Exploratory studies for the position-space approach to hadronic light-by-light scattering in the muon $g - 2$

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June 23, 2017

References

I *Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment on the Lattice*, Talk by NA at the DPG meeting, Heidelberg, March 24, 2015

II Lattice 2015 proceedings of J. Green *et al* arXiv:1510.08384

III Lattice 2016 proceedings of NA *et al* arXiv:1609.08454.

IV A. Nyffeler, talk at Workshop (J-PARC, Japan, Nov. 2016)

V H. B. Meyer, talk at first workshop of the muon $g-2$ Theory Initiative (Q-Center, IL, 6 June 2017)

HLbL contribution to $g-2$

gyromagnetic moment: $\boldsymbol{\mu} = g\frac{e}{2m}\mathbf{S}$

anomalous magnetic moment: $a_\mu = \frac{g-2}{2}$

$\approx$ 3 to 4 standard deviations discrepancy between a_μ^{exp} and a_μ^{theo}

$\rightarrow$ new physics?

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reduce uncertainties

Experiment	Theory for HLbL	
J-PARC Fermilab	**phenomenology** model uncertainties for dominant contribution (π^0 , η , η' ; $\pi\pi$) using experimental input Colangelo *et al* '14,. . . ,'17 Pauk and Vanderhaeghen '14	**lattice QCD** model independant estimates Blum *et al* '15,. . . ,'17 (talks earlier this session) our group

Lattice QCD: Current Approach to HLbL

two independent developments by

- Blum *et al* '15 '16
- our group (I-V)

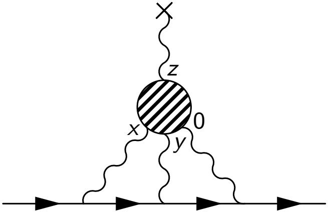
similarities

- get directly $F_2(q^2 = 0)$
- no cancellation of an $\mathcal{O}(\alpha^2)$ term
- position space
- perturbative treatment of the QED part

QED part computed in infinite volume in continuum

- Lorentz covariance is manifest in our approach
- no power law effects in the volume (an important motivation for this work)

Euclidean position-space approach to a_μ^{HLbL}



master formula (I-V)

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \left[\underbrace{\int d^4x}_{=4\pi \int_0^\pi d\beta \sin^2(\beta) \int_0^\infty d|x||x|^3} \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}} \right].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no finite-volume effects from the photons & affordable way (1d integral) to sample the integrand for the fully connected contribution.

Perturbation theory in Euclidean position-space

Scalar propagators:

$$G_0(x) = \frac{1}{4\pi^2 x^2}, \qquad G_m(x) = \frac{m}{4\pi^2 |x|} K_1(m|x|).$$

Fermion propagator:

$$S(x) \equiv \int \frac{d^4 p}{(2\pi)^4} \, \frac{-ip_\mu \gamma_\mu + m}{p^2 + m^2} \, e^{ipx} = \frac{m^2}{4\pi^2 |x|} \Big[\gamma_\mu x_\mu \, \frac{K_2(m|x|)}{|x|} + K_1(m|x|) \Big],$$

$U_n(z) =$ Chebyshev polynomials of the second kind:

$$U_0(z) = 1, \qquad U_1(z) = 2z, \qquad U_{n+1}(z) = 2z U_n(z) - U_{n-1}(z) \qquad (n \geq 1),$$

Key property: orthogonal basis on S_3; if $\hat{\epsilon}$ is a unit vector

$$\Big\langle U_n(\hat{\epsilon} \cdot \hat{x}) \; U_m(\hat{\epsilon} \cdot \hat{y}) \Big\rangle_{\hat{\epsilon}} = \frac{\delta_{nm}}{n+1} \, U_n(\hat{x} \cdot \hat{y}).$$

