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HVP contribution to $(g - 2)_\mu$ including electromagnetic corrections with twisted-mass fermions

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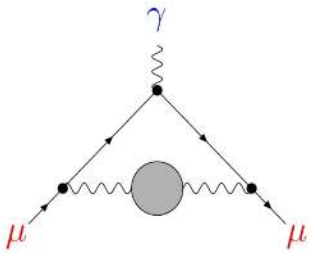
aim of the talk

* application of the RM123 approach for an evaluation of the e.m. and isospin breaking corrections to a_μ^{HVP}

* results for the strange and charm contributions (connected diagrams only) and preliminary results for the light contribution

* largest uncertainties from Hadronic Vacuum Polarization (HVP) and Hadronic Light by Light (HLbL)

HVP:

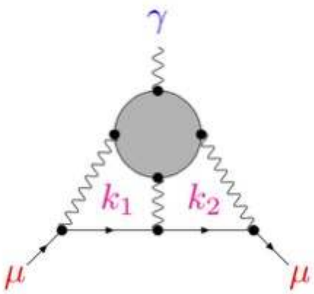


dispersion theory combined with data on $e^+e^- \to$ hadrons

$$a_\mu^{\rm HVP}\left(\alpha_{em}^2\right) = (692.3 \pm 4.2)\cdot 10^{-10} \quad \text{[Davier et al. '11]}$$

$$= (694.9 \pm 4.3)\cdot 10^{-10} \quad \text{[Hagiwara et al. '11]}$$

HLbL:



dispersion formalism much more involved for HLbL

$$a_\mu^{\rm HLbL} = (10.5 \pm 2.6)\cdot 10^{-10} \quad \text{[Hagiwara et al. '11, PDG '16]}$$

* future experiments at FermiLab [E989] and J-PARC (E34) aim at a target precision of ~ **2 10⁻¹⁰**

$$a_\mu^{HVP}\left(\alpha_{em}^3\right) \qquad \text{[Jegerlehner\&Nyffeler '09]}$$

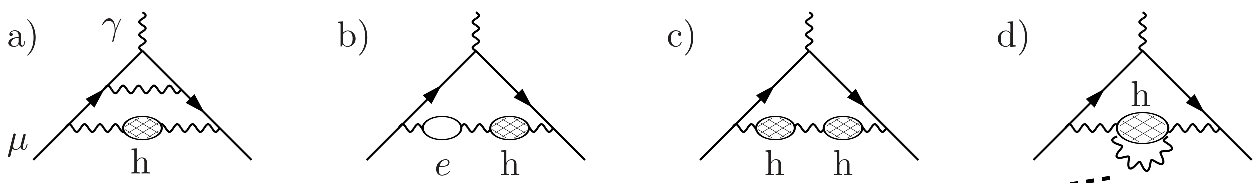
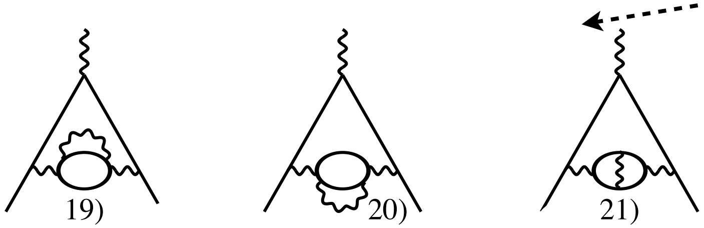


Fig. 29. Hadronic higher order VP contributions: a)-c) involving LO vacuum polarization, d) involving HO vacuum polarization (FSR of hadrons).



usually not included in $a_\mu^{HVP}\left(\alpha_{em}^3\right)$, but in $a_\mu^{HVP}\left(\alpha_{em}^2\right)$

[mainly $e^+e^- \to \pi^+\pi^-\gamma$]

$$\left.\begin{aligned} a_\mu^{\rm HVP}\left(\alpha_{em}^3\right)^{a+b+c} &= (-9.84 \pm 0.07)\cdot 10^{-10} \\ a_\mu^{\rm HLbL} &= (10.5 \pm 2.6)\cdot 10^{-10} \end{aligned}\right\} (0.7 \pm 2.6)\cdot 10^{-10}$$

Hagiwara et al. '11
PDG '16

***** lattice QCD calculations of $a_\mu^{HVP}(\alpha_{em}^2)$ and δa_μ^{HVP} are mandatory *****

* several lattice QCD calculations of $a_\mu^{\mathrm{HVP}}(\alpha_{em}^2)$ are available: RBC/UKQCD, HPQCD, ETMC, ...

* lattice calculations for the e.m. corrections to the HVP δa_μ^{HVP} are in progress

* during the last few years the issue of the electromagnetic corrections to hadron observables has been addressed on the lattice:

- hadron spectrum [BMW, RM123, RBC/UKQCD]: no IR divergencies (various techniques)

- leptonic hadron decays [RM123 + Soton]: presence of IR divergencies

******* first application of the RM123 approach to the IB corrections for a_μ^{HVP} *******

the RM123 approach is based on a double expansion in the "small" parameters α_{em} and $(m_d - m_u)$ ~ 1% [JHEP '12, PRD '13]

only isospin-symmetric QCD gauge configurations are required

master formula

$$a_\mu^{HVP} = 4\alpha_{em}^2 \int_0^\infty dQ^2 \, \frac{1}{m_\mu^2} \, f\left(\frac{Q^2}{m_\mu^2}\right) \left[\Pi\left(Q^2\right) - \Pi(0)\right]$$

$Q =$ Euclidean 4-momentum

kinematical kernel: $f(s) = \frac{1}{s}\sqrt{\frac{s}{4+s}}\left(\frac{\sqrt{4+s}-\sqrt{s}}{\sqrt{4+s}+\sqrt{s}}\right)^2$ peaked at $s = \sqrt{5} - 2 \simeq 0.24$

$\Pi\left(Q^2\right) =$ HVP form factor appearing in the covariant decomposition of the HVP tensor:

$$\Pi_{\mu\nu}(Q) = \int d^4x \, e^{iQ\cdot x} \left\langle J_\mu(x) J_\nu(0) \right\rangle = \left[\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu\right] \Pi\left(Q^2\right)$$

$$J_\mu(x) = \sum_{f=u,d,s,c,\ldots} q_f \, \bar{\psi}_f(x) \gamma_\mu \psi_f(x)$$

$$\Pi\left(Q^2\right) - \Pi(0) = 2\int_0^\infty dt \, V(t) \left[\frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2}t^2\right]$$

$$V(t) \equiv \frac{1}{3} \sum_{i=1,2,3} \int d\vec{x} \left\langle J_i(\vec{x},t) J_i(0) \right\rangle = \text{vector correlator}$$

[Bernecker&Meyer EPJA '11]

calculable on the lattice
using local vector currents

see Burger et al. [ETMC]
JHEP '15

time-momentum representation

$$a_\mu^{HVP} = 4\alpha_{em}^2 \int_0^\infty dt\, \tilde{f}(t)\, V(t)$$

[Bernecker&Meyer EPJA '11]

$$V(t) = \frac{1}{3}\sum_{i=1,2,3}\int d\vec{x} \left\langle \sum_f \bar{\psi}_f(\vec{x},t)\gamma_i\psi_f(\vec{x},t) \sum_{f'}\bar{\psi}_{f'}(0)\gamma_i\psi_{f'}(0) \right\rangle$$

$$\tilde{f}(t) \equiv 2\int_0^\infty dQ^2 \frac{1}{m_\mu^2} f\left(\frac{Q^2}{m_\mu^2}\right)\left[\frac{\cos(Qt)-1}{Q^2} + \frac{1}{2}t^2\right]$$

$$f(s) = \frac{1}{s}\sqrt{\frac{s}{4+s}}\left(\frac{\sqrt{4+s}-\sqrt{s}}{\sqrt{4+s}+\sqrt{s}}\right)^2 = \text{kinematical kernel}$$

peaked at s $\approx \sqrt{5}-2 \approx 0.24$

$\tilde{f}(t)$ is proportional to t^4 at small t and to t^2 at large t

$$\tilde{f}(t) = \frac{1}{36} m_\mu^2 t^4 + O(t^4), \qquad \tilde{f}(t) \xrightarrow[t\to\infty]{} \frac{1}{2}t^2$$

* enhancement of the large time distance behavior of the vector correlator **V(t)**

- we will limit ourselves to **connected diagrams** only (each quark flavor f contributes separately)

- the vector correlator V(t) can be calculated at discretized values of t between t = 0 and t = T / 2

$$a_\mu^{HVP} = a_\mu^{HVP}(<) + a_\mu^{HVP}(>)$$

$t \leq T_{data} < T/2$ (to avoid backward signals)

$t > T_{data} > t_{min}$ (ground-state dominance)

$$a_\mu^{HVP}(<) = 4\alpha_{em}^2 q_f^2 \sum_{\bar{t}=0}^{\bar{T}_{data}} w(\bar{t})\, \bar{f}(\bar{t})\, \bar{V}(\bar{t}) \quad \text{(directly from lattice data)}$$

$$a_\mu^{HVP}(>) = 4\alpha_{em}^2 q_f^2 \sum_{\bar{t}=\bar{T}_{data}+1}^{\infty} w(\bar{t})\, \bar{f}(\bar{t})\, \frac{\bar{G}_V}{2\bar{M}_V} e^{-\bar{M}_V \bar{t}} \quad \text{(analytic representation)}$$

overlined quantites are in lattice units

$\bar{m}_\mu = a m_\mu$, $\bar{t} = t/a$,
$\bar{M}_V = a M_V$,...

* we checked that the sum is independent on T_{data}

in what follows $\bar{T}_{data} = \bar{T}/2 - 4$
$a_\mu^{HVP}(<)$ dominates on a_μ^{HVP}
within the statistical errors

$$\bar{f}(\bar{t}) \equiv \frac{4}{\bar{m}_\mu^2} \int_0^\infty d\omega \sqrt{\frac{1}{4+\omega^2}} \left(\frac{\sqrt{4+\omega^2} - \omega}{\sqrt{4+\omega^2} + \omega} \right)^2 \left[\frac{\cos(\omega \bar{m}_\mu \bar{t}) - 1}{\omega^2} + \frac{1}{2} \bar{m}_\mu^2 \bar{t}^2 \right]$$

$$\bar{G}_V \equiv \frac{1}{3} \sum_{i=1,2,3} \left| \langle 0 | J_i(0) | V \rangle \right|^2 = \text{ coupling constant with the ground-state}$$

$w(\bar{t})$ = weights of the (cubic) Simpson formula (dropped in what follows for the sake of simplicity)

ETMC ensembles with N_f = 2+1+1

ensemble	β	V/a^4	$a\mu_{sea} = a\mu_\ell$	$a\mu_\sigma$	$a\mu_\delta$	N_{cfg}	$a\mu_s$	$a\mu_c$
$A30.32$	1.90	$32^3 \times 64$	0.0030	0.15	0.19	150	0.02363	0.27903
$A40.32$			0.0040			100		
$A50.32$			0.0050			150		
$A40.24$		$24^3 \times 48$	0.0040			150		
$A60.24$			0.0060			150		
$A80.24$			0.0080			150		
$A100.24$			0.0100			150		
$A40.20$		$20^3 \times 48$	0.0040			150		
$B25.32$	1.95	$32^3 \times 64$	0.0025	0.135	0.170	150	0.02094	0.24725
$B35.32$			0.0035			150		
$B55.32$			0.0055			150		
$B75.32$			0.0075			80		
$B85.24$		$24^3 \times 48$	0.0085			150		
$D15.48$	2.10	$48^3 \times 96$	0.0015	0.1200	0.1385	100	0.01612	0.19037
$D20.48$			0.0020			100		
$D30.48$			0.0030			100		

a = $\{0.0885, 0.0815, 0.0619\}$ fm
at
$\beta = \{1.90, 1.95, 2.10\}$

pion masses in the range
210 - 450 MeV

Table 1: *Values of the simulated sea and valence quark bare masses for the 16 ETMC gauge ensembles with $N_f = 2{+}1{+}1$ dynamical quarks adopted in this work (see Ref. [3]). The values of the strange and charm quark bare masses $a\mu_s$ and $a\mu_c$, given for each gauge ensemble, correspond to the physical strange and charm quark masses determined in Ref. [3].*

* correlators calculated in the PRACE project on “QED corrections to meson decay rates in LQCD”

* number of stochastic sources per gauge configuration not optimal for the vector correlator

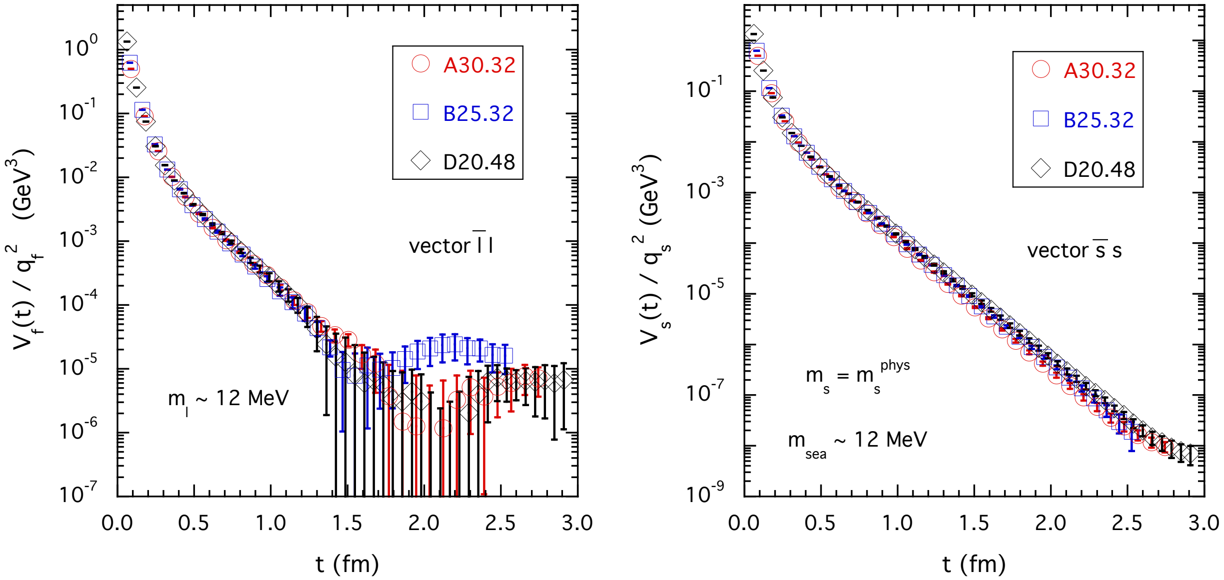


FIG. 4: *The vector correlator $V(t)/q_f^2$ (in physical units) in the case of the light (left panel) and strange (right panel) quarks for the ETMC gauge ensembles specified in the inset, which share an approximate common value of the light-quark mass $m_\ell \simeq 12$ MeV and differ in the values of the lattice spacing.*

