# Renormalization of Supersymmetric QCD on the Lattice

M. Costa* and H. Panagopoulos

University of Cyprus
* Speaker

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## Abstract

We perform a pilot study of the perturbative renormalization of a Supersymmetric gauge theory with matter fields on the lattice. As a specific example, we consider Supersymmetric $\mathcal{N}=1$ QCD (SQCD). We study the self-energies of all particles which appear in this theory, as well as the renormalization of the coupling constant. To this end we compute, perturbatively to one-loop, the relevant two-point and three-point Green's functions using both dimensional and lattice regularizations. Our lattice formulation involves the Wilson discretization for the gluino and quark fields; for gluons we employ the Wilson gauge action; for scalar fields (squarks) we use naive discretization. The gauge group that we consider is $SU(N_c)$, while the number of colors, $N_c$, the number of flavors, $N_f$, and the gauge parameter, $\alpha$, are left unspecified.
We obtain analytic expressions for the renormalization factors of the coupling constant ($Z_g$) and of the quark ($Z_\psi$), gluon ($Z_u$), gluino ($Z_\lambda$), squark ($Z_{A_\pm}$), and ghost ($Z_c$) fields on the lattice. We also compute the critical values of the gluino, quark and squark masses. Finally, we address the mixing which occurs among squark degrees of freedom beyond tree level: we calculate the corresponding mixing matrix which is necessary in order to disentangle the components of the squark field via an additional finite renormalization.
A number of current investigations of SUSY on the lattice can be found in Ref. [1], and in particular in the references quoted there.

## Lattice Action

Even though the lattice breaks supersymmetry explicitly [2], it is the only regulator which describes many aspects of strong interactions nonperturbatively. We will extend Wilson's formulation of the QCD action, to encompass SUSY partner fields as well. In this standard discretization quarks, squarks and gluinos live on the lattice sites, and gluons live on the links of the lattice: $U_\mu(x) = e^{igaT^\alpha u_\mu^\alpha(x+a\hat\mu/2)}$. This formulation leaves no SUSY generators intact, and it also breaks chiral symmetry; it thus represents a "worst case" scenario, which is worth investigating in order to address the complications [3] which will arise in numerical simulations of SUSY theories. In our ongoing investigation we plan to address also improved actions, so that we can check to what extent some of the SUSY breaking effects can be alleviated.
For Wilson-type quarks ($\psi$) and gluinos ($\lambda$), the Euclidean action $\mathcal{S}^L_{\rm SQCD}$ on the lattice becomes ($A_\pm$: squark field components):

$$\begin{aligned}\mathcal{S}^L_{\rm SQCD} &= a^4\sum_x\Big[\frac{N_c}{g^2}\sum_{\mu,\nu}\Big(1-\frac{1}{N_c}{\rm Tr}U_{\mu\nu}\Big)+\sum_\mu{\rm Tr}\left(\bar\lambda_M\gamma_\mu\mathcal{D}_\mu\lambda_M\right)-a\frac{r}{2}{\rm Tr}\left(\bar\lambda_M\mathcal{D}^2\lambda_M\right)\\ &+\sum_\mu\Big(\mathcal{D}_\mu A_+^\dagger\mathcal{D}_\mu A_+ + \mathcal{D}_\mu A_-\mathcal{D}_\mu A_-^\dagger+\bar\psi_D\gamma_\mu\mathcal{D}_\mu\psi_D\Big)-a\frac{r}{2}\bar\psi_D\mathcal{D}^2\psi_D\\ &+i\sqrt{2}g\big(A_+^\dagger\bar\lambda^\alpha_M T^\alpha P_+\psi_D-\bar\psi_D P_-\lambda^\alpha_M T^\alpha A_+ + A_-\bar\lambda^\alpha_M T^\alpha P_-\psi_D-\bar\psi_D P_+\lambda^\alpha_M T^\alpha A_-^\dagger\big)\\ &+\frac{1}{2}g^2(A_+^\dagger T^\alpha A_+ - A_- T^\alpha A_-^\dagger)^2 - m(\bar\psi_D\psi_D - mA_+^\dagger A_+ - mA_-A_-^\dagger)\Big],\end{aligned} \tag{1}$$

