Atiyah-Patodi-Singer index theorem for domain-wall fermion Dirac operator

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Outline

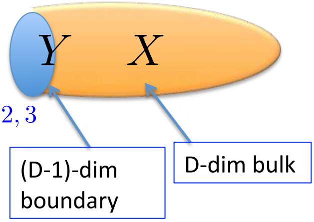
1. Introduction

a. Domain wall fermion and SPT phase

Massive Fermions in D-dim with D-1-dim boundary:

very interesting physics

- Domain-wall fermions: $D_{\rm bulk}=5, D_{\rm bound.}=4$
- topological insulators: $D_{\rm bulk}=3,4,\quad D_{\rm bound.}=2,3$



Massless fermion at boundary

for symmetry protected topological (SPT) phase

Cancellation of { T- anomaly (D_bulk=even) / pert. gauge anomaly (D_bulk=odd) } is the key.

In this work, we consider D_bulk=4 (even) case.

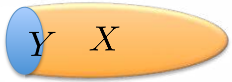
(I) SPT phase (without boundary)

$$Z \overset{\text{reg}}{=} \text{Det}\left(\frac{D^{4d}+M}{D^{4d}+\Lambda}\right) = |Z^{\text{reg}}| \exp\left(i\theta P\right) \qquad P = \int d^4x \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\text{tr}\left[F_{\mu\nu}F_{\alpha\beta}\right]$$

T symm. allows only $\theta = 0$ (normal), or $\theta = \pi$ (SPT phase)

$M/\Lambda > 0$ $\qquad$ $M/\Lambda < 0$

(II) SPT phase on manifold X boundary Y



$$P = \int_X d^4x \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\text{tr}\left[F_{\mu\nu}F_{\alpha\beta}\right]$$

P can no longer be integer → Violation of T symmetry!

However, additional massless state (edge mode) can appear on Y

$$S_{\text{eff}} = \int_Y d^3x \bar{\psi} D^{3d}\psi + \cdots \qquad \lambda_i : \text{ eigenvalue of } D^{3d}$$

Additional T-anomaly in $\text{Det}(D^{3d})$ due to T-violating regularization

$$\text{Det}(D^{3d}) = \prod_i \frac{\lambda_i}{\lambda_i + i\Lambda} = |\text{Det}(D^{3d})| \exp\left(-i\frac{\pi}{2}\eta(D^{3d})\right)$$

$$\eta(D^{3d}) = \lim_{s\to 0}\sum_k \text{sign}(\lambda_k)|\lambda_k|^{-s} : \eta\text{-invariant}$$

b. APS index theorem

Atiyah, Patodi and Singer “Spectral asymmetry and Riemanian geometry”,
Math. Proc. Cambridge Philos. Soc. 77, 43

Index theorem for massless Dirac op.
on even-dim manifold with boundary.

$x_4 = 0$

$$D^{4\mathrm{D}} = \gamma_4(\partial_4 + H), \quad H = \gamma_4 \gamma_i D^i$$

$A_4 = 0$ gauge

With non-local boundary condition (APS b.c.)

$$\frac{1}{2}\left(H + |H|\right)\psi\Big|_{x_4=0} = 0$$

$$\text{Index of } D^{4D} = -\frac{\eta(D^{3d})}{2} + P$$

$$P = \int_X d^4x \frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} \mathrm{tr}\left[F_{\mu\nu} F_{\alpha\beta}\right]$$

APS index theorem

Witten's argument based on low energy effective theory

Witten "Fermion path integrals and topological phases", Rev. Mod. Phys. 88 (2016)

Full theory is invariant under T symmetry

→Anomaly cancelled between the bulk & boundary ?

Does this really happen? → YES! (Witten)

$$Z^{\rm full}_{\rm micro} \approx Z^{\rm full}_{\rm low} = |Z^{\rm full}_{\rm low}| \exp\left(i\pi\mathcal{J}\right)$$

$$\mathcal{J} = P - \frac{\eta(D^{3d})}{2}$$

$$\mathcal{J} = \text{ APS index } = \text{ integer}$$

Recovers T symmetry!

This is the reason for the stable existence of edge modes!

