Distribution of the Dirac modes in QCD

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Introduction

- Banks-Casher relation:

$$\Sigma = -\langle \bar{q}q \rangle = \pi \rho(0)$$

- Truncated quark propagator

$$D_{tr}^{-1} = D^{-1} - \sum_{i=1}^{k} \frac{1}{\lambda_i} |\lambda_i\rangle\langle\lambda_i|$$

k = number of removed low-lying modes

- Restoration of *Chiral Symmetry*

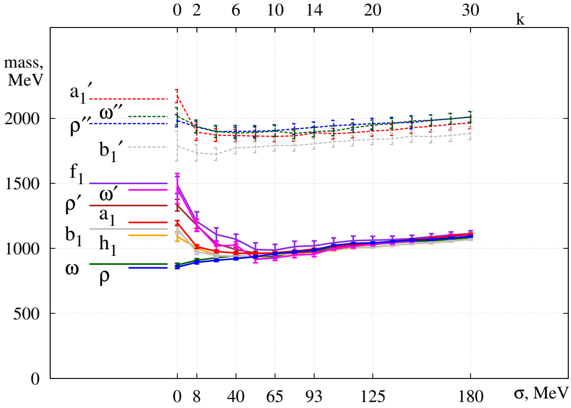
$$SU(2)_R \times SU(2)_L \times U(1)_A$$

- Emergence of

$$SU(4) \supset SU(2)_R \times SU(2)_L \times U(1)_A$$

M. Denissenya et al. Phys. Rev. D91 (2015)

$N_F = 2$ degenerate quarks:



L. Ya. Glozman Eur. Phys. J. A (2015)
L. Ya. Glozman, M. Pak Phys. Rev. D92 (2015)

The Problem

- Low-lying modes carry the information about *Chiral Symmetry Breaking*
- *ChRMT* provides an explanation of the low-lying modes distributions

  E.V. Shuryak and J.J.M. Verbaarschot Nucl. Phys. A560, (1993)
- We ask the question:

  What happens for higher eigenmodes?

QCD and ϵ regime

Effective chiral Lagrangian $o(p^2)$, coming from *Chiral Symmetry Breaking*, $SU(N_F)_L \otimes SU(N_F)_R \to SU(N_F)_V$. H. Leutwyler and A. Smilga Phys. Rev. D46 (1992):

$$L_{eff}(U) = \frac{F^2}{4} Tr\left(\partial_\mu U^\dagger(x)\partial_\mu U(x)\right) - m\Sigma Re\left\{e^{i\frac{\theta}{N_F}} Tr\left(U^\dagger(x)\right)\right\}$$

If $\Lambda_{QCD} L \gg 1$ and $m_\pi L \ll 1 \Leftarrow$ ϵ-regime

$$Z_{eff} = \int e^{-\int d^4x L_{eff}(U)} dU \to \int P(W)dW$$

$$P(W)dW = \mathcal{N}(det\ D)^{N_F} e^{-\frac{N\Sigma^2\beta}{4} Tr\left(W^\dagger W\right)} dW$$

- $D = \begin{pmatrix} m & iW \\ iW^\dagger & m \end{pmatrix}$ is the Dirac operator
- W is an $N_+ \times N_-$ matrix, $N_+ + N_- = N$
- β is the Dyson index

E.V. Shuryak and J.J.M. Verbaarschot Nucl. Phys. A560, (1993)

Lattice Setup

- lattice size: $V = L^3 \times L_t = 16^3 \times 32$
- $a = 0.1184(30)\ fm$, $L \approx 1.9\ fm$
- $N_F = 2$ dynamical fermions with same mass $m = 0.015$
- 100 configurations (JLQCD)
- $\beta = 2.30$
- Topological charge $Q = 0$
- Overlap Dirac operator $D_{ov}(m)$
- $m_\pi = 289(2)\ MeV$

S. Aoki et al. (2008)

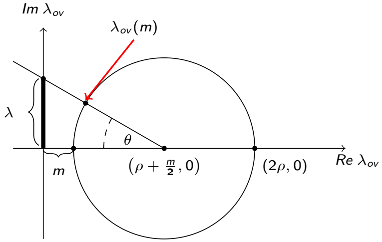
In our case the chiral condensate is $\Sigma = (251 \pm 7 \pm 11\ MeV)^3$, $V\Sigma m \simeq 4$ and $m_\pi L \simeq 3$

We are not in the ϵ-regime

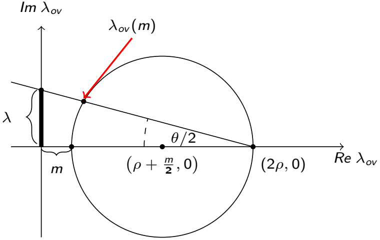
Stereographic projections

Overlap Dirac operator:

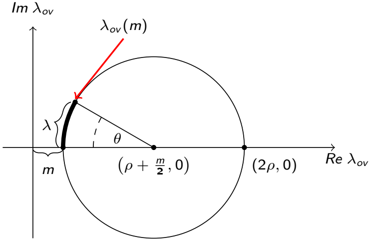
$$D_{ov}(m) =$$
$$= \left(\rho + \frac{m}{2}\right) + \left(\rho - \frac{m}{2}\right) \gamma_5 sign\left(H_w\right) =$$
$$= \left(1 - \frac{m}{2\rho}\right) D_{ov}(0) + m$$



(b)



(a)



(c)

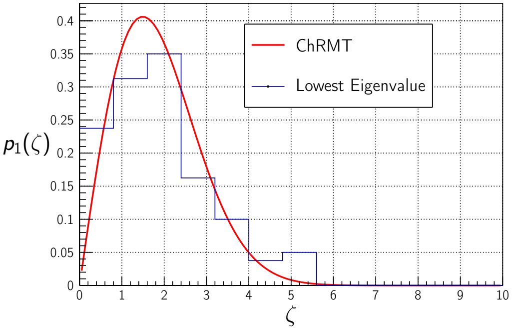
Lowest eigenvalue distribution

Defining

- $\mu = V\Sigma m$
- $\zeta = V\Sigma\lambda$
- and sorting the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

From *ChRMT* we can get

- the distribution $p_k(\zeta_k)$ for each ζ_k
- the total distribution $\rho(\zeta) = \sum_k p_k(\zeta)$



$$p_1(\zeta_1) = const \cdot \zeta_1 e^{-\frac{\zeta_1^2}{4}} \left(I_0(\tilde{\zeta}_1) - I_1(\tilde{\zeta}_1) I_2(\tilde{\zeta}_1) \right)$$

where

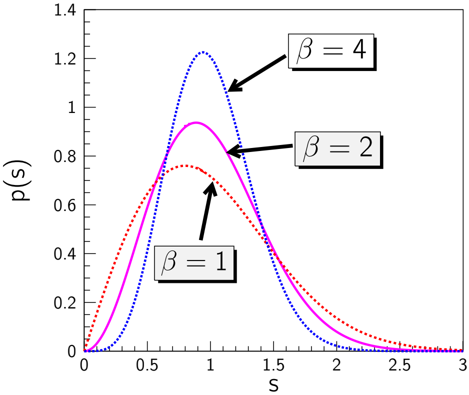
- $\tilde{\zeta}_k = \sqrt{\mu^2 + \zeta_k^2}$
- $I_n(x)$ is the modified Bessel function

NNS distribution

An important prediction of RMT is given by the nearest neighbor spacing (NNS) distribution $p(s)$:

$$p_\beta(s) = a_\beta s^\beta e^{-b_\beta s^2}$$

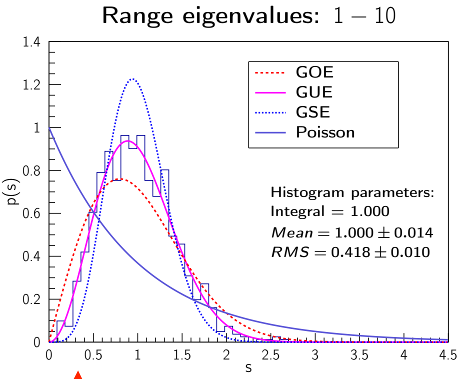
ensemble	a_β	b_β
ChGOE ($\beta = 1$)	$\pi/2$	$\pi/4$
ChGUE ($\beta = 2$)	$32/\pi^2$	$4/\pi$
ChGSE ($\beta = 4$)	$262144/729\pi^3$	$64/9\pi$

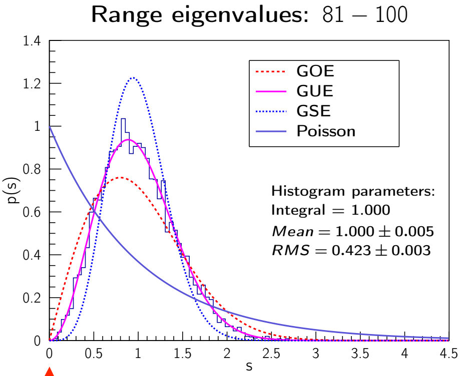


$$s_n = \xi_{n+1} - \xi_n$$

$$\xi_n \equiv \xi(\lambda_n) = \int_0^{\lambda_n} \rho(\lambda) d\lambda.$$

$p(s)$ for low and higher eigenvalues





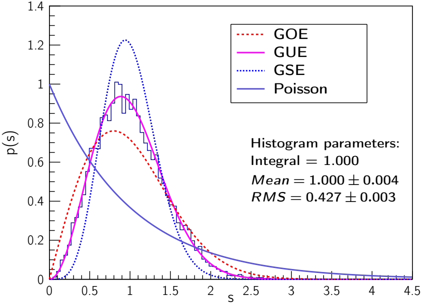
Contains information about
$SU(2)_R \otimes SU(2)_L$ and $U(1)_A$ breaking
Consistent with ChRMT and previous results