where: $U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat\mu)U^\dagger_\mu(x+a\hat\nu)U^\dagger_\nu(x)$, and a summation over flavors is understood in the last three lines of Eq. (1). The 4-vector $x$ is restricted to the values $x = na$, with $n$ being an integer 4-vector. Thus the momentum integration, after a Fourier transformation, is restricted to the first Brillouin zone (BZ) $[-\pi/a,\pi/a]^4$ and the sum over $x$ leads to momentum conservation in each vertex. The terms proportional to the Wilson parameter, $r$, eliminate the problem of fermion doubling, at the expense of breaking chiral invariance.
The definitions of the covariant derivatives are as follows:

$$\mathcal{D}_\mu\lambda_M(x) \equiv \frac{1}{2a}\Big[U_\mu(x)\lambda_M(x+a\hat\mu)U^\dagger_\mu(x)-U^\dagger_\mu(x-a\hat\mu)\lambda_M(x-a\hat\mu)U_\mu(x-a\hat\mu)\Big] \tag{2}$$

$$\mathcal{D}^2\lambda_M(x) \equiv \frac{1}{a^2}\sum_\mu\Big[U_\mu(x)\lambda_M(x+a\hat\mu)U^\dagger_\mu(x)-2\lambda_M(x)+U^\dagger_\mu(x-a\hat\mu)\lambda_M(x-a\hat\mu)U_\mu(x-a\hat\mu)\Big] \tag{3}$$

$$\mathcal{D}_\mu\psi_D(x) \equiv \frac{1}{2a}\Big[U_\mu(x)\psi_D(x+a\hat\mu)-U^\dagger_\mu(x-a\hat\mu)\psi_D(x-a\hat\mu)\Big] \tag{4}$$

$$\mathcal{D}^2\psi_D(x) \equiv \frac{1}{a^2}\sum_\mu\Big[U_\mu(x)\psi_D(x+a\hat\mu)-2\psi_D(x)+U^\dagger_\mu(x-a\hat\mu)\psi_D(x-a\hat\mu)\Big] \tag{5}$$

$$\mathcal{D}_\mu A_+(x) \equiv \frac{1}{a}\Big[U_\mu(x)A_+(x+a\hat\mu)-A_+(x)\Big] \tag{6}$$

$$\mathcal{D}_\mu A_+^\dagger(x) \equiv \frac{1}{a}\Big[A_+^\dagger(x+a\hat\mu)U^\dagger_\mu(x)-A_+^\dagger(x)\Big] \tag{7}$$

$$\mathcal{D}_\mu A_-(x) \equiv \frac{1}{a}\Big[A_-(x+a\hat\mu)U^\dagger_\mu(x)-A_-(x)\Big] \tag{8}$$

$$\mathcal{D}_\mu A_-^\dagger(x) \equiv \frac{1}{a}\Big[U_\mu(x)A_-^\dagger(x+a\hat\mu)-A_-^\dagger(x)\Big] \tag{9}$$

A gauge-fixing term, together with the compensating ghost field term, must be added to the action, in order to avoid divergences from the integration over gauge orbits; these terms are the same as in the non-supersymmetric case. Similarly, a standard "measure" term must be added to the action, in order to account for the Jacobian in the change of integration variables: $U_\mu \to u_\mu$.