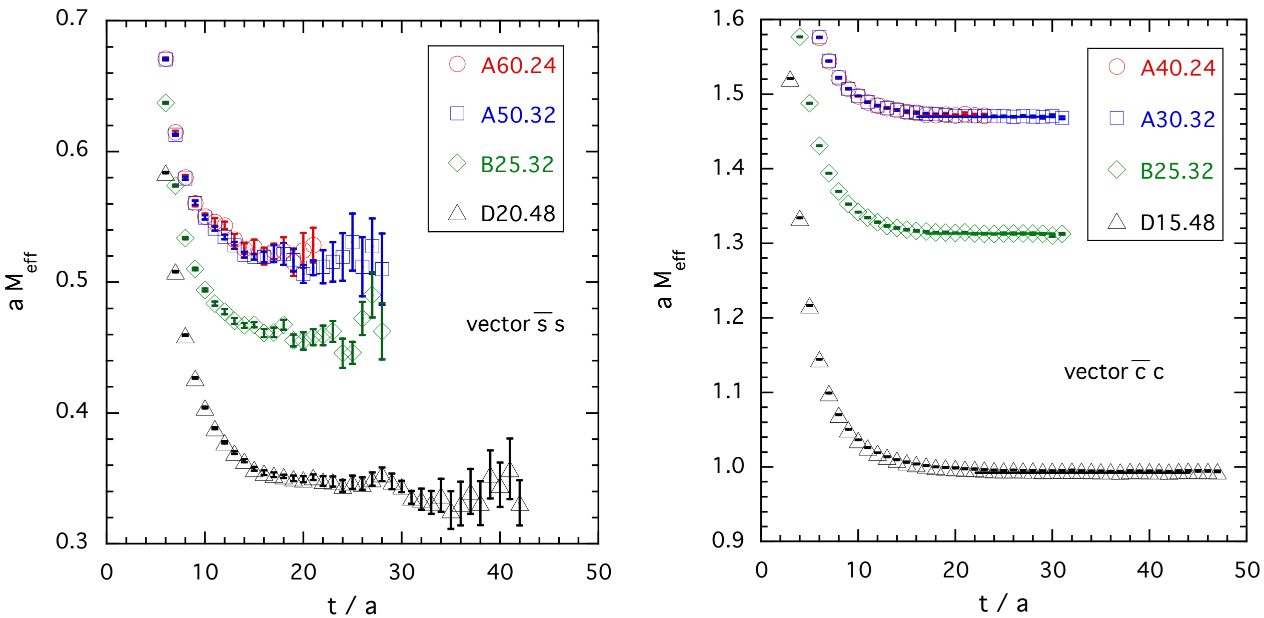


FIG. 5: *Effective mass of the vector correlator $\overline{V}(\bar{t})$ in the case of the strange (left panel) and charm (right panel) contributions for the ETMC gauge ensembles specified in the insets.*

ground-state identification

number of stochastic sources per gauge configuration not optimal for the vector correlator

↓

not OK for the light contribution

~ OK for the strange contribution

OK for the charm contribution

$$\bar{M}_{eff} = \log \frac{\bar{V}(\bar{t})}{\bar{V}(\bar{t}-1)}$$

Effective Lepton Mass (ELM) procedure

instead of am_μ^{phys} :

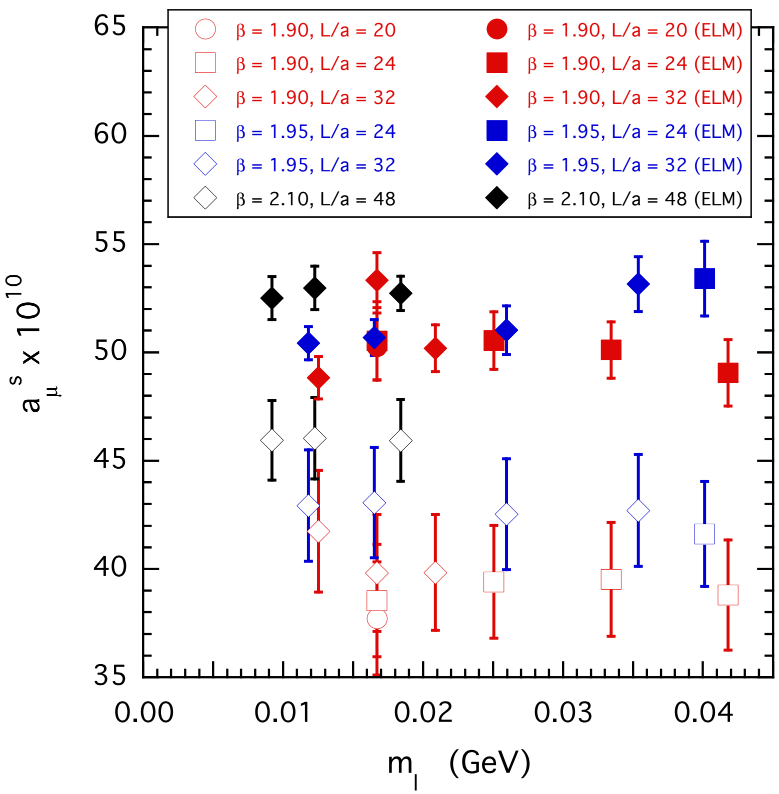
$$am_\mu^{ELM} = aM_V \frac{m_\mu^{phys}}{M_V^{phys}}$$

[ETMC JHEP '14]

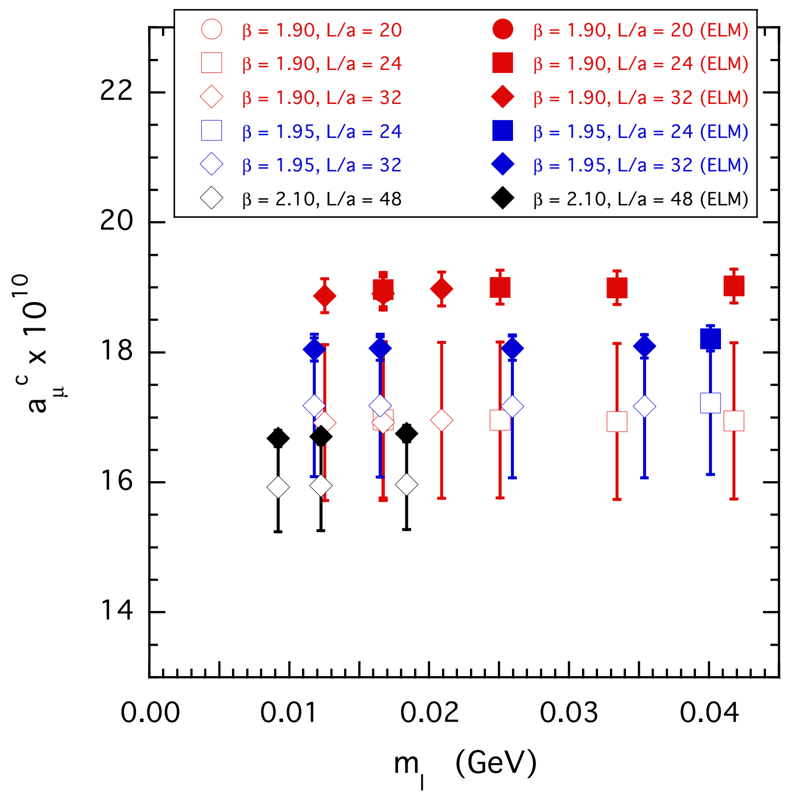
- no need of the value of the lattice spacing (no sensitivity to the lattice scale setting)

- sensitivity to the precision of the vector meson mass aM_V

strange contribution M_φ= 1.095 GeV

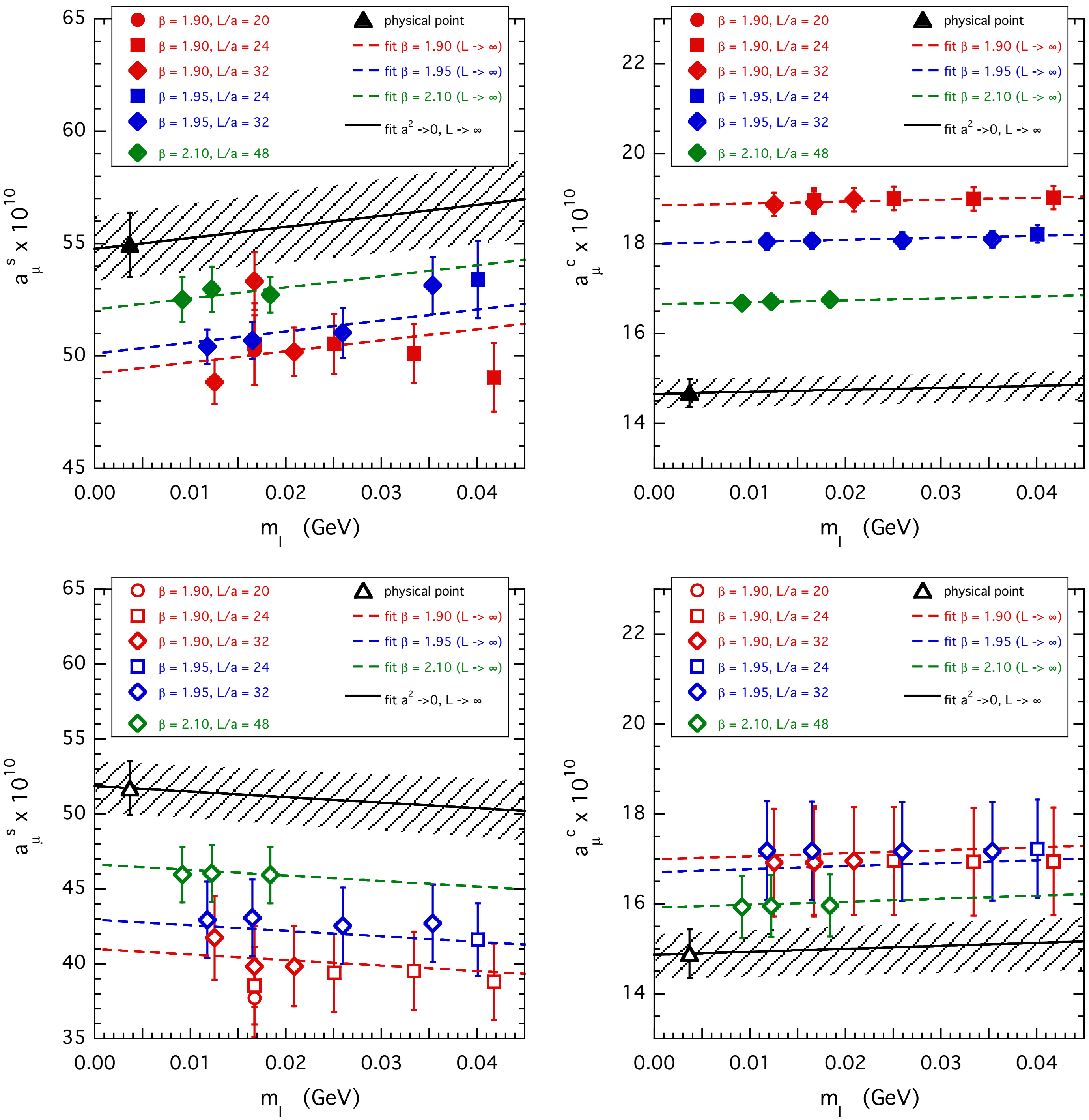


charm contribution $M_{J/\psi}$= 3.0969 GeV



much better precision with the ELM procedure strange contribution

charm contribution



with ELM

fitting functions:

$$a_\mu^{s,c} = A_0^{s,c}\left[1 + A_1^{s,c}\xi + D^{s,c}\, a^2 + F^{s,c}\,\xi\,\frac{e^{-M_\pi L}}{M_\pi L}\right]$$

$$\xi = \frac{M_\pi^2}{(4\pi f_0)^2}$$

without ELM

fitting functions: $a_\mu^{s,c} = A_0^{s,c}\left[1 + A_1^{s,c}\xi + D^{s,c}\, a^2 + F^{s,c}\, \xi \frac{e^{-M_\pi L}}{M_\pi L}\right]$ $\qquad \xi = \frac{M_\pi^2}{(4\pi f_0)^2}$

error budget: a^2: with/without the ELM procedure

FSE: F = 0 or F ≠ 0

chir: $A_1 = 0$ or $A_1 \neq 0$

input: uncertainties due to the scale setting, to the physical quark masses, ...

stat+fit: statistical + fitting procedure errors

strange contribution:

$$a_\mu^s(phys) = \left(53.1 \pm 1.6_{stat+fit} \pm 1.5_{input} \pm 1.3_{a^2} \pm 0.2_{FSE} \pm 0.1_{chiral}\right)\cdot 10^{-10}$$
$$= (53.1 \pm 2.5)\cdot 10^{-10}$$

$$a_\mu^s(phys) = (53.41 \pm 0.59)\cdot 10^{-10} \quad [\text{HPQCD '14}, N_f = 2+1+1]$$
$$= \left(53.1 \pm 0.9^{+0.1}_{-0.3}\right)\cdot 10^{-10} \quad [\text{RBC/UKQCD '16}, N_f = 2+1]$$
$$= (51.1 \pm 1.7 \pm 0.4)\cdot 10^{-10} \quad [\text{CLS/Mainz '17}, N_f = 2]$$

charm contribution:

$$a_\mu^c(phys) = \left(14.75 \pm 0.42_{stat+fit} \pm 0.36_{input} \pm 0.10_{a^2} \pm 0.03_{FSE} \pm 0.01_{chir}\right)\cdot 10^{-10}$$
$$= (14.75 \pm 0.56)\cdot 10^{-10}$$

$$a_\mu^c(phys) = (14.42 \pm 0.39)\cdot 10^{-10} \quad [\text{HPQCD '14}, N_f = 2+1+1]$$
$$= (14.3 \pm 0.2 \pm 0.1)\cdot 10^{-10} \quad [\text{CLS/Mainz '17}, N_f = 2]$$

leading-order e.m. corrections

For each quark flavor f one has :

$$\delta V(t) = \delta V^{self}(t) + \delta V^{exch}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t)$$