Sensitive to confinement physics
$SU(4)$ symmetric regime

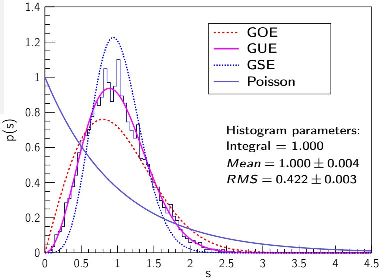
They are identical!

$p(s)$ distribution for 100 eigenvalues

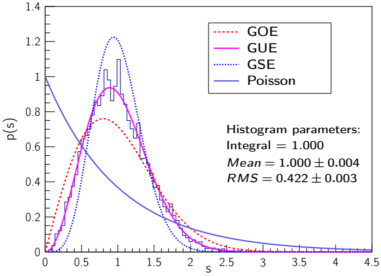
Results do not depend on definition of λ



(a) $\lambda = 2\rho tg(\theta/2)$



(b) $\lambda = \left(\rho + \frac{m}{2}\right) tg(\theta)$



(c) $\lambda = \left(\rho - \frac{m}{2}\right) \theta$

Conclusions

- The nearest neighbor spacing (NNS) distribution $p(s)$ is the same for
  1. low eigenmodes $1-10$
  2. higher eigenmodes $81-100$
  3. All possible range of eigenvalues
  4. different stereographic projections
- In all these cases $p(s)$ follows the Wigner distribution

  predicted for the *GUE*
- The Wigner distribution that is consistent with the *Chiral Symmetry Breaking* is connected in reality to something of more general

Backup

Chiral Symmetry on the lattice

$$\{\gamma_5, D_{ov}(0)\} = \frac{1}{\rho} D_{ov}(0)\gamma_5 D_{ov}(0)$$

Ginsparg-Wilson equation (1982)

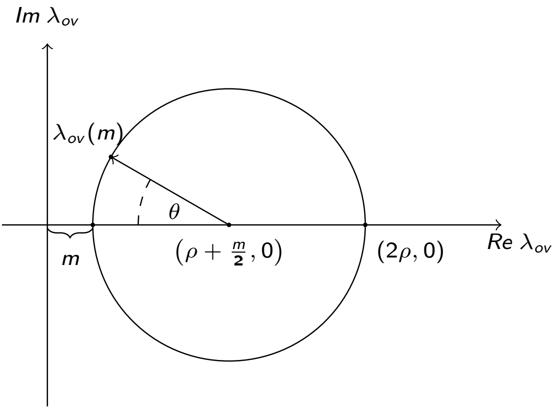
The eigenvalues:

$$D_{ov}(m)|\lambda\rangle = \lambda_{ov}(m)|\lambda\rangle$$

are on the cirlce:

- Radius: $R = \rho - \frac{m}{2}$
- Centre: $(\rho + \frac{m}{2}, 0)$

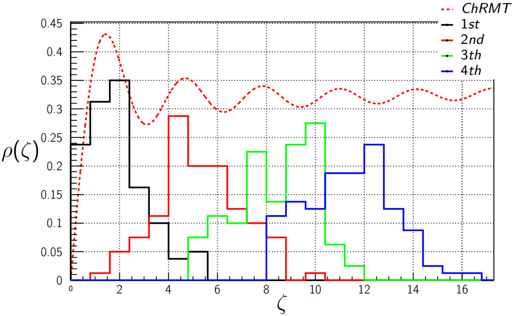
and they come in pairs $(\lambda_{ov}(m), \lambda^*_{ov}(m))$