## One-loop Feynman Diagrams

We calculate perturbatively the 2-pt and 3-pt Green's functions up to one loop, both in the continuum and on the lattice. The quantities that we study are the self-energies of the quark ($\psi$), gluon ($u_\mu$), squark ($A$), gluino ($\lambda$), and ghost ($c$) fields, using both dimensional regularization (DR) and lattice regularization (L). In addition we calculate the gluon-antighost-ghost Green's function in order to renormalize the coupling constant ($g$). The Green's functions leading to self-energies of squarks exhibit also mixing among $A_+$ and $A_-^\dagger$; we calculate the elements of the corresponding $2\times 2$ mixing matrix.
The one-loop Feynman diagrams (one-particle irreducible (1PI)) contributing to the quark propagator, $\langle\psi(x)\bar\psi(y)\rangle$, are shown in Fig. 1, those contributing to the squark propagator, $\langle A_+(x)A_+^\dagger(y)\rangle$, in Fig. 2. Identical results are obtained for $\langle A_+(x)A_+^\dagger(y)\rangle$ and $\langle A_-^\dagger(x)A_-(y)\rangle$. The one-loop Feynman diagrams contributing to the gluon propagator, $\langle u^\alpha_\mu(x)u^\beta_\nu(y)\rangle$, and gluino propagator, $\langle\lambda^\alpha(x)\bar\lambda^\beta(y)\rangle$, are shown in Fig. 3 and Fig. 4, respectively. Lastly, the 1PI Feynman diagram which contributes to the ghost propagator, $\langle c(x)\bar c(y)\rangle$, is shown in Fig. 5. In this paper we also calculate the gluon-antighost-ghost Green's function in order to renormalize the coupling constant. In Fig. 6 we have drawn the corresponding lattice 1PI Feynman diagrams for the 3-pt function.
As is usually done, we will work in a mass-independent scheme, and thus all of our calculations, in the continuum as well as on the lattice, will be done at zero renormalized masses for all particles.

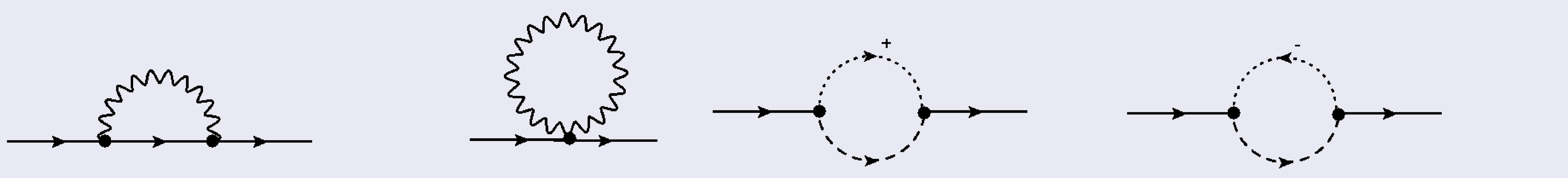

Figure 1: One-loop Feynman diagrams contributing to the 2-pt Green's function $\langle\psi(x)\bar\psi(y)\rangle$. A wavy (solid) line represents gluons (quarks). A dotted (dashed) line corresponds to squarks (gluinos). Squark lines are further marked with a $+(-)$ sign, to denote an $A_+$ $(A_-)$ field. A squark line arrow entering (exiting) a vertex denotes a $A_+$ $(A_+^\dagger)$ field; the opposite is true for $A_-$ $(A_-^\dagger)$ fields.

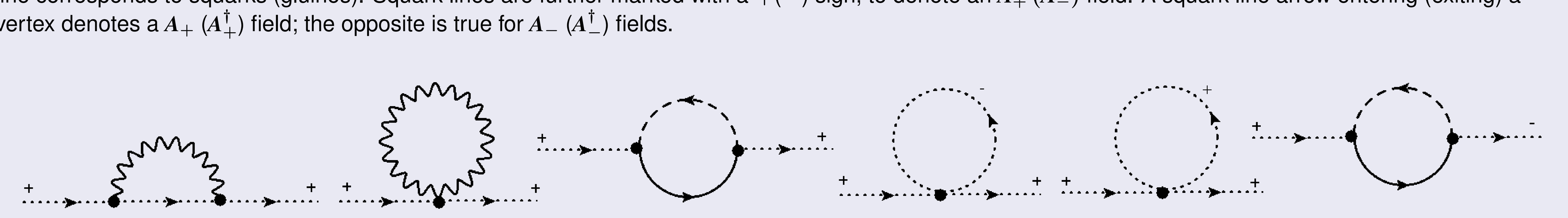

Figure 2: The first 5 Feynman diagrams contribute to the 2-pt Green's function $\langle A_+(x)A_+^\dagger(y)\rangle$. The case of $\langle A_-^\dagger(x)A_-(y)\rangle$ is completely analogous. The last diagram is responsible for mixing between $A_+$ and $A_-^\dagger$.