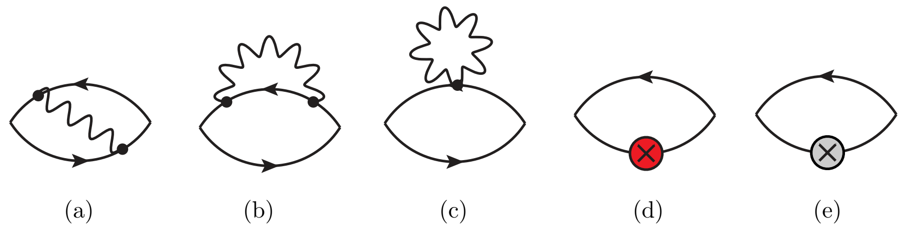


FIG. 8: *Fermionic connected diagrams contributing to the e.m. corrections to a_μ^{had}: exchange (a), self energy (b), tadpole (c), pseudoscalar (d) and scalar (e) insertions. Solid lines represent quark propagators.*

$$\delta V^{self}(t) + \delta V^{exch}(t) = \frac{4\pi\alpha_{em}}{3} \sum_{i=1,2,3} \sum_{\vec{x},y_1,y_2} \langle 0|T\left\{ J_i^\dagger(\vec{x},t) \sum_\mu J_\mu^C(y_1) J_\mu^C(y_2)\ J_i(0) \right\}|0\rangle ,$$

$$\delta V^{tad}(t) = \frac{4\pi\alpha_{em}}{3} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0|T\left\{ J_i^\dagger(\vec{x},t) \sum_\nu T_\nu(y)\ J_i(0) \right\}|0\rangle ,$$

$$\delta V^{PS}(t) = \frac{2\delta m_f^{crit}}{3} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0|T\left\{ J_i^\dagger(\vec{x},t)\ i\overline{\psi}_f(y)\gamma_5\psi_f(y)\ J_i(0) \right\}|0\rangle ,$$

$$\delta V^{S}(t) = -\frac{2m_f}{3Z_m\mathcal{Z}_f} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0|T\left\{ J_i^\dagger(\vec{x},t)\ \overline{\psi}_f(y)\psi_f(y)\ J_i(0) \right\}|0\rangle ,$$

$$q_f^{sea} = 0$$

quenched QED

lattice conserved current

$$J_\mu^C(x) = q_f\ \frac{1}{2}\left[\bar{\psi}_f(x)(\gamma_\mu - i\tau^3\gamma_5)U_\mu(x)\psi_f(x+a\hat{\mu}) + \bar{\psi}_f(x+a\hat{\mu})(\gamma_\mu + i\tau^3\gamma_5)U_\mu^\dagger(x)\psi_f(x)\right] .$$

$\delta m_f^{crit} \propto \alpha_{em} q_f^2$ = e.m. shift of the critical mass (breaking of chiral symmetry)

$\frac{1}{Z_m} = Z_P\left(\overline{MS},\mu\right) \rightarrow$ mass RC (maximally twisted LQCD)

tadpole operator

$$T_\nu(y) = q_f^2\ \frac{1}{2}\left[\bar{\psi}_f(y)(\gamma_\nu - i\tau^3\gamma_5)U_\nu(y)\psi_f(y+a\hat{\nu}) - \bar{\psi}_f(y+a\hat{\nu})(\gamma_\nu + i\tau^3\gamma_5)U_\nu^\dagger(y)\psi_f(y)\right] .$$

$\frac{1}{Z_f}\left(\overline{MS},\mu\right) = \frac{\alpha_{em} q_f^2}{4\pi}\left[6\log(a\mu) - 22.596\right] \rightarrow$ mass RC (LO in QED)

* **separation of QCD and QED effects is prescription dependent** **[see Gasser et al. EPJC '03]**

- mass anomalous dimensions in QCD and QCD+QED are different
- matching possible only at a given renormalization scale μ^* [our choice: $\mu^* = 2$ GeV]

$$\hat{m}_f\left(\overline{MS},\mu^*\right) = m_f\left(\overline{MS},\mu^*\right)$$

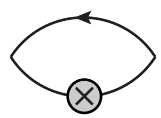
renormalized mass in QCD+QED ← → renormalized mass in QCD only

- bare mass difference: $\hat{\mu}_f - \mu_f = \frac{\hat{m}_f}{\hat{Z}_{m,f}} - \frac{m_f}{Z_m}$ Z_m = mass RC in QCD

$$\frac{1}{\hat{Z}_{m,f}} = \frac{1}{Z_m}\left(1 + \frac{1}{Z_f}\right) \Rightarrow \hat{\mu}_f - \mu_f \simeq \frac{1}{Z_m}\left(\hat{m}_f - m_f\right) + \frac{1}{Z_m Z_f} m_f$$

$\frac{1}{Z_f}$ → $O(\alpha_{em})$

$$\left(\hat{\mu}_f - \mu_f\right) \times \left(\text{insertion of scalar density}\right) \xrightarrow[f=s,c]{\mu=\mu^*} \frac{m_f}{Z_m Z_f^{em} Z_m^{fact}}$$



for details see PRD95 ('17) 114504 (arXiv:1704.0656)

LO in QED and $O(\alpha_s^0)$ in QCD: $\frac{1}{Z_f^{em}}\left(\overline{MS},\mu^*\right) = \frac{\alpha_{em} q_f^2}{4\pi}\left[6\log\left(a\mu^*\right) - 22.596\right]$ **[Martinelli&Zhang '82, Aoki et al. '98]**

Z_m^{fact} is $O\left(\alpha_{em}\alpha_s\right)$ $Z_m^{fact} = 1$ ⟶ "factorization approximation" between QED and QCD vertex corrections

* e.m. corrections to the renormalization of the (local) e.m. current:

we have adopted a maximally twisted-mass setup with quarks and anti-quarks regularized with opposite values of the Wilson r-parameter: the vector current renormalizes multiplicatively with Z_A

$$Z_A = Z_A^{(0)} + \alpha_{em} Z_A^{(1)} + O\left(\alpha_{em}^2\right) = Z_A^{(0)} \left(1 - 2.51406\, \alpha_{em} q_f^2 Z_A^{fact}\right) + O\left(\alpha_{em}^2\right)$$

perturbative estimate at LO

[Martinelli&Zhang PLB '82]

$Z_A^{fact} = 0.9 \pm 0.1$ ⟶ correction to the "factorization approximation" between QED and QCD vertex corrections based on WI

doublet of mass- and charge-degenerate TM quarks $\Rightarrow \partial_\mu A_\mu^{1p-split}(x) = 2mP_5(x)$

$$\begin{cases} Z_V\, \partial_\mu A_\mu^{TM,local}(x) = 2mP_5^{TM}(x) + O\left(a^2\right) \\ Z_V \langle 0 | A_0^{TM,local} | PS \rangle = Z_A \langle 0 | A_0^{OS,local} | PS \rangle + O\left(a^2\right) \end{cases}$$

β	$Z_V^{(fact)}$	$Z_A^{(fact)}$
1.90	1.027 (5)	0.85 (5)
1.95	1.033 (4)	0.93 (5)
2.10	1.034 (3)	0.87 (6)

⟶ $Z_V^{(fact)} = 1.03 \pm 0.01\,, \quad Z_A^{(fact)} = 0.9 \pm 0.1$

relevant uncertainty

* addition of a further contribution: $\delta V(t) = \delta V^{self}(t) + \delta V^{exch}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t) + \delta V^{Z_A}(t)$

$$\delta V^{Z_A}(t) = -2.51406\, \alpha_{em} q_f^2\, Z_A^{fact}\, V(t)$$

master formula

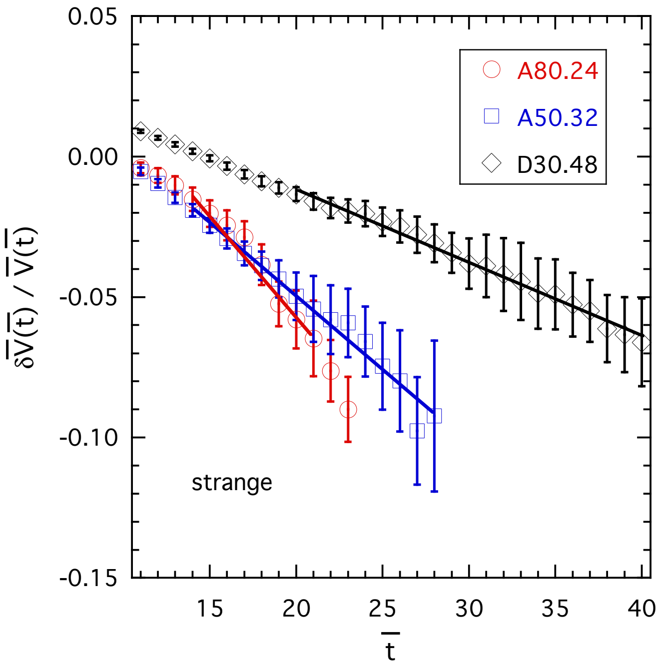
$$\delta a_\mu^{HVP} = \delta a_\mu^{HVP}(<) + \delta a_\mu^{HVP}(>)$$

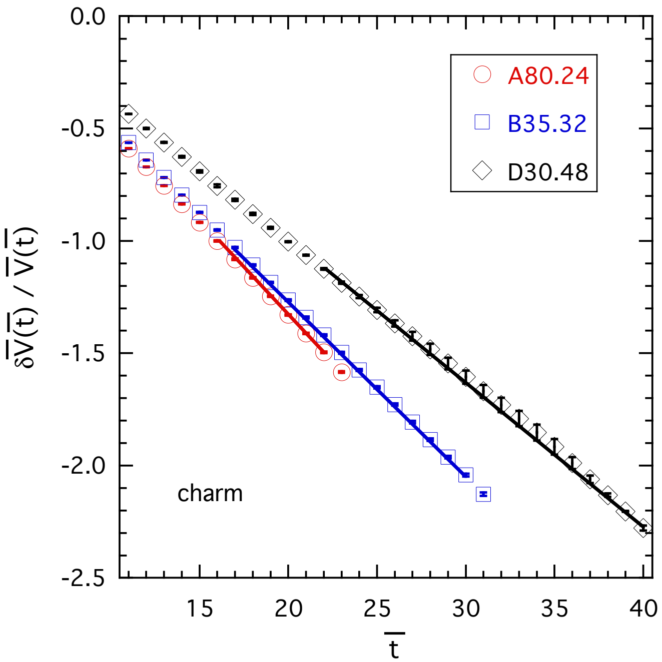
$$\delta a_\mu^{HVP}(<) = 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=0}^{N_{data}} \bar{f}(\bar{t})\, \delta\bar{V}(\bar{t})$$ → directly from lattice data

$$\delta a_\mu^{HVP}(>) = 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=N_{data}+1}^{\infty} \bar{f}(\bar{t})\, \delta\left[\frac{\bar{G}_V}{2\bar{M}_V} e^{-\bar{M}_V \bar{t}}\right]$$ → analytic representation

$$= 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=N_{data}+1}^{\infty} \bar{f}(\bar{t}) \frac{\bar{G}_V}{2\bar{M}_V}\left[\frac{\delta\bar{G}_V}{\bar{G}_V} - \frac{\delta\bar{M}_V}{\bar{M}_V}(1+\bar{M}_V \bar{t})\right] e^{-\bar{M}_V \bar{t}}$$

stat. errors only





RM123 approach

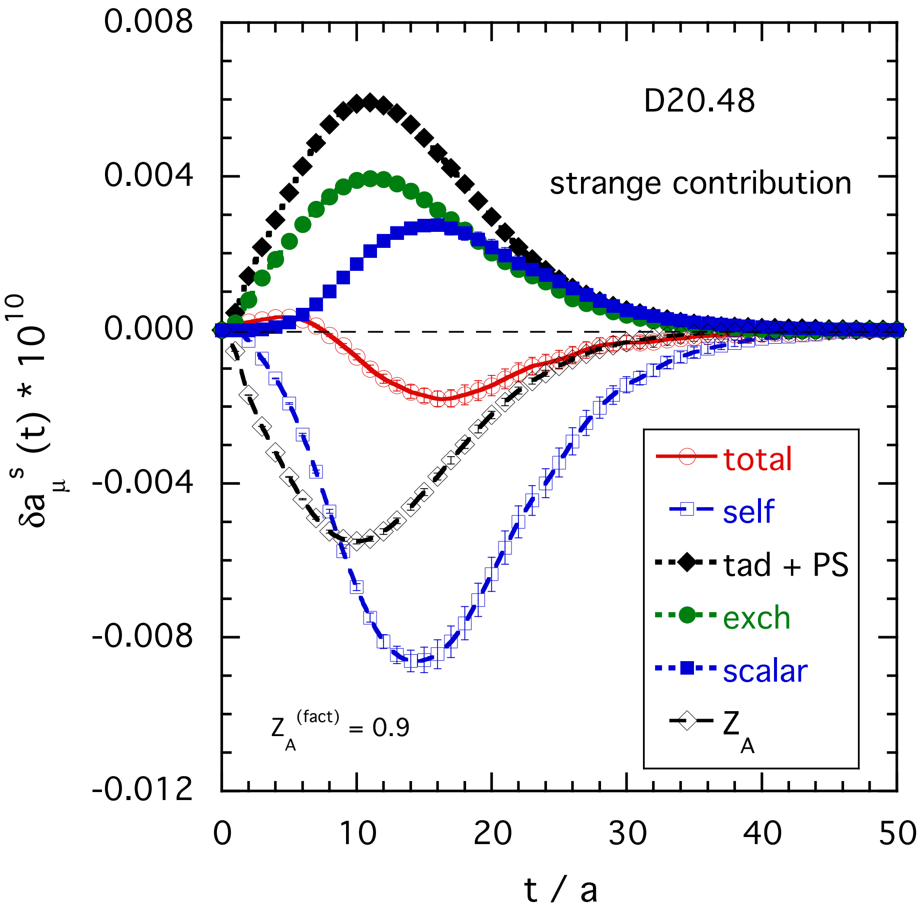
[JHEP '12, PRD '13]

