual partition function is a sum over configurations of the expansion indices, i.e. the **dual variables**.

For several abelian lattice field theories this strategy results in a representation of the partition sum at $\mu > 0$ which has only real and positive contributions.
[Mercado, Gattringer and Schmidt, Phys.Rev.Lett. **111** (2013)]

Abelian Color Cycle method

- Approach developed to dualize non-abelian lattice gauge theories.
- Key is to rewrite the trace and matrix multiplications of the Wilson action as explicit sums of complex numbers, so-called Abelian Color Cycles (ACCs).
- The ACC decomposition of the action allows to proceed with the dualization as in the abelian case.
- The ACCs represent paths in color space around plaquettes.
- They allow to construct a worldsheet representation of non-abelian lattice gauge theories.
- This approach is very general, and it can be applied to theories with matter as well.

[Gattringer and CM, Nucl. Phys. B **916** (2017), [1609.00124]]

SU(3) dualization with ACCs

Partition sum

$$Z = \int D[U] e^{-S_G[U]} \,, \qquad \int D[U] = \prod_{x,\mu} \int_{\mathrm{SU}(3)} dU_{x,\mu} \,, \qquad U_{x,\mu} \in \mathrm{SU}(3)$$

SU(3) lattice gauge action

$$S_G[U] = -\frac{\beta}{3} \sum_{x,\mu<\nu} \mathrm{Re}\,\mathrm{Tr}\, U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu}$$

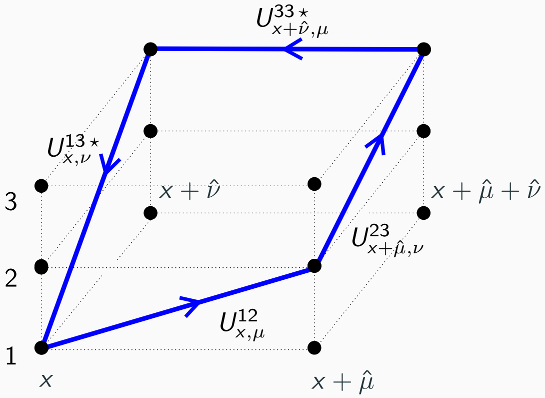
$$= -\frac{\beta}{6} \sum_{x,\mu<\nu} \sum_{a,b,c,d=1}^{3} \left[U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu} + U^{ab\,\star}_{x,\mu} U^{bc\,\star}_{x+\hat{\mu},\nu} U^{dc}_{x+\hat{\nu},\mu} U^{ad}_{x,\nu} \right]$$

$U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu} \in \mathbb{C}$ **Abelian Color Cycles (ACCs)**

Solves reordering problem $\Rightarrow$ "abelian"

Abelian color cycles...

Abelian color cycles $U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}$: paths in color space along plaquettes.



For SU(3) there are a total of 3^4 abelian color cycles (ACC).

...enable the reordering of link elements

Expand the Boltzmann factor and reorder the link elements:

$$Z = \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d=1}^{3} e^{\frac{\beta}{6} U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}} e^{\frac{\beta}{6} U^{ab\,\star}_{x,\mu} U^{bc\,\star}_{x+\hat{\mu},\nu} U^{dc}_{x+\hat{\nu},\mu} U^{ad}_{x,\nu}}$$

$$= \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d=1}^{3} \sum_{n^{abcd}_{x,\mu\nu}=0}^{\infty} \sum_{\bar{n}^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{(\beta/6)^{n^{abcd}_{x,\mu\nu}+\bar{n}^{abcd}_{x,\mu\nu}}}{n^{abcd}_{x,\mu\nu}\,!\,\bar{n}^{abcd}_{x,\mu\nu}\,!}$$

$$\times \left(U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}\right)^{n^{abcd}_{x,\mu\nu}} \left(U^{ab\,\star}_{x,\mu} U^{bc\,\star}_{x+\hat{\mu},\nu} U^{dc}_{x+\hat{\nu},\mu} U^{ad}_{x,\nu}\right)^{\bar{n}^{abcd}_{x,\mu\nu}}$$

$n^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$ and $\overline{n}^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$ **cycle occupation numbers**

Integration of the conventional degrees of freedom

To perform the Haar integration

$$\int dU \prod_{a,b=1}^{3} \left(U^{ab}\right)^{N^{ab}} \left(U^{ab\,\star}\right)^{\bar{N}^{ab}}$$

we choose an explicit parametrization for the SU(3) matrices U.

[Bronzan, Phys. Rev. **D38** (1988)]

The elements U^{ab} of the matrix appear in two forms

- $U^{ab} = r^{ab} e^{\varphi^{ab}}$
- $U^{ab} = \rho^{ab} e^{\alpha^{ab}} + \omega^{ab} e^{\beta^{ab}}$

For the latter we make use of the binomial theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\Rightarrow$ introduction of additional dual variables b, $\overline{b}$ for those matrix elements

Dual representation of SU(3)

Dual partition sum

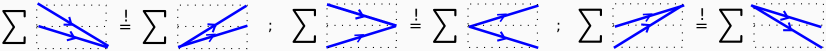
$$Z \;=\; \sum_{\{c\}} W[c]\; C_H[c]\; \mathrm{sign}[c]$$

- sum over configurations $\{c\} = \{n, \overline{n}, b, \overline{b}\}$
- $W[c]$ weight of a given configuration of the dual variables
- $\mathrm{sign}[c]$ sign of a given configuration
- $C_H[c]$ constraints on the cycle occupation numbers

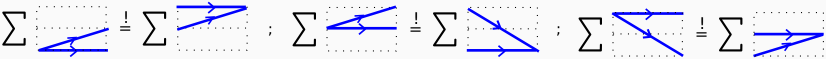
Constraints

The constraints $C_H[c]$ enforce relations between color fluxes on every link.

- The flux out of a color has to equal the flux into that color



- Relations that governs the exchange of flux between colors



Allowed configurations for the cycle occupation numbers are generalized surfaces.

Conclusions

- Using Abelian Color Cycles in SU(3) pure gauge theory we obtain an exact worldsheet representation of the system.
- The dual partition function is a sum over the admissible configurations of the dual variables, corresponding to worldsheets.
- The partition function can be organized in powers of β, and all terms of the expansion are known in closed form.
- Configurations with negative sign are not excluded by the constraints.
- ACC method is general and can be used for any non-abelian group
$\Rightarrow$ lattice QCD with staggered fermions is exactly rewrite to a theory of worldsheets for the gauge fields and worldlines for the fermions.

Thank you for your attention!