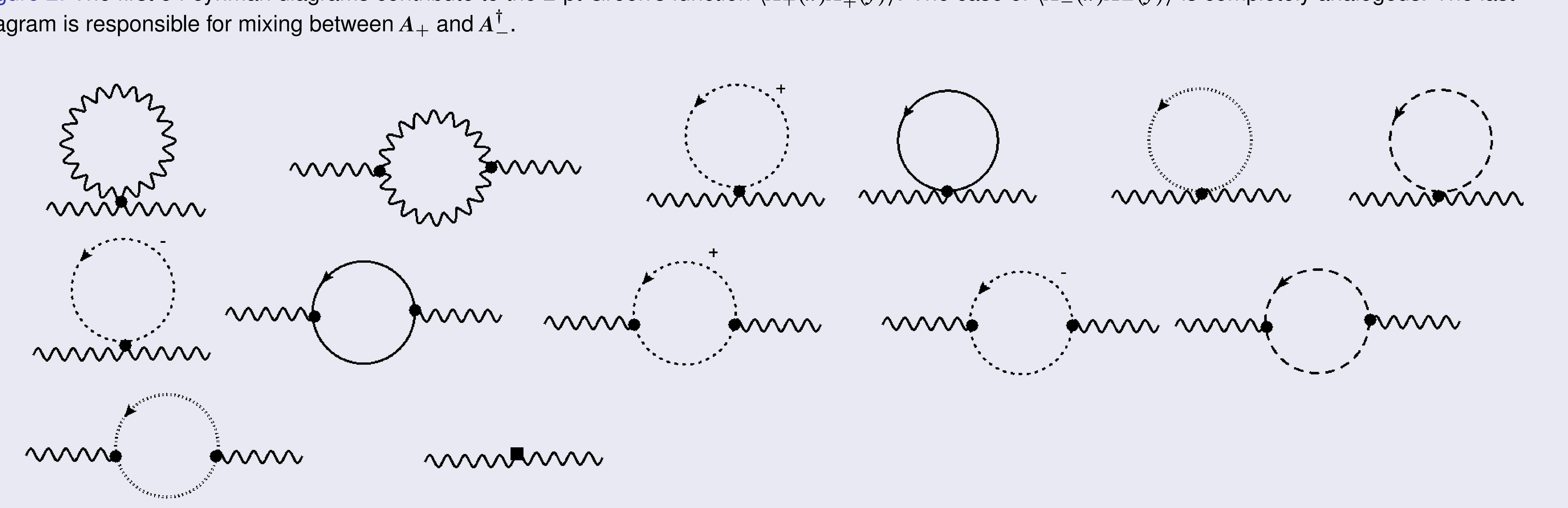

Figure 3: One-loop Feynman diagrams contributing to the 2-pt Green's function $\langle u^\alpha_\mu(x)u^\beta_\nu(y)\rangle$. The "double dashed" line is the ghost field and the solid box in the bottom right vertex comes from the measure part of the lattice action.

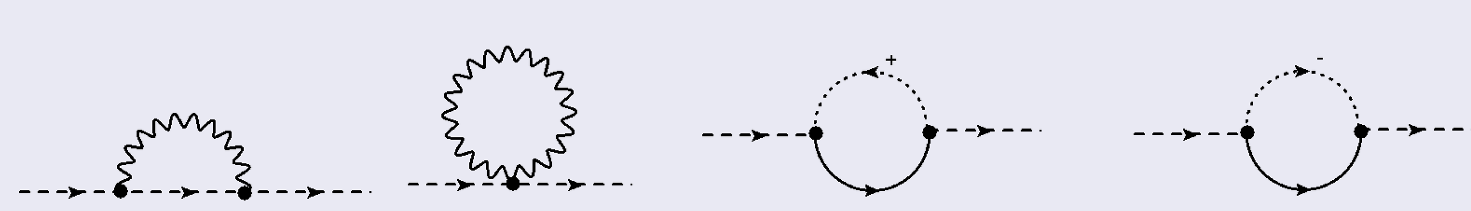

Figure 4: One-loop Feynman diagrams contributing to the 2-pt Green's function $\langle\lambda^\alpha(x)\bar\lambda^\beta(y)\rangle$.