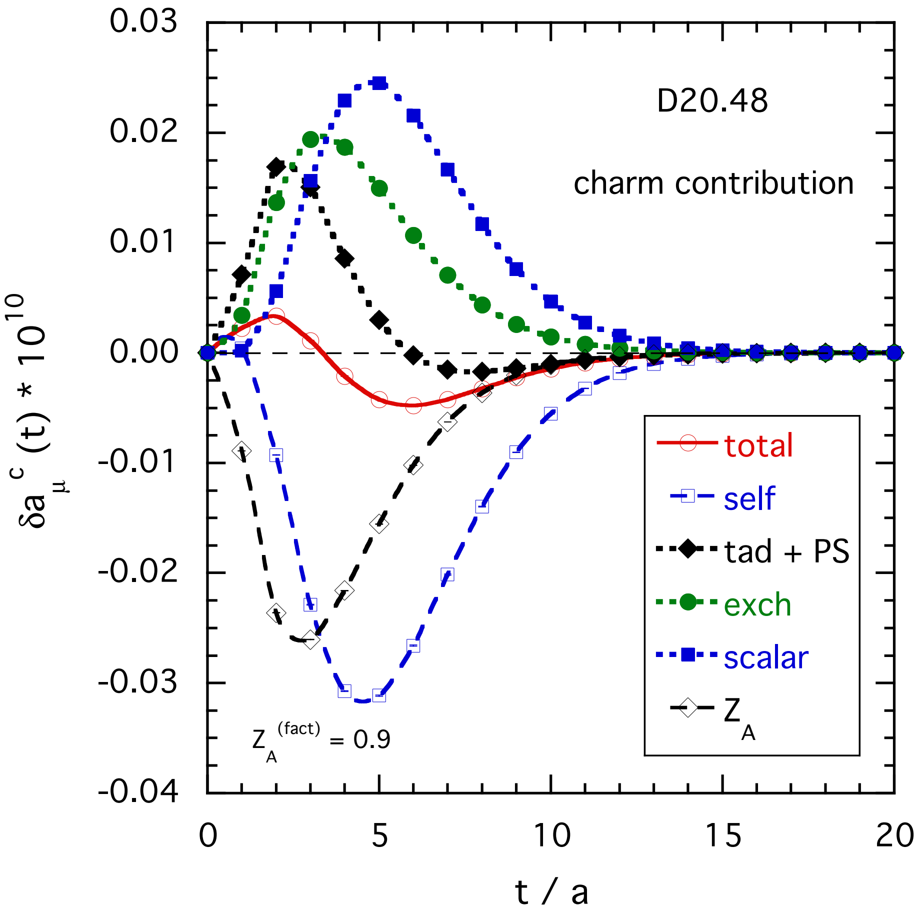
$$\frac{\delta\bar{V}(\bar{t})}{\bar{V}(\bar{t})} \xrightarrow{\bar{t}>>1} \frac{\delta\bar{G}_V}{\bar{G}_V} - \frac{\delta\bar{M}_V}{\bar{M}_V}(1+\bar{M}_V \bar{t})$$

$$\delta a_\mu^{HVP}(<) = 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=0}^{\bar{T}_{data}} \bar{f}(\bar{t})\,\delta\bar{V}(\bar{t})$$ (directly from lattice data)

$$\delta\bar{V}(\bar{t}) = \delta\bar{V}^{self}(\bar{t}) + \delta\bar{V}^{tad}(\bar{t}) + \delta\bar{V}^{PS}(\bar{t}) + \delta\bar{V}^{exch}(\bar{t}) + \delta\bar{V}(\bar{t})^{scalar} + \delta\bar{V}(\bar{t})^{ZA}$$

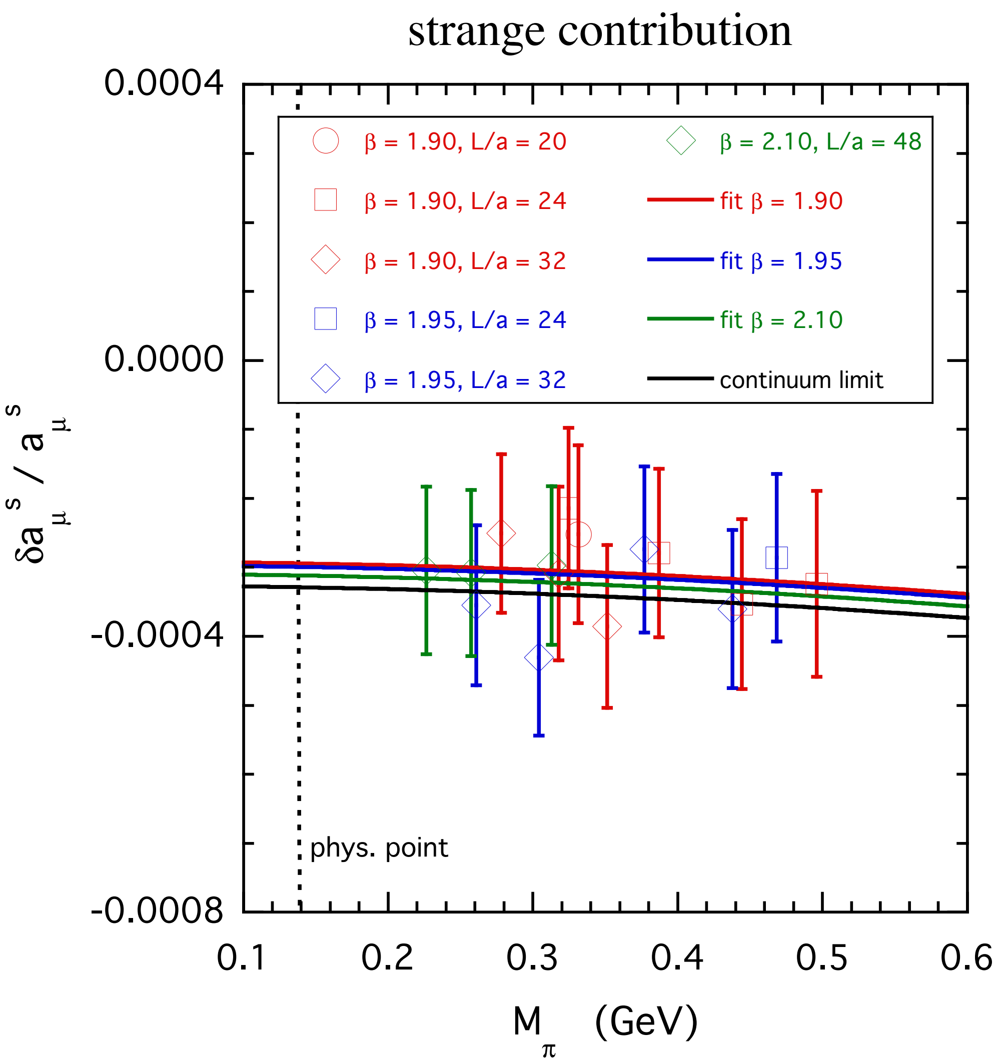
stat. errors only

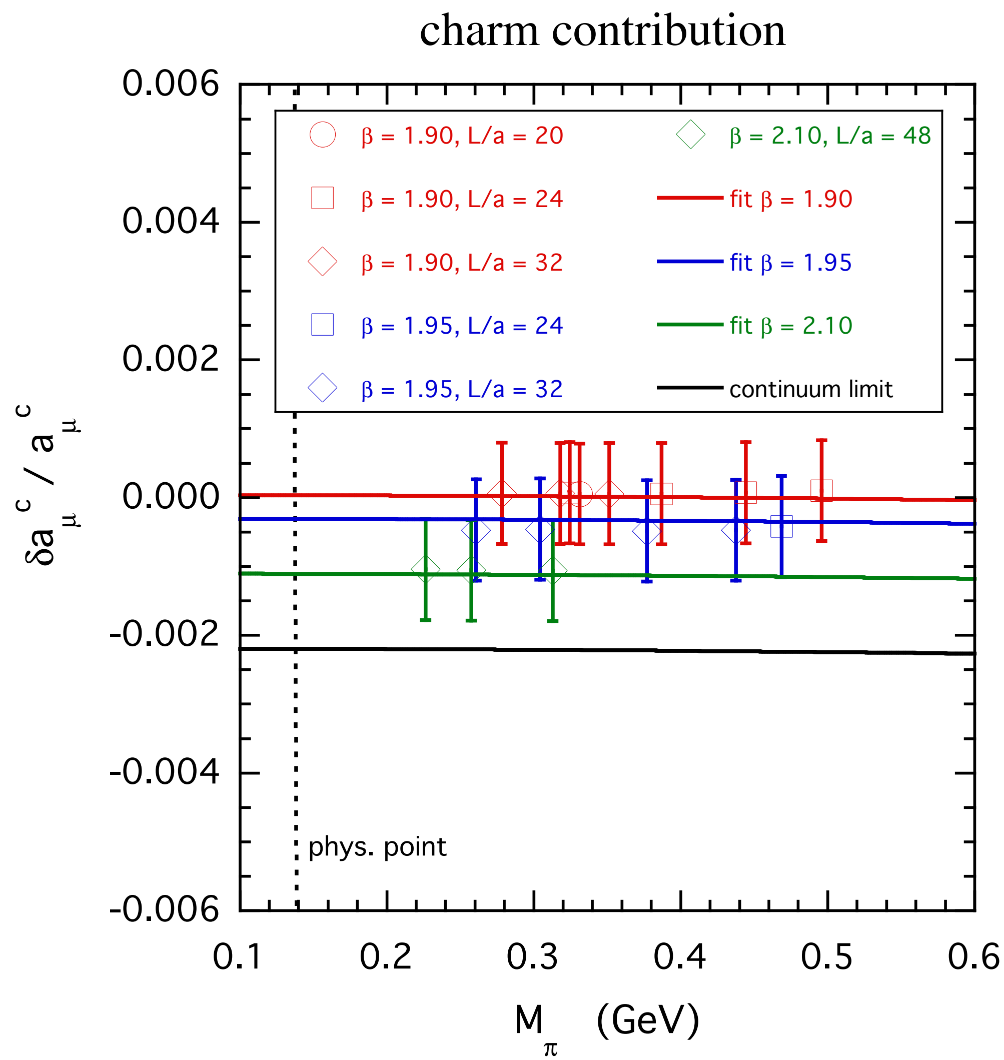




- contributions with different signs → total sum smaller than the separate terms

- partial cancellations among the various terms → large sensitivity to the uncertainty on Z_A





- in the ratios $\delta a_\mu^{HVP} / a_\mu^{HVP}$ various systematics cancel out

- errors dominated by the uncertainty of Z_A^{fact}

- the ELM procedure does not improve the precision

- no FSEs are visible

fitting functions:

$$\delta a_\mu^{s,c} = \delta A_0^{s,c} \left[1 + \delta A_1^{s,c} M_\pi^2 + \delta D^{s,c} \, a^2 \right]$$

need of a precise, non-perturbative determination of Z_A^{fact}

* **results at the physical pion mass and in the continuum limit**:

$$\frac{\delta a_\mu^s}{a_\mu^s} = -0.033\,(2)_{stat+fit}\,(1)_{syst}\,(21)_{Z_A}\ \%$$

$$\frac{\delta a_\mu^c}{a_\mu^c} = -0.205\,(3)_{stat+fit}\,(4)_{syst}\,(86)_{Z_A}\ \%$$

using our lowest order results

$$a_\mu^s = (53.1 \pm 2.5)\cdot 10^{-10}$$

$$a_\mu^c = (14.75 \pm 0.56)\cdot 10^{-10}$$

we have

$$\delta a_\mu^s = -(1.8 \pm 1.1)\cdot 10^{-12}$$

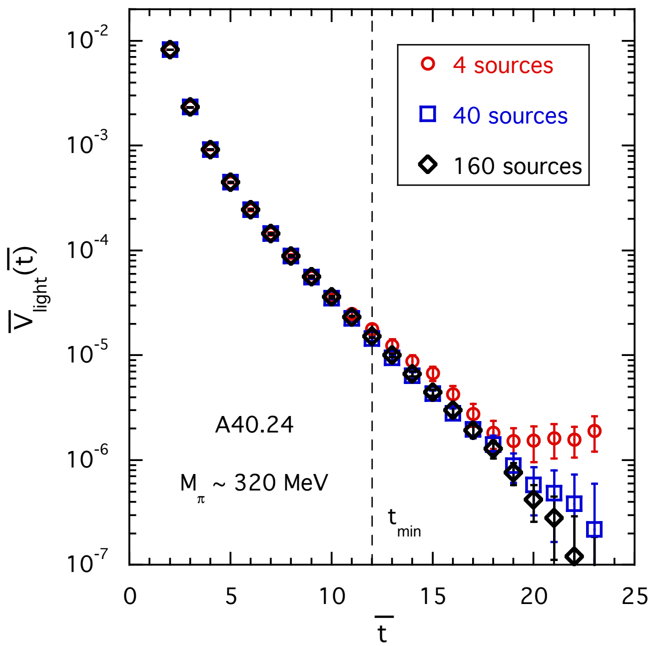
$$\delta a_\mu^c = -(3.0 \pm 1.3)\cdot 10^{-12}$$

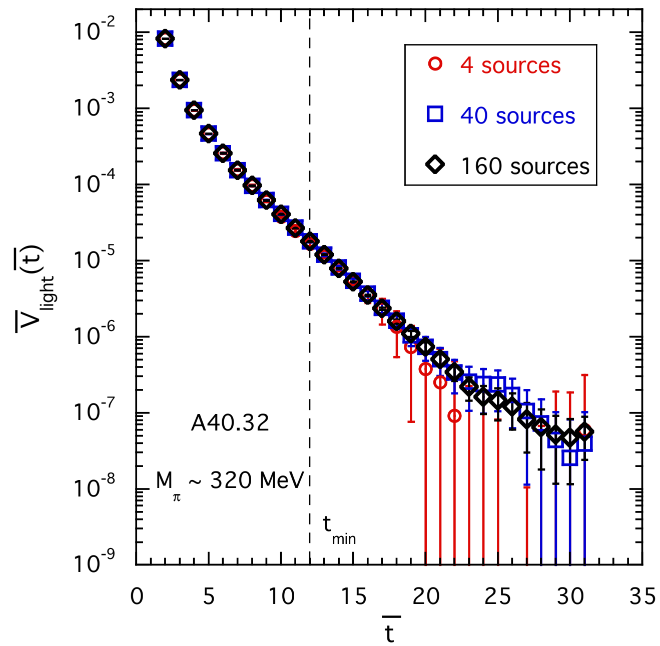
* need of a precise, non-perturbative determination of the e.m. corrections to the RCS of bilinear operators through procedures like RI-MOM