Distribution of the first 4 lowest eigenvalues

The distribution of ζ for $V\Sigma m \gg 1$, using *ChRMT*, is given by

$$\rho(\zeta) = \frac{\zeta}{2}(J_0^2(\zeta) + J_1^2(\zeta)).$$

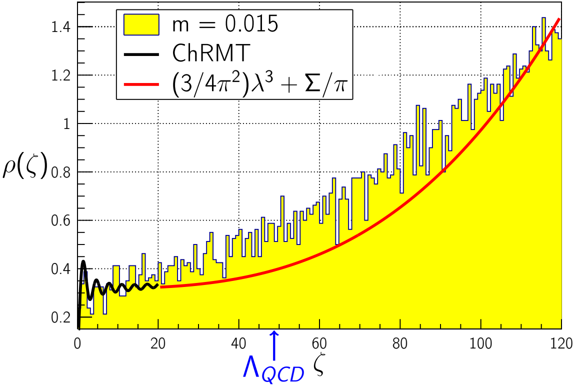


k/j	$\langle\lambda_k\rangle/\langle\lambda_j\rangle$	σ	RMT
2/1	2.72	0.19	2.70
3/1	4.35	0.28	4.46
4/1	5.92	0.38	6.22
4/2	2.17	0.08	2.30
4/3	1.36	0.03	1.40
3/2	1.60	0.06	1.65

Ratio $\langle\lambda_k\rangle/\langle\lambda_j\rangle$ for all $1 \leq j \leq k \leq 4$

Distribution of the first 100 eigenvalues of the Dirac operator

$\Sigma = (232.18 \pm 0.85 MeV)^3$



From theoretical arguments the distribution for $\lambda \ll \Lambda_{QCD}$ is given by

$$\rho(\lambda) = \frac{\Sigma}{\pi} + \frac{\Sigma^2(N_f^2 - 4)}{32\pi^2 N_f F^4}|\lambda| + o(\lambda)$$

and for $\lambda \gg \Lambda_{QCD}$:

$$\rho(\lambda) \simeq \frac{N_c}{4\pi^2}|\lambda|^3.$$

A. V. Smilga (1995)

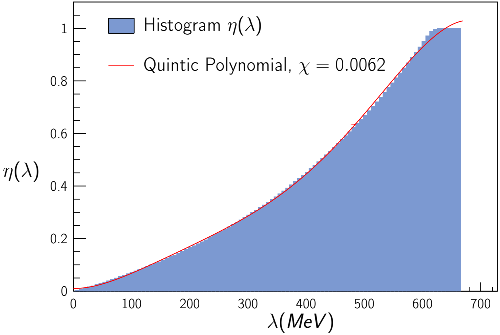
Unfolding Procedure

$$S(\lambda) = \frac{1}{M}\langle\sum_{k=1}^{M}\delta(\lambda-\lambda_k)\rangle$$

$$\eta(\lambda) = \int_0^\lambda S(\lambda)d\lambda = \frac{1}{M}\langle\sum_{k=1}^{M}\theta(\lambda-\lambda_k)\rangle$$

$$\eta(\lambda) = \xi(\lambda) + \eta_{fl}(\lambda)$$

- $M = 100$ is the number of eigenvalues studied
- the smooth part is $\xi(\lambda) = \int_0^\lambda \rho(\lambda)d\lambda$
- $\eta_{fl}(\lambda)$ is the fluctuating part



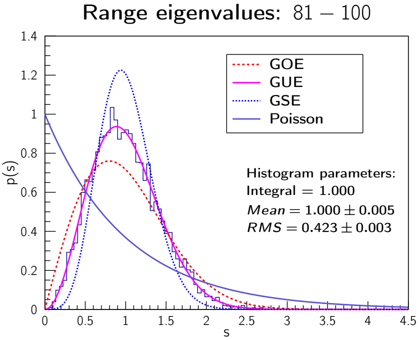
Consequences:

- The density distribution $\tilde{\rho}(\xi) = 1$
- in a small region of the spectrum such that in the limit $M \to \infty$ it contains a lot of levels, then $\rho(\lambda) = \frac{1}{D}$, with $D = \langle\lambda_{n+1} - \lambda_n\rangle$, therefore $\xi(\lambda) = \frac{\lambda}{D}$.

Higher eigenvalues for different stereographic projections

For higher eigenvalues the distribution

$p(s)$ doesn't change!



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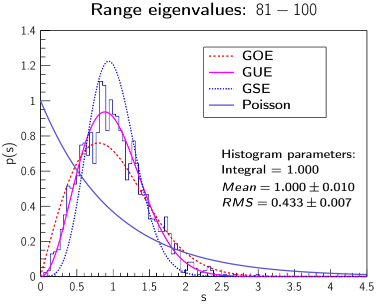

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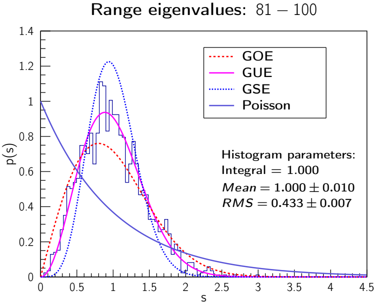

(c) $\lambda = \left(\rho - \frac{m}{2}\right) \theta$