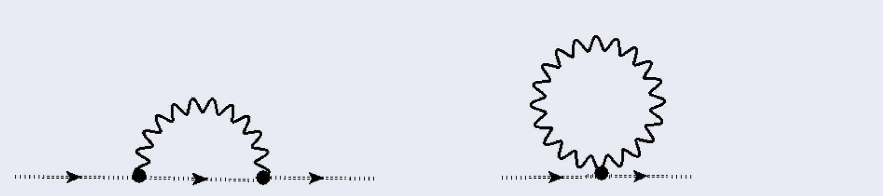

Figure 5: One-loop Feynman diagram contributing to the 2-pt Green's function $\langle c(x)\bar c(y)\rangle$.

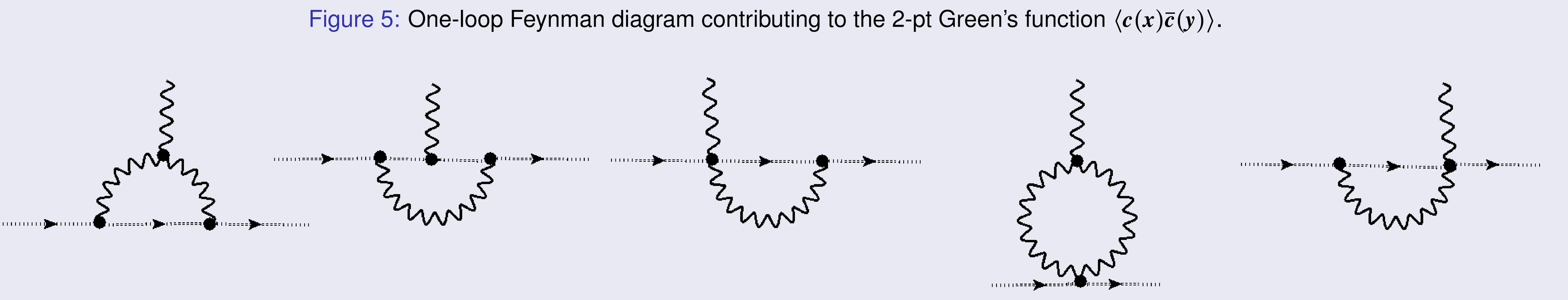

Figure 6: One-loop Feynman diagrams contributing to $\langle c^\alpha(x)\bar c^\beta(y)u^\gamma_\mu(z)\rangle$.

## $\overline{\rm MS}$-Renormalized Green's Functions

The first step in our perturbative procedure is to calculate the 2-pt and 3-pt Green's functions in the continuum, where we regularize the theory in $D$ Euclidean dimensions ($D = 4-2\,\epsilon$). From this computation we obtain the $\overline{\rm MS}$-renormalized Green's functions by elimination of the pole part of the continuum bare Green's functions. The renormalized Green's functions are relevant for the ensuing calculation of the corresponding Green's functions using lattice regularization and $\overline{\rm MS}$ renormalization. Depending on the prescription used to define $\gamma_5$ in $D$-dimensions, mixing between squarks may appear also in dimensional regularization. For the continnum calculations we adopt the t'Hooft-Veltman (HV) scheme. Other prescriptions are related among themselves via finite conversion factors.
Here we collect all $\overline{\rm MS}$-renormalized results for the 2-pt and 3-pt Green's functions; the first result which we present, is the inverse quark renormalized propagator in momentum space:

$$\langle\tilde\psi^R(q)\tilde{\bar\psi}^R(q')\rangle^{\overline{\rm MS}}_{\rm inv} = (2\pi)^4\delta(q-q')i\,\not{q}\left[1+\frac{g^2\,C_F}{16\,\pi^2}\left(4+\alpha+(2+\alpha)\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\right]. \tag{10}$$