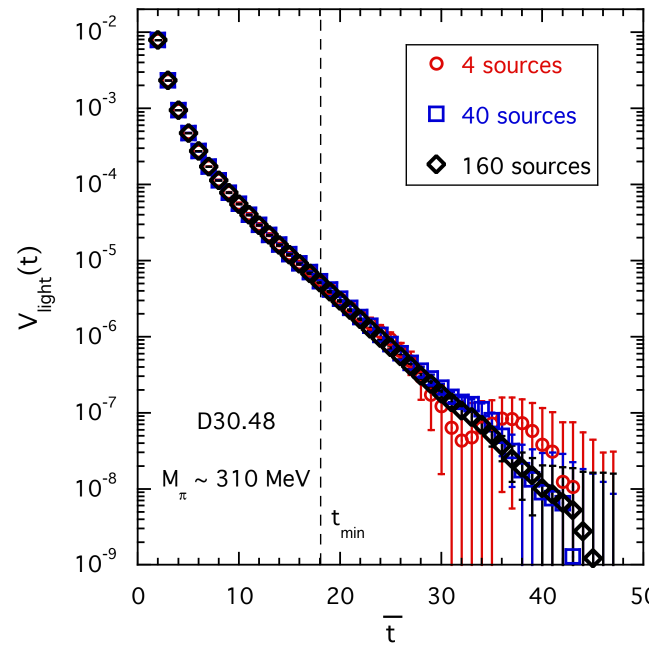
* for the strange and charm contributions: **e.m. corrections << uncertainties of the lowest order**

light (u, d) contribution

* number of stochastic sources / gauge configuration



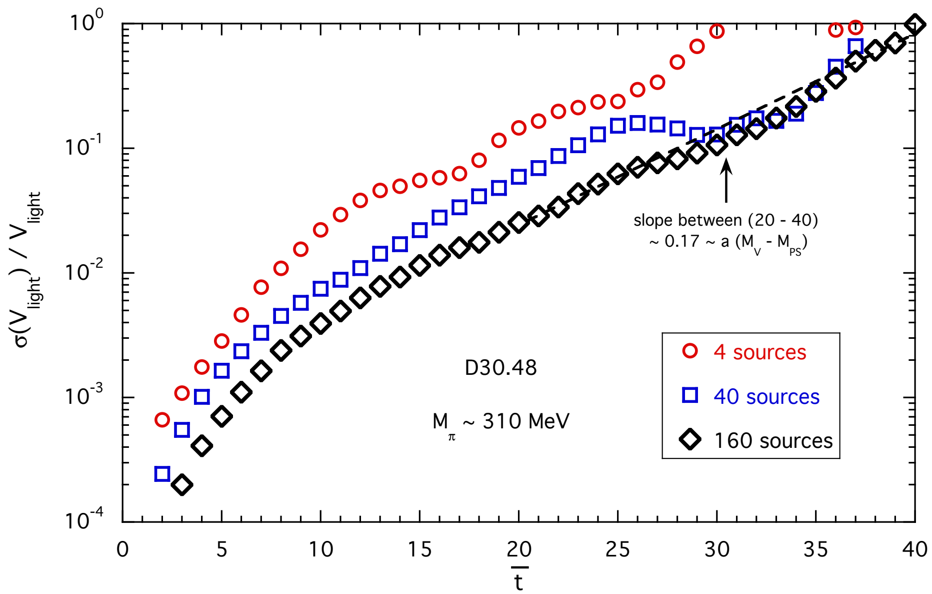




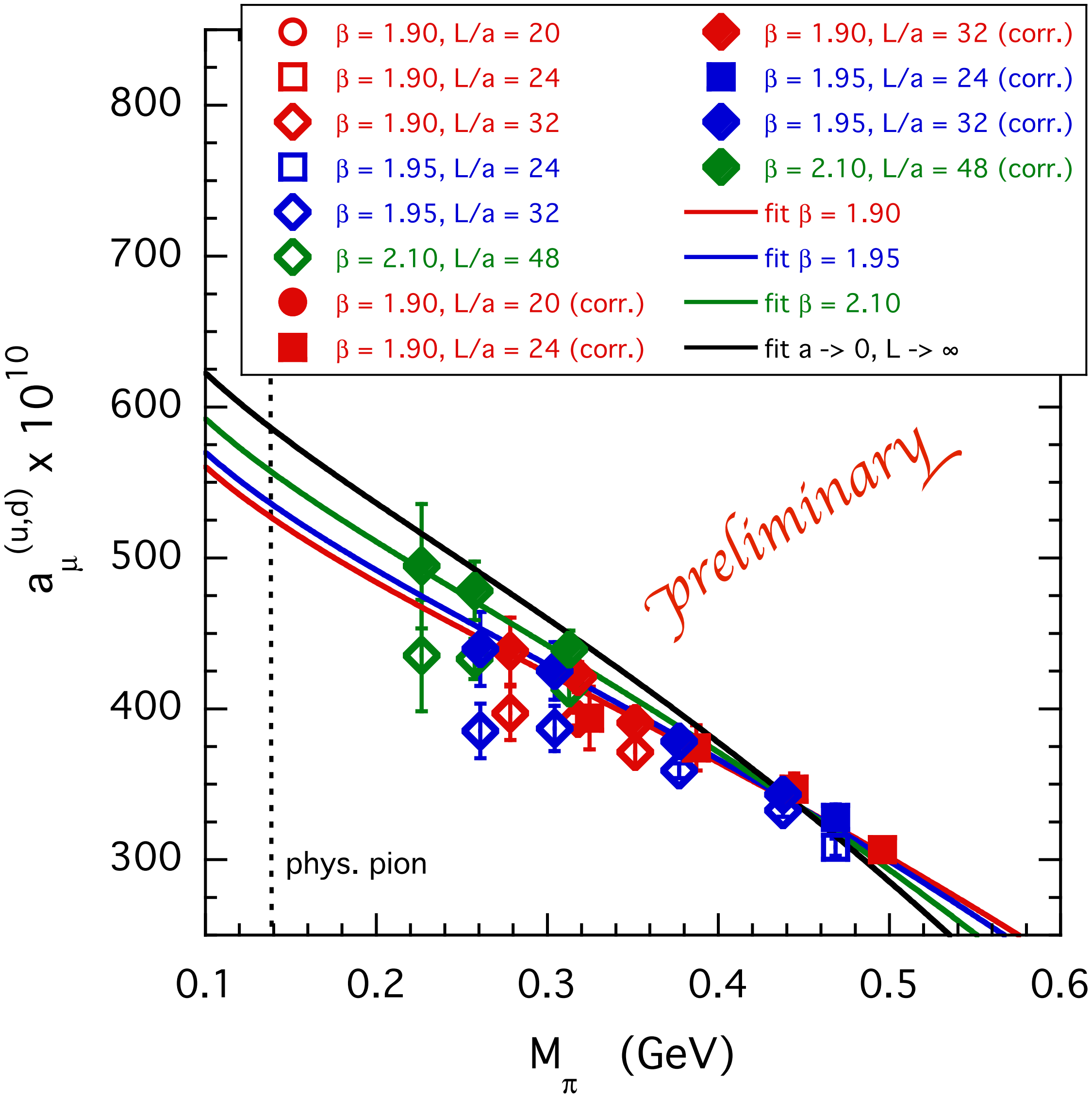
variance: $\propto e^{-2M_\pi t}$

noise / signal: $\propto e^{(M_\rho - M_\pi)t}$

similar to the nucleon case **[Parisi, Lepage]**



- thanks to improvements in the Dirac inverter (DDαAMG versus CG) we are increasing the number of sources per gauge configuration for all the ETMC ensembles



empty markers: lattice data

filled markers: data corrected by the FSE of the fitting procedure

fitting Ansätz

leading chiral log **[Golterman et al '17]**

$$a_\mu^{(u,d)} = -\frac{\alpha_{em}^2}{12\pi^2}\log\left(\frac{M_\pi^2}{m_\mu^2}\right) + A\, M_\pi^2 + a^2\left(D + D_m M_\pi^2\right)$$
$$+ F\left[a_\mu^{(2\pi)}(L) - a_\mu^{(2\pi)}(\infty)\right]$$

FSEs inspired from non-interacting pions **[Francis et al '13]**

A, D, D_m, F : free parameters

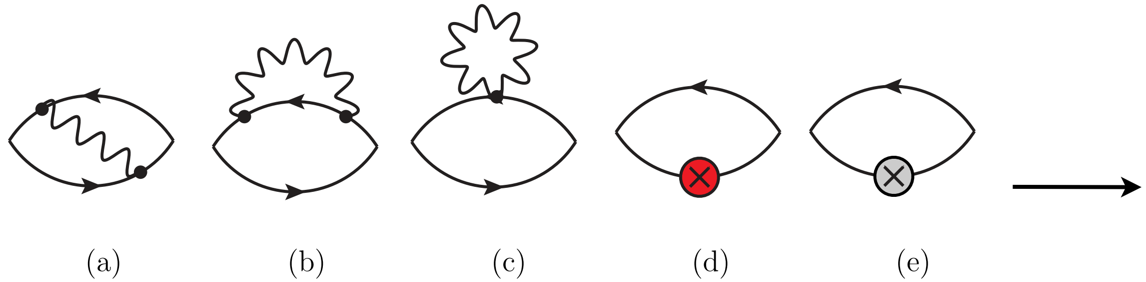
* impact of finite size + a² effects ~ 15 % *

other lattice results

$$a_\mu^{(u,d)}(phys) = (572 \pm 11)\cdot 10^{-10} \quad [\text{ETMC '14}]$$
$$= (598 \pm 11)\cdot 10^{-10} \quad [\text{HPQCD '16}]$$
$$= (588 \pm 36)\cdot 10^{-10} \quad [\text{CLS/Mainz '17}]$$

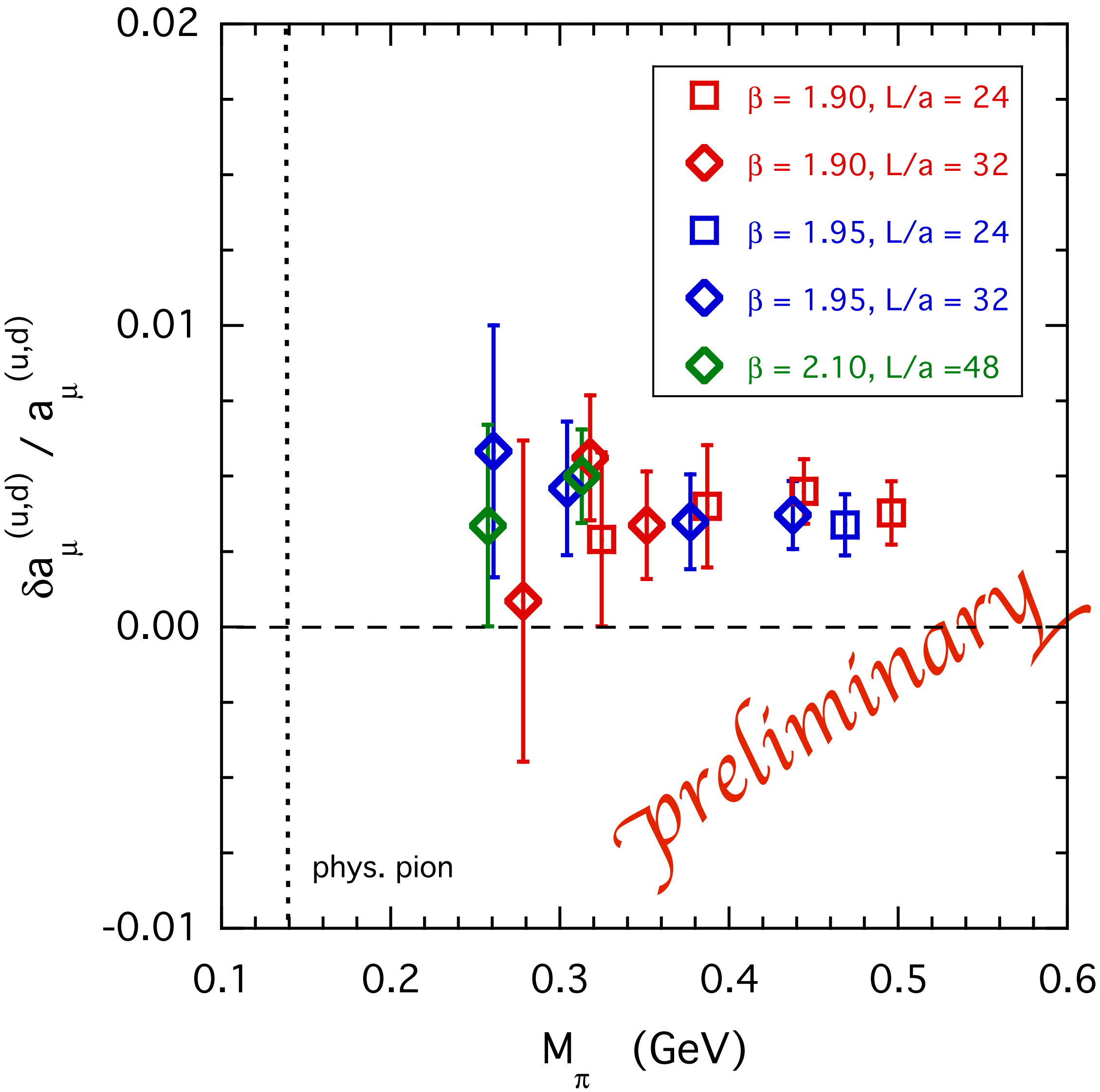
$$a_\mu^{(u,d)}(phys) = \left(589 \pm 21_{stat+fit}\right)\cdot 10^{-10}$$

*** light (u, d) contribution:**



also a contribution proportional to $(m_d - m_u)$

Figure 1: *Fermionic connected diagrams contributing at $O(e^2)$ and $O(m_d - m_u)$ to the IB corrections to meson masses: exchange (a), self energy (b), tadpole (c), pseudoscalar insertion (d) and scalar insertion (e).*



$$\frac{\delta a_\mu^{(u,d)}}{a_\mu^{(u,d)}}(phys) \sim 0.5 \pm 0.5\ \%$$

$$\delta a_\mu^{(u,d)}(phys) \sim (3 \pm 3) \cdot 10^{-10}$$

* more statistics is needed *

CONCLUSIONS

* the HVP contribution, a_μ^{HVP}, is presently one of the major sources of the theoretical uncertainty to the muon (g-2)

* in the past few years several lattice results have been obtained and many more are expected in the next future at the lowest order $O(\alpha_{em}^2)$

* lattice results are coming up for the e.m. and strong isospin-breaking corrections to a_μ^{HVP} at order $O(\alpha_{em}^3)$

* we have adopted the RM123 approach, which is based on a double expansion in the "small" parameters α_{em} and $(m_d - m_u)$ and has been already applied successfully to the calculation of the charged **meson masses** and to the **leptonic decay rates** of pseudoscalar mesons

* using the time-momentum representation of the HVP form factor both the lowest-order $a_\mu(\alpha_{em}^2)$ and the e.m. and strong isospin-breaking corrections $a_\mu(\alpha_{em}^3)$ have been determined **at the physical point** in the case of the strange and charm contributions:

$$a_\mu^s\left(\alpha_{em}^2\right) = (53.1 \pm 2.5) \cdot 10^{-10} \qquad a_\mu^c\left(\alpha_{em}^2\right) = (14.75 \pm 0.56) \cdot 10^{-10}$$

$$a_\mu^s\left(\alpha_{em}^3\right) = -(1.8 \pm 1.1) \cdot 10^{-12} \qquad a_\mu^c\left(\alpha_{em}^3\right) = -(3.0 \pm 1.3) \cdot 10^{-12}$$

* in the case of u- and d-quarks more statistics is required and we have reached till now **preliminary results**:

$$a_\mu^{(u,d)}(\alpha_{em}^2) = (589 \pm 21_{stat}) \cdot 10^{-10} \qquad a_\mu^{(u,d)}(\alpha_{em}^3) = (3 \pm 3) \cdot 10^{-10}$$

open issues

* **non-perturbative determinations of the e.m. corrections to the RCs of bilinear operators**

* **removal of the quenched QED approximation** (effects of the sea-quark electric charges)

* **non-diagonal flavor contributions to a_μ^{HVP}**

$$J_\mu(x) J_\nu(y) \subset q_f \bar{\Psi}_f(x) \gamma_\mu \Psi_f(x) \, q_{f'} \bar{\Psi}_{f'}(y) \gamma_\nu \Psi_{f'}(y)$$

- preliminary lattice estimates **[HPQCD, RBC/UKQCD, CLS/Mainz]** are in the range - (1 – 2 %)

↓

evaluation of fermionic disconnected diagrams

work is in progress ...