Sketch of the derivation

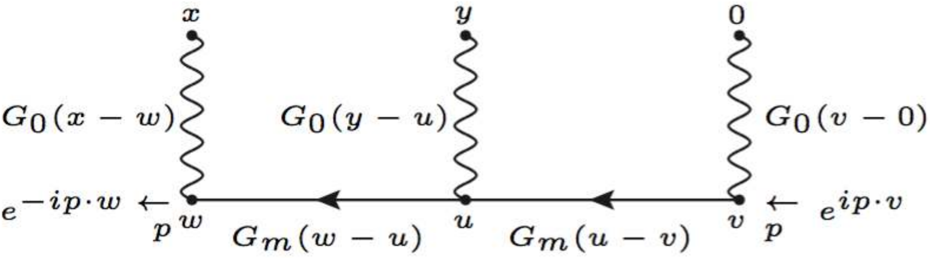
$$\hat{F}_2(0) = -\frac{i}{48m}\mathrm{Tr}\left\{[\gamma_\rho,\gamma_\tau]\,(-i\not{p}+m)\Gamma_{\rho\tau}(p,p)(-i\not{p}+m)\right\},$$

$$\Gamma_{\rho\sigma}(p,p) = -e^6\int_{x_1,x_2} K_{\mu\nu\lambda}(x_1,x_2,p)\,\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x_1,x_2),$$

$$K_{\mu\nu\lambda}(x_1,x_2,p) = \gamma_\mu(i\not{p}+\not{\partial}^{(x_1)}-m)\gamma_\nu(i\not{p}+\not{\partial}^{(x_1)}+\not{\partial}^{(x_2)}-m)\gamma_\lambda\,\mathcal{I}(\hat{\epsilon},x_1,x_2),$$

$$\mathcal{I}(\hat{\epsilon},x,y) = \int_{q,k}\frac{1}{q^2k^2(q+k)^2}\,\frac{1}{(p-q)^2+m^2}\,\frac{1}{(p-q-k)^2+m^2}\,e^{-i(qx+ky)}.$$

With $p = im\hat{\epsilon}$. From [III]. Diagrammatic representation of $\mathcal{I}(\hat{\epsilon},x,y)$:



The scalar function $\mathcal{I}(\hat{\epsilon}, x, y)$

$$\mathcal{I}(\hat{\epsilon}, x, y) = \int_u G_0(u-y)\, J(\hat{\epsilon}, u)\, J(\hat{\epsilon}, x-u),$$

$$J(\hat{\epsilon}, y) \equiv \int_u G_0(y-u)\, e^{m\hat{\epsilon}\cdot u} G_m(u) = \sum_{n\geq 0} z_n(y^2)\, U_n(\hat{\epsilon}\cdot\hat{y}),$$

$$\begin{aligned} S(x,y) &= \int_u G_0(u-y) s(x,u), \qquad \text{(IR regulated)} \\ V_\delta(x,y) &= \int_u G_0(u-y) v_\delta(x,u), \\ T_{\beta\delta}(x,y) &= \int_u G_0(u-y) t_{\beta\delta}(x,u), \end{aligned}$$

where

$$\begin{aligned} s(x,u) &= \Big\langle J(\hat{\epsilon}, u) J(\hat{\epsilon}, x-u) \Big\rangle_{\hat{\epsilon}} = \sum_{n=0}^{\infty} z_n(u^2) z_n((x-u)^2)\, \frac{U_n(\hat{u}\cdot\widehat{x-u})}{n+1}, \\ v_\delta(x,u) &= \Big\langle \epsilon_\delta\, J(\hat{\epsilon}, u) J(\hat{\epsilon}, x-u) \Big\rangle_{\hat{\epsilon}} = \ldots \end{aligned}$$

Explicit form of the QED kernel

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}\, T^{(A)}_{\alpha\beta\delta}(x,y),$$

with e.g.

$$\mathcal{G}^{\mathrm{I}}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda} \equiv \frac{1}{8}\mathrm{Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma] + 2(\delta_{\delta\sigma}\gamma_\rho - \delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},$$

$$\begin{aligned}
T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha(\partial^{(x)}_\beta + \partial^{(y)}_\beta)V_\delta(x,y),\\
T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\Big)\\
T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta)\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\Big),
\end{aligned}$$

$$\begin{aligned}
S(x,y) &= \bar{g}^{(0)}(|x|,\hat{x}\cdot\hat{y},|y|),\\
V_\delta(x,y) &= x_\delta\bar{\mathfrak{g}}^{(1)}(|x|,\hat{x}\cdot\hat{y},|y|) + y_\delta\bar{\mathfrak{g}}^{(2)}(|x|,\hat{x}\cdot\hat{y},|y|),\\
T_{\alpha\beta}(x,y) &= (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}.
\end{aligned}$$

The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six weight functions.