In matrix notation, our results for the renormalized 2-pt Green's functions with external squark legs are:

$$\langle\tilde A^R(q)\tilde A^{R\,\dagger}(q')\rangle^{\overline{\rm MS}}_{\rm inv} = (2\pi)^4\delta(q-q')\left[q^2\begin{pmatrix}1&0\\0&1\end{pmatrix}+q^2\frac{g^2\,C_F}{16\,\pi^2}\left(\frac{16}{3}+(1+\alpha)\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+q^2\frac{g^2\,C_F}{16\,\pi^2}\frac{4}{3}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right], \tag{11}$$

where $A^R$ is a 2-component column which contains the renormalized squark fields: $A^R = \begin{pmatrix}A_+^R\\ A_-^{R\,\dagger}\end{pmatrix}$.
We now turn to the gluon renormalized propagator. The contributions from the diagrams of Fig. 3, taken separately, are not transverse. But, their sum has this property, and it is found to take the following form:

$$\begin{aligned}\langle\tilde u^R_\mu(q)\tilde u^R_\nu(q')\rangle^{\overline{\rm MS}}_{\rm inv} &= (2\pi)^4\delta(q+q')\Big\{\frac{1}{\alpha}q_\mu q_\nu+(q^2\delta_{\mu\nu}-q_\mu q_\nu)\Big[1+\frac{g^2\,N_f}{16\,\pi^2}\left(2+\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\\ &-\frac{g^2\,N_c}{16\,\pi^2}\frac{1}{2}\left(\frac{19}{6}+\alpha+\frac{\alpha^2}{2}+(3-\alpha)\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\Big]\Big\}.\end{aligned} \tag{12}$$

The result for the inverse gluino renormalized propagator to one-loop order is:

$$\langle\tilde\lambda^R(q)\tilde{\bar\lambda}^R(q')\rangle^{\overline{\rm MS}}_{\rm inv} = (2\pi)^4\delta(q-q')\frac{i}{2}\not{q}\left[1+\frac{g^2\,N_f}{16\,\pi^2}\left(2+\log\left(\frac{\bar\mu^2}{q^2}\right)\right)+\frac{g^2\,N_c}{16\,\pi^2}\left(\alpha+\alpha\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\right]. \tag{13}$$

The ghost propagator is the same as in the non-supersymmetric case:

$$\langle\tilde c^R(q)\tilde{\bar c}^R(q')\rangle^{\overline{\rm MS}}_{\rm inv} = (2\pi)^4\delta(q-q')q^2\left[1-\frac{g^2\,N_c}{16\,\pi^2}\left(1+\frac{1}{4}(3-\alpha)\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\right]. \tag{14}$$

The 3-pt amputated Green's function, at zero antighost momentum, in $\overline{\rm MS}$ renormalization scheme, gives:

$$\langle\tilde c^{R\,\alpha}(q)\tilde{\bar c}^{R\,\beta}(0)\tilde u^{R\,\gamma}_\mu(q')\rangle^{\overline{\rm MS}}_{\rm amp} = (2\pi)^4\delta(q+q')f^{\alpha\,\beta\,\gamma}(igq_\mu)\left[1+\frac{g^2\,N_c}{16\,\pi^2}\frac{\alpha}{2}\left(1+\log\left(\frac{\bar\mu^2}{q^2}\right)\right)\right]. \tag{15}$$

## Green's Functions on the lattice

The first result presented here, Eq. (16), is the lattice inverse quark propagator up to one loop. In all lattice expressions the systematic errors, coming from numerical loop integration, are smaller than the last digit we present.