BACKUP SLIDES

motivations

* the muon magnetic anomaly $a_\mu = (g - 2) / 2$ is measured to 0.5 ppm **[Muon G-2 Coll. ’06]**

* within the SM a_μ is known to 0.4 ppm **[PDG ’16]**

* tension with SM prediction at ~ **3.5 standard deviations**:

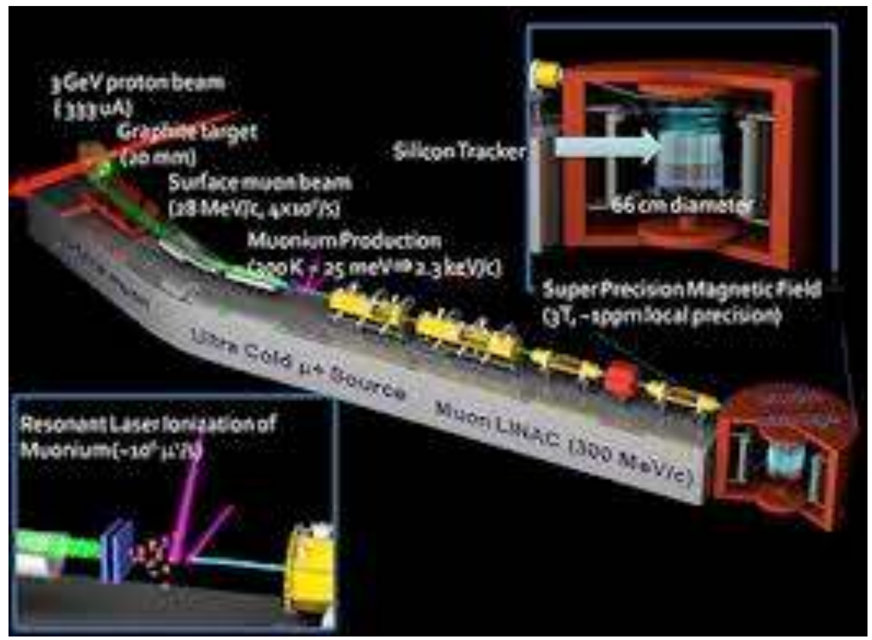
$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = (27.6 \pm 8.0) \cdot 10^{-10} \quad \text{[Muon G-2 Coll. '06]}$$

$$= (28.8 \pm 8.0) \cdot 10^{-10} \quad \text{[PDG '16]}$$

* future experiments at FermiLab [E989] and J-PARC (E34) aim at a target precision of ~ 2 10^{-10}

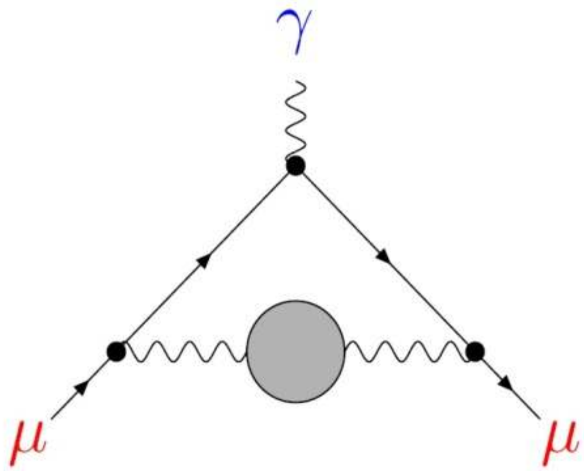
storage ring (FNAL)

ultra cold muon beam (JPARC)



* largest uncertainties from Hadronic Vacuum Polarization (HVP) and Hadronic Light by Light (HLbL)

HVP:

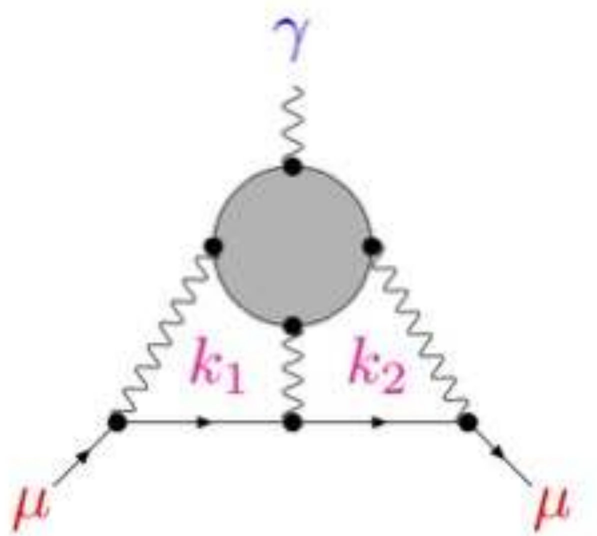


dispersion theory combined with data on $e^+e^- \to$ hadrons

$$a_\mu^{\mathrm{HVP}}\left(\alpha_{em}^2\right) = \left(692.3 \pm 4.2\right)\cdot 10^{-10} \quad \text{[Davier et al. '11]}$$

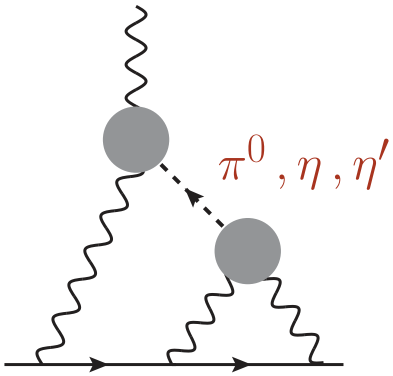
$$= \left(694.9 \pm 4.3\right)\cdot 10^{-10} \quad \text{[Hagiwara et al. '11]}$$

HLbL:



$$a_\mu^{\mathrm{HLbL}} = \left(10.5 \pm 2.6\right)\cdot 10^{-10} \quad \text{[Hagiwara et al. '11, PDG '16]}$$

important processes like



* dispersion formalism much more involved for HLbL

$a_\mu^{HVP}\left(\alpha_{em}^3\right)$ [Jegerlehner&Nyffeler '09]

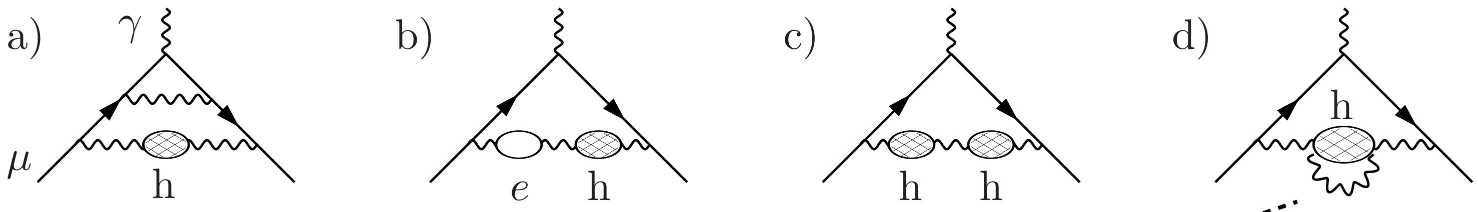
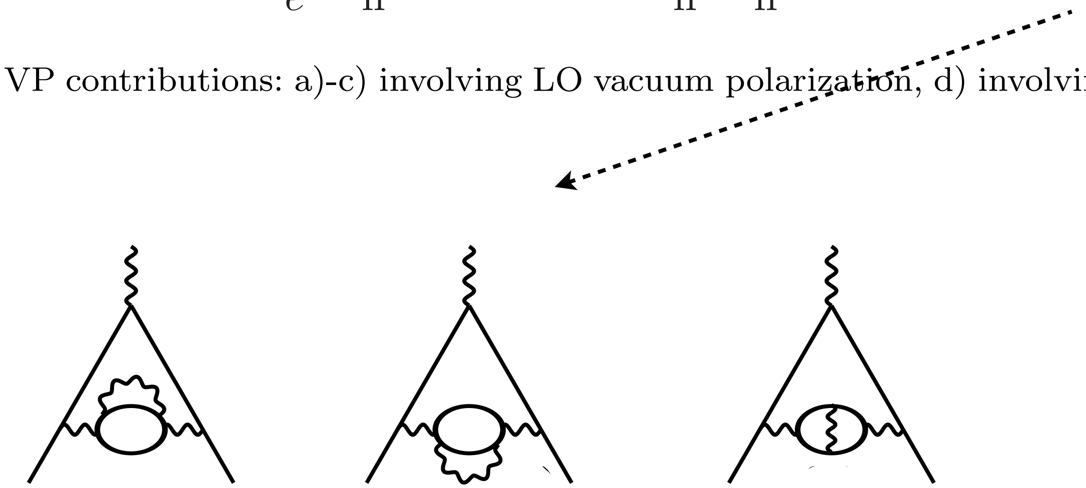


Fig. 29. Hadronic higher order VP contributions: a)-c) involving LO vacuum polarization, d) involving HO vacuum polarization (FSR of hadrons).



usually not included in $a_\mu^{HVP}\left(\alpha_{em}^3\right)$, but taken into account in the experimental determination of $a_\mu^{HVP}\left(\alpha_{em}^2\right)$

[mainly $e^+e^- \to \pi^+\pi^-\gamma$]

NOTE:

$$\left.\begin{aligned} a_\mu^{\mathrm{HVP}}\left(\alpha_{em}^3\right)^{a+b+c} &= \left(-9.84 \pm 0.07\right)\cdot 10^{-10} \\ a_\mu^{\mathrm{HLbL}} &= \left(10.5 \pm 2.6\right)\cdot 10^{-10} \end{aligned}\right\} \left(0.7 \pm 2.6\right)\cdot 10^{-10}$$

[Hagiwara et al. '11, PDG '16]

* **hybrid method**: lattice evaluation of the HVP form factor $\Pi(Q^2)$ as FT (periodic momenta)

- low Q^2: parameterization using lattice data (Padé approximants, conformal polynomials, VMD, ...)

- mid Q^2: direct integration of the lattice data in Q^2

ETMC ’14, RBC/UKQCD ’16, HPQCD ’14 and ’16, CLS ’17...

- high Q^2: matching with pQCD

* alternatively, the sine-cardinal method: direct FT at arbitrary Q [exp. suppressed finite-T effects]

RBC/UKQCD ’16

* **time moments** **[HPQCD ’14]**

- HVP form factor $\Pi(Q^2)$ reconstructed from the time behavior of the vector correlator V(t)

$$\Pi(Q^2) - \Pi(0) = \sum_{j=1}^{\infty} \Pi_j \, Q^{2j}$$

$$\Pi_j = (-)^{j+1} \frac{V_{2j+2}}{(2j+2)!}$$

$$V_{2j+2} = a^4 \sum_t t^{2j+2} \, V(t)$$

- few moments (j≤4) and Padé approximants

local e.m. current

at maximal twist: $J^L_\mu(x) = q_f\, Z_V\, \bar{\psi}_f(x)\gamma_\mu \psi_f(x)$ $\qquad Z_V$ = vector RC

thanks to the findings of **Burger et al. [ETMC] JHEP '15**

$$\int d^4x\, e^{iQ\cdot x} \left\langle J^L_\mu(x) J^L_\nu(0) \right\rangle = \Pi_{\mu\nu}(Q) + \delta_{\mu\nu} Z_1 \left(\frac{1}{a^2} - S_6 + \frac{S_5^2}{2} \right) + \delta_{\mu\nu} Z_m m^2 + \delta_{\mu\nu} Z_L Q^2 + \delta_{\mu\nu} Z_T \left(\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu \right) + O\left(a^2\right)$$

$\Pi_{\mu\nu}(Q)$ = transverse polarization tensor

$S_{5(6)}$ = v.e.v. of dim-5(6) terms of Symanzik expansion of twisted-mass action

$Z_{1,m,L,T}$ = non-perturbative mixing coefficients

$$\int d^4x \left(e^{iQ\cdot x} - 1 \right) \left\langle J^L_\mu(x) J^L_\nu(0) \right\rangle = \Pi_{\mu\nu}(Q) + \delta_{\mu\nu} Z_L Q^2 + \delta_{\mu\nu} Z_T \left(\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu \right) + O\left(a^2\right)$$

↓ $\quad \Pi_{\mu\nu}(Q) = \left[\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu \right] \Pi\left(Q^2\right)$

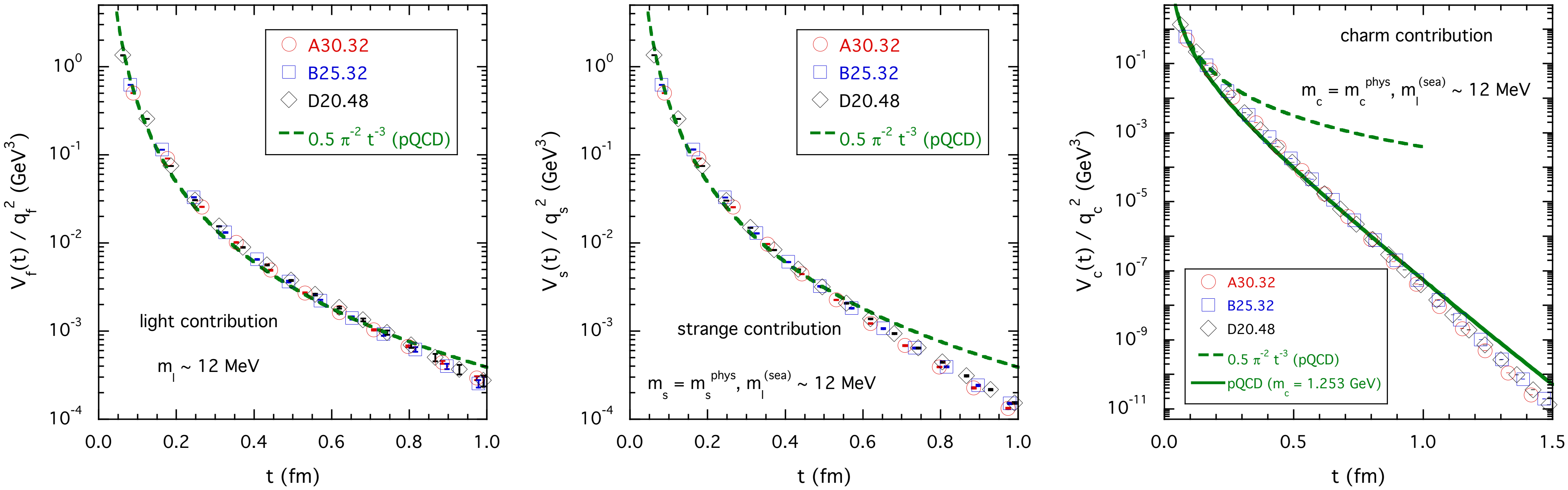
$Q = \{Q, \vec{0}\}$

$$\int dt \left(\frac{\cos(Qt) - 1}{Q^2} \right) \int d\vec{x}\, \frac{1}{3} \sum_{i=1,2,3} \left\langle J^L_i(\vec{x},t) J^L_i(0) \right\rangle = \Pi\left(Q^2\right) + \left(Z_L + Z_T \right) + O\left(a^2\right)$$