Example: Form Factor $g^{(2)}$

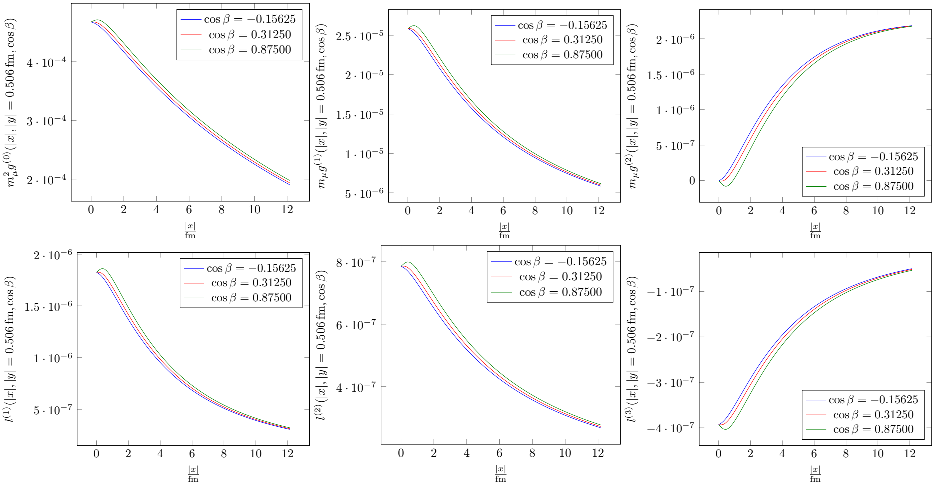
$$g^{(2)}(x^2, x\cdot y, y^2) = \frac{1}{8\pi y^2 |x| \sin^3\beta} \int_0^\infty du\, u^2 \int_0^\pi d\phi_1$$
$$\left\{ 2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1 \right) \frac{\log\chi}{\sin\phi_1} \right\} \sum_{n=0}^{\infty}$$
$$\Big\{ z_n(|u|) z_{n+1}(|x-u|) \left[|x-u|\cos\phi_1 \frac{U_n}{n+1} + (|u|\cos\phi_1 - |x|)\frac{U_{n+1}}{n+2} \right]$$
$$+ z_{n+1}(|u|) z_n(|x-u|) \left[(|u|\cos\phi_1 - |x|)\frac{U_n}{n+1} + |x-u|\cos\phi_1 \frac{U_{n+1}}{n+2} \right] \Big\}$$

where

$$x\cdot y = |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1}$$
$$\chi = \frac{y^2+u^2-2|u||y|\cos(\beta-\phi)}{y^2+u^2-2|u||y|\cos(\beta+\phi)}\,, \quad U_n = U_n\Big(\frac{|x|\cos\phi_1 - |u|}{|u-x|}\Big)$$
$$z_n = \text{linear combination of products of two modified Bessel functions.}$$

From [I]. Reminder: $V_\delta(x,y) = x_\delta \bar{\mathfrak{g}}^{(1)}(|x|, \hat{x}\cdot\hat{y}, |y|) + y_\delta \bar{\mathfrak{g}}^{(2)}(|x|, \hat{x}\cdot\hat{y}, |y|)$.

Complete set of weight functions: $|x|$ dependence



$\bar{\mathfrak{g}}^{(0)}(|x|, \hat{x}\cdot\hat{y}, |y|)$ contains an arbitrary additive constant (due to the IR divergence in $I(\hat{\epsilon}, x, y)$), which does not contribute to $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y)$.