$$\langle\tilde\psi^B(q)\tilde{\bar\psi}^B(q')\rangle^L_{\rm inv} = (2\pi)^4\delta(q-q')\left\{i\,\not{q}\left[1-\frac{g^2\,C_F}{16\,\pi^2}\left[12.8025-4.7920\alpha+(2+\alpha)\log\left(a^2\,q^2\right)\right]\right]+\frac{g^2\,C_F}{16\,\pi^2}\frac{1}{a}51.4347\,r\right\}. \tag{16}$$

The inverse squark propagator, is:

$$\langle A^B A^{B\,\dagger}\rangle^L_{\rm inv} = q^2\mathbb{1}+\frac{g^2\,C_F}{16\,\pi^2}\left\{q^2\left[11.0173-3.7920\alpha+(1+\alpha)\log(a^2\,q^2)\right]\begin{pmatrix}1&0\\0&1\end{pmatrix}+1.0087q^2\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{a^2}\begin{pmatrix}65.3930&75.4031\\75.4031&65.3930\end{pmatrix}\right\}. \tag{17}$$

The gluino inverse propagator is:

$$\begin{aligned}\langle\tilde\lambda^B(q)\tilde{\bar\lambda}^B(q')\rangle^L_{\rm inv} &= (2\pi)^4\delta(q-q')\Big\{\frac{i}{2}\,\not{q}\Big[1+\frac{g^2\,N_f}{16\,\pi^2}\left(1.9209-\log\left(a^2\,q^2\right)\right)-\frac{g^2\,N_c}{16\,\pi^2}\left(16.6444-4.7920\,\alpha+\alpha\log\left(a^2\,q^2\right)\right)\Big]\\ &+\frac{g^2}{16\,\pi^2}\frac{N_c}{2}\frac{1}{a}51.4347\,r\Big\}.\end{aligned} \tag{18}$$

The gluon inverse propagator is given by:

$$\begin{aligned}\langle\tilde u^B_\mu(q)\tilde u^B_\nu(q')\rangle^L_{\rm inv} &= (2\pi)^4\delta(q+q')\Big\{\frac{1}{\alpha}q_\mu q_\nu+(q^2\delta_{\mu\nu}-q_\mu q_\nu)\Big[1-\frac{g^2}{16\,\pi^2}\Big[-19.7392\frac{1}{N_c}+N_f\left(-2.9622+\log\left(a^2\,q^2\right)\right)\\ &-N_c\left(20.1472-0.8863\,\alpha+\frac{\alpha^2}{4}+\left(\frac{\alpha}{2}-\frac{3}{2}\right)\log\left(a^2\,q^2\right)\right)\Big]\Big]\Big\}.\end{aligned} \tag{19}$$

The ghost field renormalization, $Z_c$, which enters the evaluation of $Z_g$ can be extracted from the ghost propagator:

$$\langle\tilde c^B(q)\tilde{\bar c}^B(q')\rangle^L_{\rm inv} = (2\pi)^4\delta(q-q')q^2\left[1+\frac{g^2\,N_c}{16\,\pi^2}\left(4.6086-1.2029\alpha-\frac{1}{4}(3-\alpha)\log\left(a^2\,q^2\right)\right)\right]. \tag{20}$$

Lastly, the amputated gluon-antighost-ghost Green's function is:

$$\langle\tilde c^{B\,\alpha}(q)\tilde{\bar c}^{B\,\beta}(0)\tilde u^{B\,\gamma}_\mu(q')\rangle^L_{\rm amp} = (2\pi)^4\delta(q+q')f^{\alpha\,\beta\,\gamma}\,(igq_\mu)\left[1+\frac{g^2\,N_c}{16\,\pi^2}\left(2.3960\alpha-\frac{1}{2}\alpha\log\left(a^2\,q^2\right)\right)\right]. \tag{21}$$

The critical masses for the quark, squark and gluino can be read off (up to a minus sign) from the $\mathcal{O}(q^0)$ parts of Eqs. (16), (17), (18), respectively.