Unsatisfactory points in Witten's argument:

- Reasonable argument but not derivation
  Imposing theoretical consistency :
  "symmetry (microscopic) = symmetry (low energy)"
  Similar to anomaly matching condition

- Why APS index ?
  - massless 4d Dirac op. does not appear in DW fermion
  - APS index theorem do not deal with localized edge modes
  - Non-local b.c. in APS ←??→ local b.c. in DW
  → Physics of "APS index" and "DW fermion" are quite different

- Simple and Direct Derivation like Fujikawa's method?

C. Our goal

We directly compute the index for domain-wall fermion.

If successful,

- Microscopic derivation of the determinant phase of the Domain-wall fermion in a "physicist-friendly" way, similar to the Fujikawa method.
- Better understanding of anomaly descent equations.

Our main result:

✓ index for domain-wall fermion = APS index

✓ T-anomaly cancellation using Fujikawa's method.

2. Index for domain wall Dirac operator

a. Determinant phase of Domain-wall Dirac op.

Domain-wall fermion in 4-dim with 3-dim boundary

$$\det\left(\frac{D+M\epsilon(x_4)}{D-M}\right)$$

Domain-wall with kink mass

Pauli-Villars regulator

Defining
$$H_{\mathrm{DW}} = \gamma_5\left(D+M\epsilon(x_4)\right)$$
$$H_{\mathrm{PV}} = \gamma_5\left(D-M\right)$$

Phase can be evaluated as

$$\det\left(\frac{D+M\epsilon(x_4)}{D-M}\right)$$
$$=\left|\det\left(\frac{D+M\epsilon(x_4)}{D-M}\right)\right|\exp\left(\frac{i\pi}{2}\left(\eta(H_{\mathrm{DW}})-\eta(H_{\mathrm{PV}})\right)\right)$$

4-dim Hamiltoinan $H_{\rm DW}, H_{\rm PV}$: Hermitian

Det. should be real if properly regularized.

$$\frac{1}{2}\left(\eta(H_{\rm DW}) - \eta(H_{\rm PV})\right) = \text{integer} \equiv \mathcal{J}^{\rm DW}$$

We propose to define this quantity $\mathcal{J}_{\rm DW}$
as "the index of Domain-wall Dirac op."

Computation of eta-invariant

$$\eta(H) = \lim_{s\to 0} \sum_k \frac{\text{sign}(\frac{\lambda_k}{M})}{|\frac{\lambda}{M}|^s} = \lim_{s\to 0} \text{Tr}\left[\frac{\frac{H}{M}}{\sqrt{\frac{H^2}{M^2}}^{1+s}}\right]$$

$$= \lim_{s\to 0} \frac{1}{\Gamma(\frac{1+s}{2})} \int_0^\infty dt\ t^{\frac{s-1}{2}} \text{Tr}\left[\frac{H}{M} \exp\left(-t\frac{H^2}{M^2}\right)\right]$$

b. Computation of $\eta(H_{\rm PV})$

$$\frac{H_{\rm PV}}{M} = -\gamma_5 + \gamma_5 \frac{D^{4d}}{M}, \quad \frac{H_{\rm PV}^2}{M^2} = 1 + \frac{(D^{4d})^2}{M^2}$$

$$\eta(H_{\rm PV}) = \lim_{s\to 0} \int_0^\infty dt \frac{t^{\frac{s-1}{2}}}{\Gamma(\frac{1+s}{2})} e^{-t} {\rm Tr}\left[\left(-\gamma_5 + \gamma_5 \frac{D^{4d}}{M}\right) \exp\left(-t\frac{(D^{4d})^2}{M^2}\right)\right]$$

Does not contributue

Calculation of trace using plane wave gives

$$\eta(H_{\rm PV}) = -\frac{1}{32\pi^2} \int_{-\infty}^{\infty} dx_4 \int d^3x \epsilon^{\mu\nu\alpha\beta} {\rm tr}\left[F_{\mu\nu} F_{\alpha\beta}\right] + O(\frac{1}{M^2})$$

Essentially the same calculation as Fujikawa's method.

c. Computation of $\eta(H_{\rm DW})$

DW fermion: feels γ_4 -dependent potential at the origin

$$\frac{H_{\rm DW}}{M} = \gamma_5 \epsilon(x_4) + \gamma_5 \frac{D^{4d}}{M}, \quad \frac{H^2_{\rm DW}}{M^2} = 1 + \frac{(D^{4d})^2 - 2M\gamma_4\delta(x_4)}{M^2}$$

Complete set of plane wave states should be modified!