The π^0 pole contribution

Assume a vector-meson-dominance transition form factor (parameters: m_V, m_π and overall normalization)

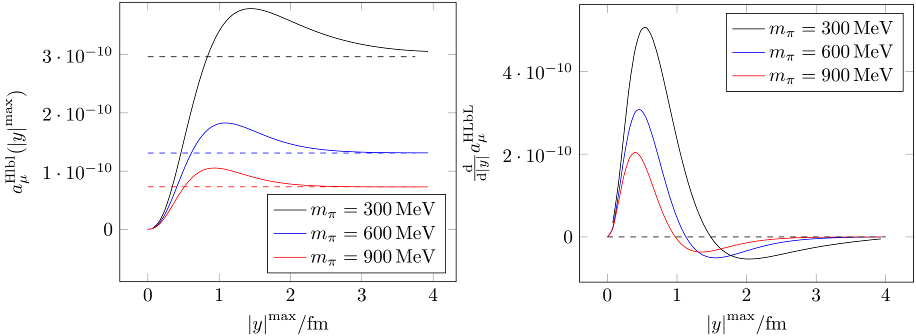
$$\mathcal{F}(-q_1^2, -q_2^2) = \frac{c}{(q_1^2 + m_V^2)(q_2^2 + m_V^2)}, \qquad c = -\frac{N_c m_V^4}{12\pi^2 F_\pi}.$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \frac{c^2}{m_V^2(m_V^2 - m_\pi^2)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\beta} \Big\{ \epsilon_{\mu\nu\alpha\beta}\epsilon_{\sigma\lambda\rho\gamma} \Big(\frac{\partial}{\partial x_\gamma} + \frac{\partial}{\partial y_\gamma} \Big) K_\pi(x,y)$$
$$+ \epsilon_{\mu\lambda\alpha\beta}\epsilon_{\nu\sigma\gamma\rho} \frac{\partial}{\partial y_\gamma} K_\pi(y-x, y) + \epsilon_{\mu\sigma\alpha\rho}\epsilon_{\nu\lambda\beta\gamma} \frac{\partial}{\partial x_\gamma} K_\pi(x, x-y) \Big\}.$$

where

$$K_\pi(x,y) \equiv \int d^4u \Big(G_{m_\pi}(u) - G_{m_V}(u) \Big) G_{m_V}(x-u) G_{m_V}(y-u) = K_\pi(y,x).$$

Contribution of the π^0 to a_μ^{HLbL}



Dashed line = result from momentum-space integration

- Contribution is perhaps surprisingly long-range.

The lepton loop: fully analytic result for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$

$$
\begin{aligned}
i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad = \quad & \widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \\
& +\widehat{\Pi}^{(1)}_{\rho;\nu\lambda\mu\sigma}(y-x,-x) + x_\rho\,\Pi^{(r,1)}_{\nu\lambda\mu\sigma}(y-x,-x) \\
& +\widehat{\Pi}^{(1)}_{\rho;\lambda\nu\mu\sigma}(-x,y-x) + x_\rho\,\Pi^{(r,1)}_{\lambda\nu\mu\sigma}(-x,y-x).
\end{aligned}
$$

$$
\begin{aligned}
&\Pi^{(r,1)}_{\mu\nu\lambda\sigma}(x,y) \\
&= 2\Big(\frac{m}{2\pi}\Big)^8\Big[\frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot I_{\gamma\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot p(|y|)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot p(|y|)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\cdot q_\gamma(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\cdot q_\gamma(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\cdot q_\delta(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\cdot q_\delta(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot I_{\gamma\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}
$$

The lepton loop (continued)

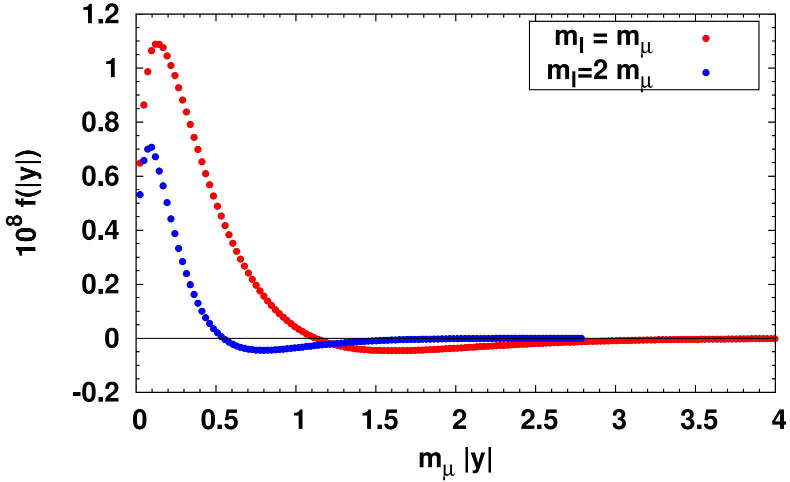
$$\begin{aligned}
\widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) &= 2\Big(\frac{m}{2\pi}\Big)^8 \Big[\frac{(-x_\alpha)(x-y)_\beta K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\, h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\, h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\, \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\, \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}$$