## Renormalizarion Factors

Renormalization factors relate bare quantities ($B$) on the lattice to their renormalized ($R$) continuum counterparts:

$$\psi^R = \sqrt{Z_\psi}\,\psi^B,\;\; A^R = Z_A^{1/2}A^B, u^R_\mu = \sqrt{Z_u}\,u^B_\mu,\;\; \lambda^R = \sqrt{Z_\lambda}\,\lambda^B,\;\; c^R = \sqrt{Z_c}\,c^B,\;\; g^R = Z_g\,\mu^{-\epsilon}\,g^B. \tag{22}$$

Combining the Green's functions on the lattice with the corresponding results from the continuum, we extract $Z_\psi^{L,\overline{\rm MS}}$, $Z_u^{L,\overline{\rm MS}}$, $Z_\lambda^{L,\overline{\rm MS}}$, $Z_{A_\pm}^{L,\overline{\rm MS}}$, $Z_c^{L,\overline{\rm MS}}$ and $Z_g^{L,\overline{\rm MS}}$ in the $\overline{\rm MS}$ scheme and on the lattice.

$$Z_\psi^{L,\overline{\rm MS}} = 1+\frac{g^2\,C_F}{16\,\pi^2}\left(-16.8025+3.7920\alpha-(2+\alpha)\log\left(a^2\,\bar\mu^2\right)\right). \tag{23}$$

$$Z_\lambda^{L,\overline{\rm MS}} = 1-\frac{g^2}{16\,\pi^2}\left[N_c\left(16.6444-3.7920\alpha+\alpha\log\left(a^2\,\bar\mu^2\right)\right)+N_f\left(0.07907+\log\left(a^2\,\bar\mu^2\right)\right)\right]. \tag{24}$$

$$\left(Z_A^{1/2}\right)^{L,\overline{\rm MS}} = \mathbb{1}-\frac{g^2\,C_F}{16\,\pi^2}\left\{\left[8.1753-1.8960\alpha+\frac{1}{2}(1+\alpha)\log\left(a^2\,\bar\mu^2\right)\right]\begin{pmatrix}1&0\\0&1\end{pmatrix}-0.1623\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}. \tag{25}$$

$$Z_u^{L,\overline{\rm MS}} = 1+\frac{g^2}{16\,\pi^2}\left[19.7392\frac{1}{N_c}-N_c\left(18.5638-1.3863\alpha+\left(-\frac{3}{2}+\frac{\alpha}{2}\right)\log\left(a^2\,\bar\mu^2\right)\right)+N_f\left(0.9622-\log\left(a^2\,\bar\mu^2\right)\right)\right]. \tag{26}$$

$$Z_c^{L,\overline{\rm MS}} = 1-\frac{g^2N_c}{16\pi^2}\left[3.6086-1.2029\alpha-\frac{1}{4}(3-\alpha)\log\left(a^2\,\bar\mu^2\right)\right]. \tag{27}$$

$$Z_g^{L,\overline{\rm MS}} = 1+\frac{g^2}{16\,\pi^2}\left[-9.8696\frac{1}{N_c}+N_c\left(12.8904-\frac{3}{2}\log\left(a^2\,\bar\mu^2\right)\right)-N_f\left(0.4811-\frac{1}{2}\log(a^2\,\bar\mu^2)\right)\right]. \tag{28}$$

From the calculation of $Z_g^{L,\overline{\rm MS}}$ one can extract the Callan-Symanzik beta-function for SQCD. On the lattice the bare beta-function is defined as: $\beta_L(g^B) = -adg^B/da|_{g^R,\,\bar\mu}$. In the asymptotic limit for SQCD, the expansion of the beta-function is done in powers of the bare coupling constant. The first term in this expansion is:

$$\beta_L(g) = \frac{g^3}{16\,\pi^2}\left(-3N_c+N_f\right)+\mathcal{O}(g^5). \tag{29}$$

For $N_f < 3N_c$, the $\mathcal{O}(g^3)$ term is negative, in other words, the theory is asymptotically free. Our finding for the beta function agrees with what is obtained in the supersymmetric Yang-Mills theory [4].

## References

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