↓

$$2\int_0^\infty dt \left(\frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2} t^2 \right) V(t) = \Pi\left(Q^2\right) - \Pi(0) + O\left(a^2\right)$$

matching with pQCD



A30.32, **B25.32**, **D20.48** share a common value of the light-quark mass ($m_l \sim 12$ MeV) and differ in the value of the lattice spacing (a ~ **0.089**, **0.082**, **0.062** fm)

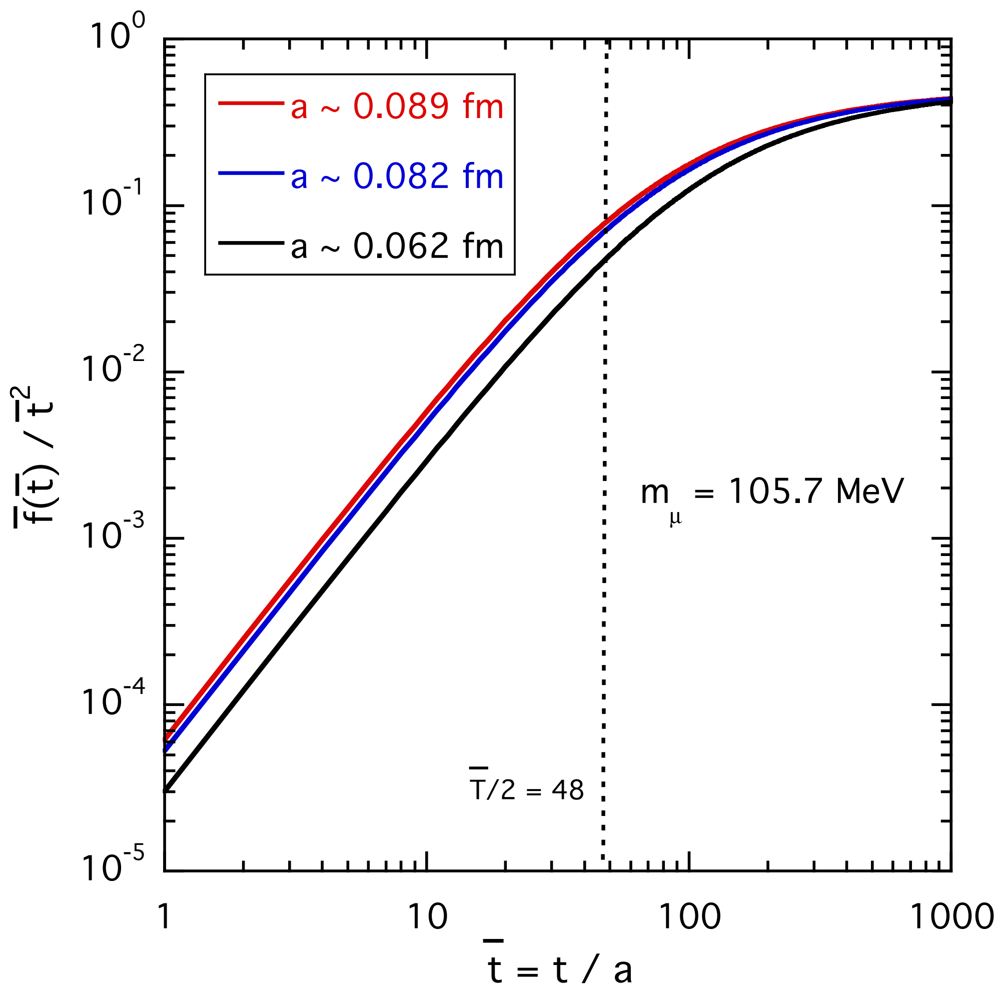
dashed green lines: massless pQCD behavior ($\sim 1 / t^3$)

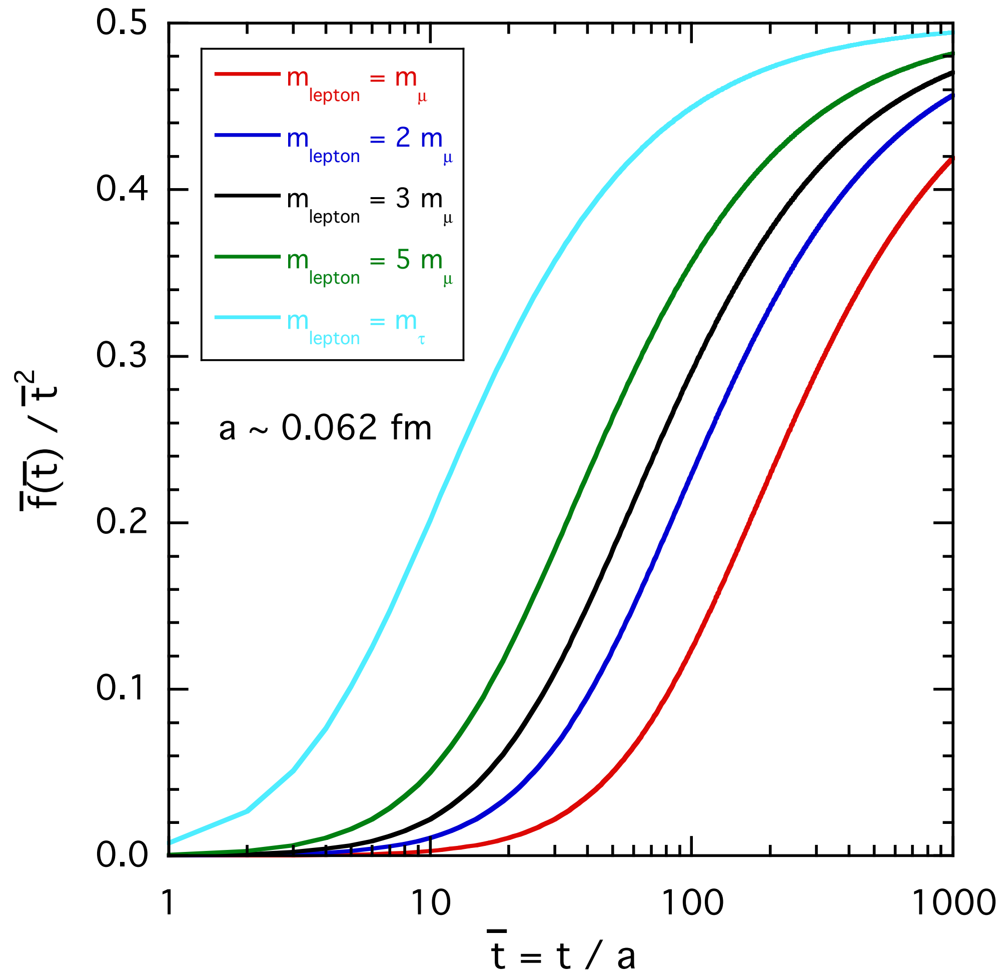
solid green line: massive pQCD behavior for the charm case

* possible issue: contributions from t < a (or equivalently $Q^2 > 1 / a^2$)

thanks to our values of the lattice spacing, which correspond to $Q^2 \geq 5$ GeV^2, the contribution from t < a turns out always to be well within the uncertainties

the kernel function f(t)





$$\frac{\bar{f}(\bar{t})}{\bar{t}^2} = \frac{1}{36} \bar{m}_\mu^2 \bar{t}^2 + O\left(\bar{t}^4\right), \qquad \frac{\bar{f}(\bar{t})}{\bar{t}^2} \xrightarrow[\bar{t} \to \infty]{} \frac{1}{2}$$

* some sensitivity to the lattice spacing

* sensitivity to the mass of the lepton:

enhancement of the large time distances for light leptons

* local vector current with maximally twisted-mass setup

$$V_\mu = Z\,\bar{\psi}_{f'}(\vec{x},t)\,\gamma_\mu \psi_f(\vec{x},t)$$

$m_{f'} = m_f$ same mass

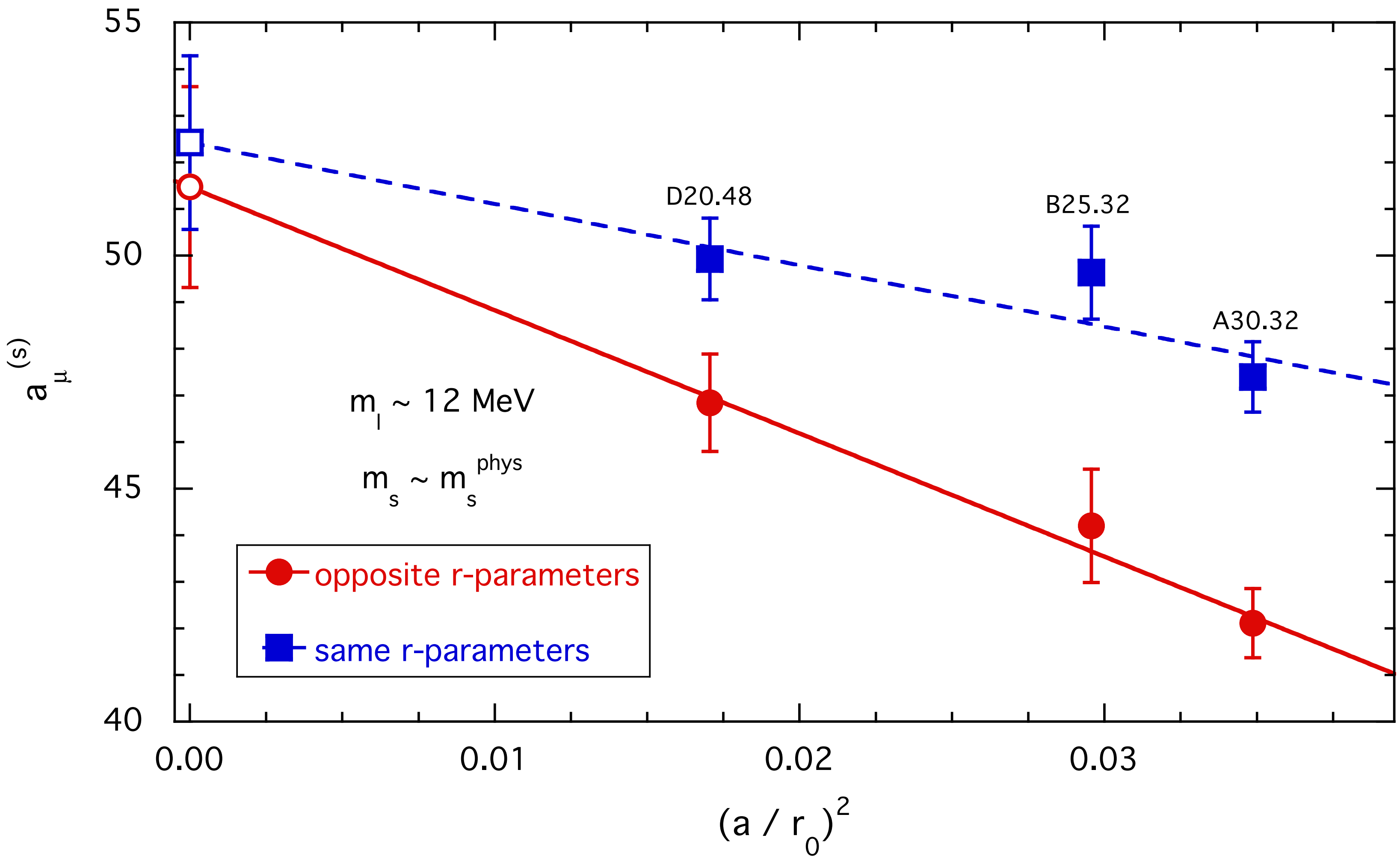
$q_{f'} = q_f$ same electric charge

* two choices of the Wilson r-parameters:

$r_{f'} = r_f \quad Z = Z_V$

$r_{f'} = -r_f \quad Z = Z_A$

* the two currents differ by O(a²) *

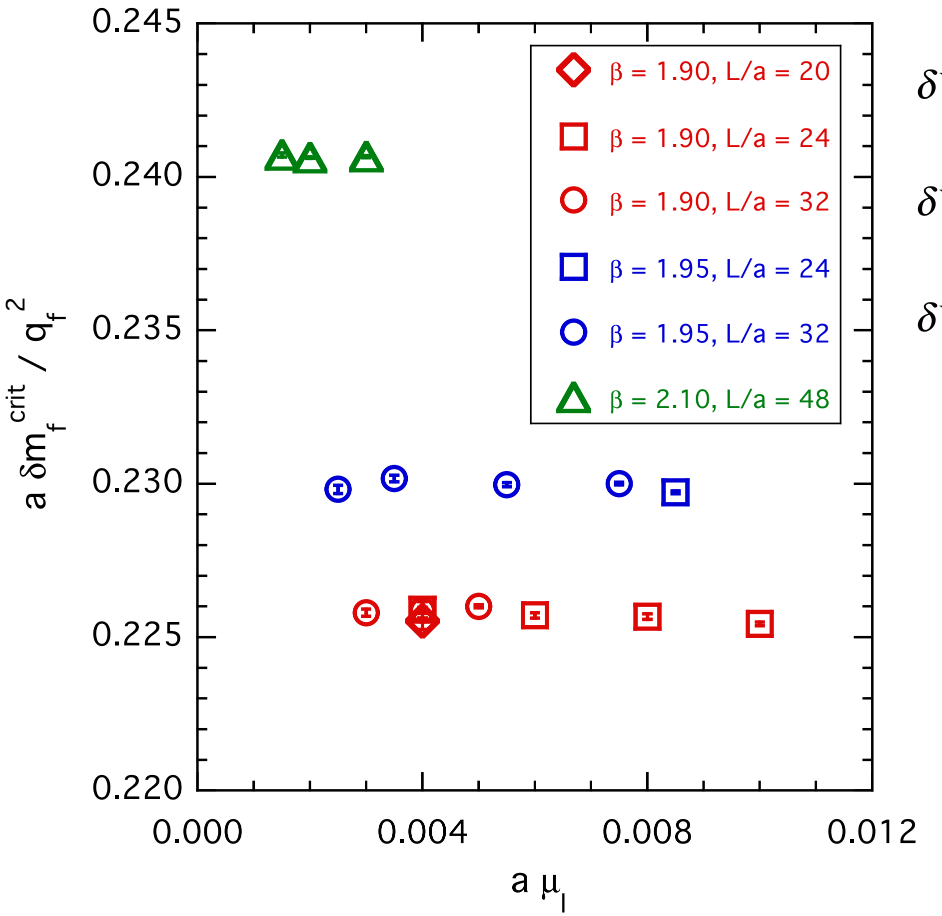


* lattice formulations of QCD which break chiral symmetry ⟹ additive mass renormalization

(twisted) Wilson term ⟹ power-divergent (1/a) mass counterterm ⟹ critical mass m_{crit}

QED contribution: δm_f^{crit} = e.m. shift of the critical mass

vector WT identity: $$\delta m_f^{crit} = -\frac{\nabla_0 \left[\delta V_f^J(t) + \delta V_f^T(t) \right]}{\nabla_0 \, \delta V_f^{P_f}(t)}$$