$$I_{\gamma\delta}(y) = \frac{2\pi^2}{m^2}\Big(\hat{y}_\gamma\hat{y}_\delta\ K_2(m|y|) - \delta_{\gamma\delta}\ \frac{K_1(m|y|)}{m|y|}\Big), \quad h_{\rho\gamma}(y) = \frac{\pi^2}{m^3}\Big(\hat{y}_\gamma\hat{y}_\rho\ m|y|\ K_1(m|y|) - \delta_{\gamma\rho}K_0(m|y|)\Big),$$

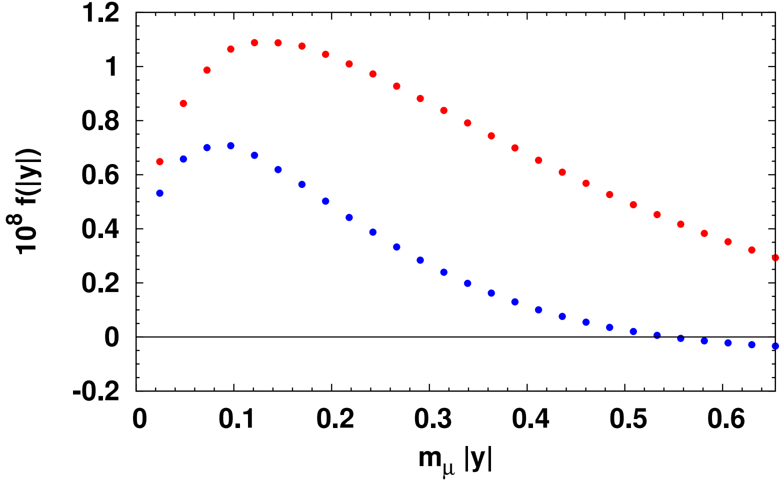
$$\hat{f}_{\rho\delta}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\rho\hat{y}_\delta\ m|y|K_1(m|y|) + \delta_{\rho\delta}K_0(m|y|)\Big\} \qquad q_\gamma(y) = \frac{2\pi^2}{m^2}\ \hat{y}_\gamma\ K_1(m|y|),$$

$$f_{\rho\delta\gamma}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\gamma\hat{y}_\delta\hat{y}_\rho\ m|y|K_2(m|y|) + (\delta_{\rho\delta}\hat{y}_\gamma - \delta_{\gamma\rho}\hat{y}_\delta - \delta_{\gamma\delta}\hat{y}_\rho)\,K_1(m|y|)\Big\}, \quad p(|y|) = \frac{2\pi^2}{m^2}\ K_0(m|y|).$$

Lepton loop integrand contribution to a_μ^{HLbL}



Lepton loop integrand contribution to a_μ^{HLbL} (zoom)



- numerically compatible with $f(|y|) \propto m_\mu |y| \log^2(m_\mu |y|)$ for small $|y|$
- analytic result reproduced at the percent level

Conclusion

- The covariant position-space method looks like a promising approach. We plan to make the QED kernel publicly available.
- Tests of the QED kernel: π^0 pole, lepton loop.
- The analytic result for the lepton loop is reproduced at the percent level
- The π^0 contribution is very long-range, but with the π^0 contribution calculated, we hope to be able to correct for the finite-size effects on this contribution, by computing the transition form factor on the same ensemble (Gérardin '16).
- A parallel activity: analysis of the eight forward light-by-light scattering amplitudes, constraining the resonance transition form factors (with V. Pascalutsa; next talk by A. Gérardin).