Taking the basis $\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$
and solving eigenstates for the free part of H_{DW}^2

$$H_{\mathrm{DW}}^2 = -\partial_4^2 - \sum_{i=1}^{3} \partial_i^2 + M^2 - 2M\gamma_4\delta(x_4)$$

Complete set of states

$$\varphi_o^{\omega,\vec{k}}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} \phi \\ 0 \end{pmatrix},$$

$$\varphi_e^{\omega,\vec{k}}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega - M)e^{i\omega|x_4|} + (i\omega + M)e^{-i\omega|x_4|}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} \phi \\ 0 \end{pmatrix},$$

$$\varphi_e^{\mathrm{edge}}(x_4) = \sqrt{M} e^{-M|x_4|} \begin{pmatrix} \phi \\ 0 \end{pmatrix},$$

$$\varphi_{-,o}^{\omega}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} 0 \\ \chi \end{pmatrix},$$

$$\varphi_{-,e}^{\omega}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega + M)e^{i\omega|x_4|} + (i\omega - M)e^{-i\omega|x_4|}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} 0 \\ \chi \end{pmatrix},$$

$$\eta(H_{\rm DW}) = \lim_{s\to 0}\int_0^\infty dt \frac{t^{\frac{s-1}{2}}}{\Gamma(\frac{1+s}{2})} e^{-t}{\rm Tr}\left[\left(\gamma_5\epsilon(x_4) + \gamma_5\frac{D^{4d}}{M}\right)\exp\left(-t\frac{H_{\rm DW}^2}{M^2}\right)\right]$$

(1) (2)

Also contributue due to
γ_4 -dep. complete set of states

Calculation of (1) using complete set of plane wave states gives

$$\eta^{(1)}(H_{\rm DW}) = \frac{1}{32\pi^2}\int_{-\infty}^{\infty} dx_4 \int d^3x\epsilon(x_4)\epsilon^{\mu\nu\alpha\beta}{\rm tr}\left[F_{\mu\nu}F_{\alpha\beta}\right] + O(\frac{1}{M^2})$$

Essentially the same calculation as Fujikawa's method.

Calculation of (2) using complete set of states.

by differentiating & integrating over gauge field
(integer part can be dropped)

After technical calculation
using gaussian integral and error functions

$$\eta^{(2)}(H_{\rm DW}) = 2\left(-\frac{\eta(D^{3d})}{2} + {\rm mod}(\text{integer})\right)$$

$$\eta(H_{\rm DW}) = \eta^{(1)}(H_{\rm DW}) + \eta^{(2)}(H_{\rm DW})$$
$$= \frac{1}{32\pi^2}\int d^4x \epsilon(x_4)\epsilon^{\mu\nu\alpha\beta}{\rm tr}[F_{\mu\nu}F_{\alpha\beta}] + 2\left(-\frac{\eta(D^{3d})}{2} + {\rm mod}(\text{integer})\right)$$

d. Final results

Combining PV and DW contributions, we obtain

$$\mathcal{J}^{\rm DW} \equiv \frac{1}{2}\left(\eta(H_{\rm DW}) - \eta(H_{\rm PV})\right)$$
$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x {\rm tr}[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}] - \frac{\eta(D^{3d})}{2} ({\rm mod\ integer})$$

This agrees with the result by APS index theorem!

Direct macroscopic derivation of
Domain-Wall fermion determinant phase,

No fictitious massless Dirac op. needed as a mathematica tool.

3. Summary

- We have succeeded in direct derivation of the determinant phase for Domain-wall fermion using Fujikawa's method
- We reformulated APS index theorem with domain-wall Dirac op.
- T-anomaly cancellation was understood from microscopic theory.

$$\mathfrak{I}(D^{4\mathrm{D}})|_{\mathrm{APS\ boundary}} \quad = \eta(H^{4\mathrm{D}}(m))|_{\mathrm{SO(3)\ symmetirc\ boundary}}$$
$$= \int_{x_4>0} d^4 x F \wedge F - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

What is next?

1. Generalization to Shamir-type domain-wall fermions,
2. Generalization to odd dimensions,
3. Non-perturbative formulation of APS index theorem on a lattice,
4. Application to 6D formulation of lattice chiral gauge theory.