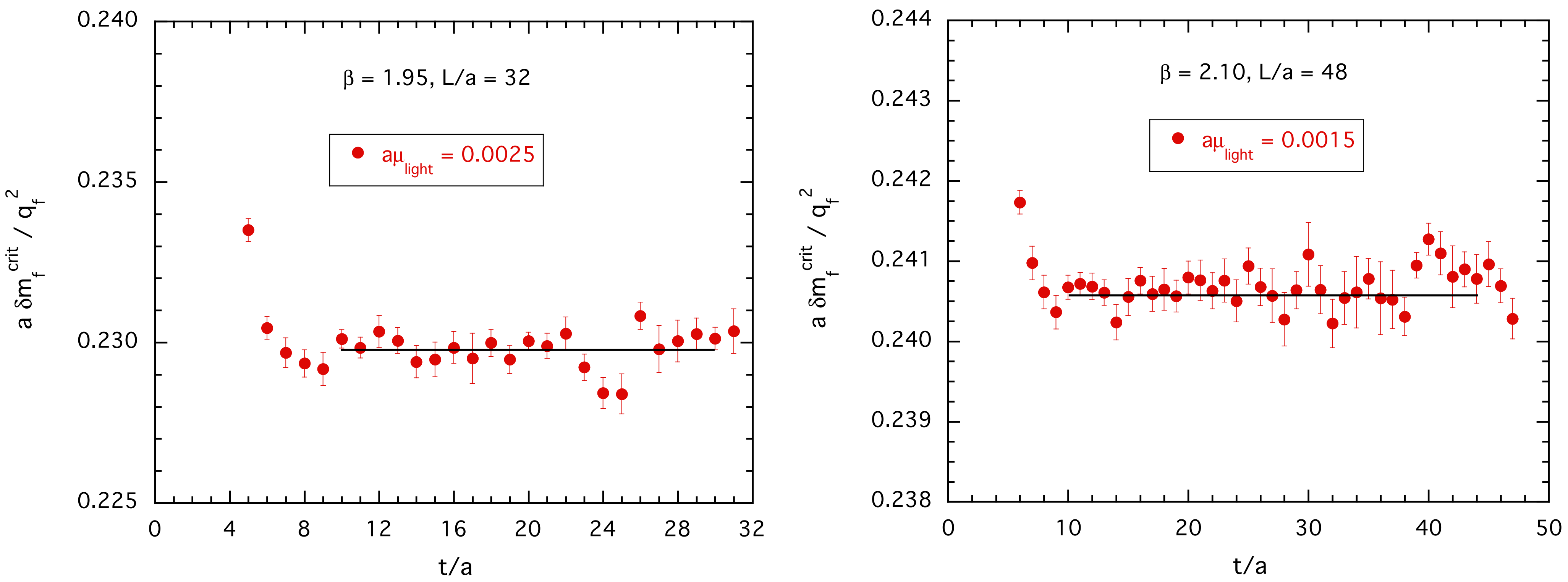
$$\delta V_f^J(t) = \frac{1}{L^6} \sum_{\vec{x}, y_1, y_2} \langle 0 | T\left\{ \bar{\psi}_f(\vec{x},t)\gamma_0\psi_f(\vec{x},t) J_\mu^C(y_1) J_\mu^C(y_2) i\bar{\psi}_f(0)\gamma_5\psi_f(0) \right\} | 0 \rangle$$

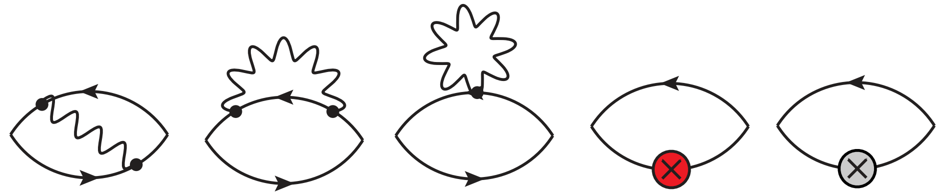
$$\delta V_f^T(t) = \frac{1}{L^3} \sum_{\vec{x}, y} \langle 0 | T\left\{ \bar{\psi}_f(\vec{x},t)\gamma_0\psi_f(\vec{x},t) T(y) i\bar{\psi}_f(0)\gamma_5\psi_f(0) \right\} | 0 \rangle$$

$$\delta V_f^{P_f}(t) = \frac{1}{L^3} \sum_{\vec{x}, y} \langle 0 | T\left\{ \bar{\psi}_f(\vec{x},t)\gamma_0\psi_f(\vec{x},t) i\bar{\psi}_f(y)\gamma_5\psi_f(y) i\bar{\psi}_f(0)\gamma_5\psi_f(0) \right\} | 0 \rangle$$

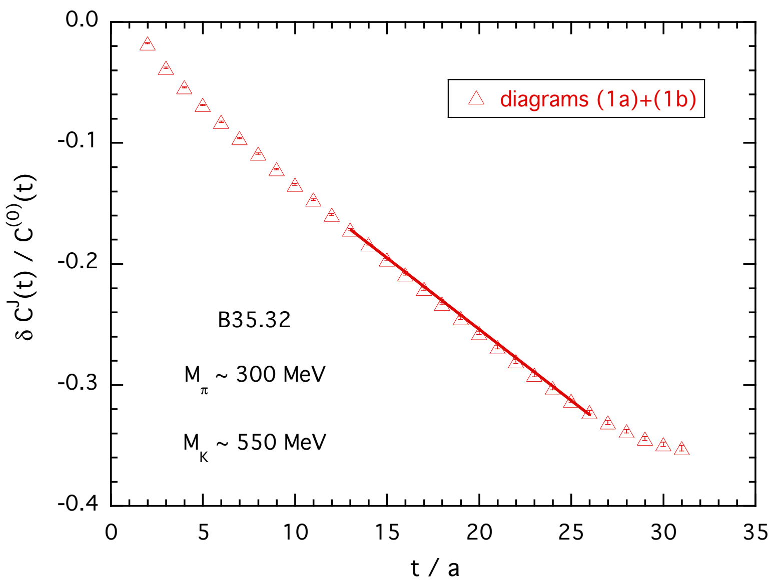
for details see PRD95 (2017) 114504 (arXiv:1704.06561)

- plateaux for δm_f^{crit} :





(a) (b) (c) (d) (e)



diagrams (1a)+(1b)
B35.32
M_π ~ 300 MeV
M_K ~ 550 MeV
δ C^J(t) / C^(0)(t)
t / a

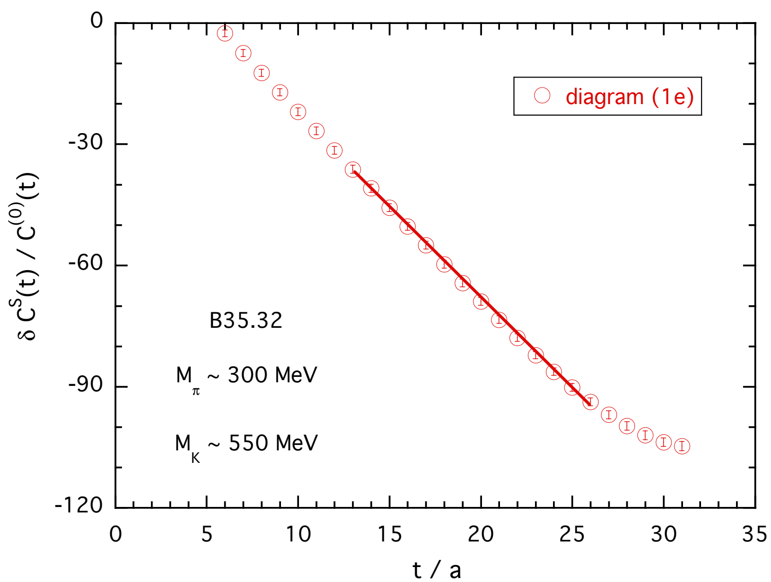
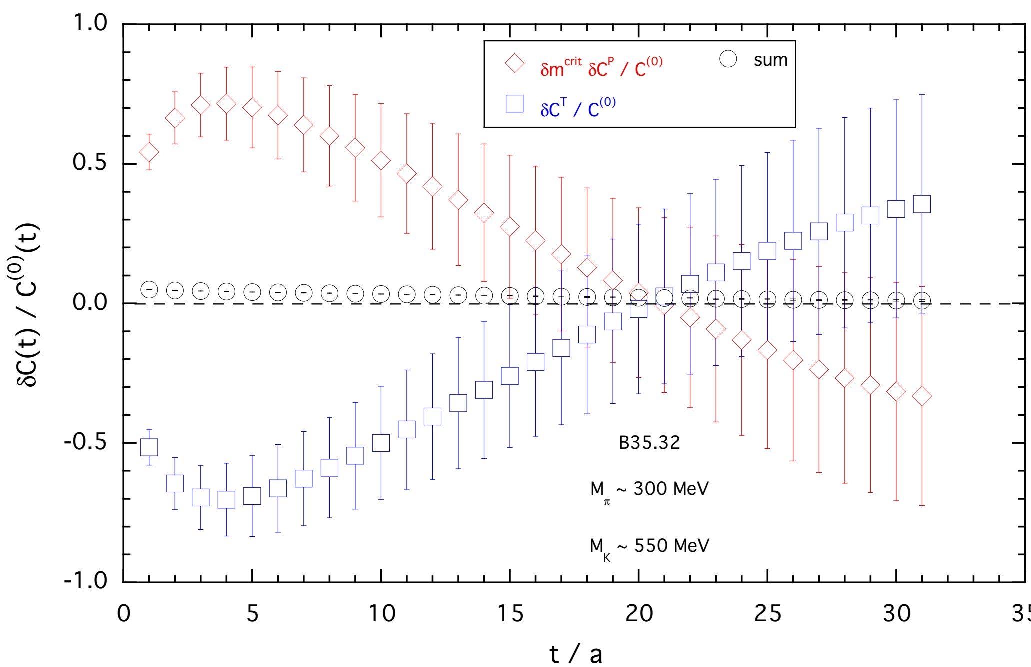


diagram (1e)
B35.32
M_π ~ 300 MeV
M_K ~ 550 MeV
δ C^S(t) / C^(0)(t)
t / a



δm^crit δC^P / C^(0)
sum
δC^T / C^(0)
B35.32
M_π ~ 300 MeV
M_K ~ 550 MeV
δC(t) / C^(0)(t)
t / a

Finite Size Effects

* infinite volume limit of two-pion states [Meyer '11]

$$V_{2\pi}(t)\xrightarrow[L\to\infty]{}\frac{1}{24\pi^2}\int_{4M_\pi^2}^{\infty} ds\,\sqrt{s}\,e^{-\sqrt{s}t}R_{2\pi}(s)=\frac{1}{24\pi^2}\int_{4M_\pi^2}^{\infty} ds\,\sqrt{s}\left(1-\frac{4M_\pi^2}{s}\right)^{\frac{3}{2}}e^{-\sqrt{s}t}\left|F_\pi(s)\right|^2$$

time-like pion form factor

* the case of finite volume [Meyer '11]

$$V_{2\pi}(t;L)=\sum_n\left|A_n\right|^2e^{-\omega_n t} \qquad \omega_n\equiv 2\sqrt{M_\pi^2+k_n^2}$$

- interacting pions [Luscher '91]

$$k_n:\quad \delta_{11}(k_n)+\phi\left(\frac{k_nL}{2\pi}\right)=n\pi$$

δ_{11} = scattering phase shift (p-wave, T=1)
ϕ = known kinematic function

$$\left|A_n\right|^2:\qquad \left|F_\pi(\omega_n^2)\right|^2=\left\{k_n\frac{\partial\delta_{11}(k_n)}{\partial k_n}+\frac{k_nL}{2\pi}\phi'\left(\frac{k_nL}{2\pi}\right)\right\}\frac{3\pi\omega_n^2}{2k_n^2}\left|A_n\right|^2$$

- non-interacting pions [Francis et al '13]: $k_n=2n\pi/L$

$$V_{2\pi}(t;L)-V_{2\pi}(t;\infty)=\frac{M_\pi^4}{3\pi^2}t\sum_{\vec n\neq 0}\left\{\frac{K_2\left[M_\pi\sqrt{L^2\vec n^2+4t^2}\right]}{M_\pi^2\left(L^2\vec n^2+4t^2\right)}-\frac{1}{M_\pi L|\vec n|}\int_1^\infty dy\,K_0\left[M_\pi y\sqrt{L^2\vec n^2+4t^2}\right]\sinh\left[M_\pi L|\vec n|(y-1)\right]\right\}$$

$$a_\mu^{(2\pi)}(L)-a_\mu^{(2\pi)}(\infty)\xrightarrow[K_i(z)\xrightarrow[z>>1]{}\sqrt{\frac{\pi}{2z}}e^{-z}]{}\propto\left(M_\pi L\right)^2e^{-M_\pi L}$$

sizable effects expected at the physical pion for current lattice volumes [CLS/Mainz '17]

strange contribution

stat. errors only — ensemble A40.24 — in units of 10^{-10}

$\bar{s}s$	$(\bar{t}_{min}+2)$	$(\bar{t}_{min}+\bar{t}_{max})/2$	$(\bar{t}_{max}-2)$	$(\bar{T}/2-4)$
$a_\mu^{had}(<)$	38.03 (28)	38.65 (29)	39.10 (29)	39.67 (30)
$a_\mu^{had}(>)$	1.97 (13)	1.41 (10)	1.00 (8)	0.49 (5)
a_μ^{had}	40.00 (32)	40.06 (31)	40.10 (31)	40.16 (31)

ensemble A30.32

$\bar{s}s$	$(\bar{t}_{min}+2)$	$(\bar{t}_{min}+\bar{t}_{max})/2$	$(\bar{t}_{max}-2)$	$(\bar{T}/2-4)$
$a_\mu^{had}(<)$	40.44 (19)	42.77 (23)	43.26 (25)	43.32 (25)
$a_\mu^{had}(>)$	3.15 (18)	0.63 (5)	0.11 (1)	0.05 (1)
a_μ^{had}	43.59 (30)	43.40 (25)	43.37 (25)	43.37 (25)

ensemble B25.32

$\bar{s}s$	$(\bar{t}_{min}+2)$	$(\bar{t}_{min}+\bar{t}_{max})/2$	$(\bar{t}_{max}-2)$	$(\bar{T}/2-4)$
$a_\mu^{had}(<)$	40.83 (14)	43.18 (17)	44.05 (18)	44.16 (19)
$a_\mu^{had}(>)$	3.52 (14)	1.11 (6)	0.23 (1)	0.11 (1)
a_μ^{had}	44.35 (22)	44.29 (19)	44.28 (19)	44.27 (19)

ensemble D15.48

$\bar{s}s$	$(\bar{t}_{min}+2)$	$(\bar{t}_{min}+\bar{t}_{max})/2$	$(\bar{t}_{max}-2)$	$(\bar{T}/2-4)$
$a_\mu^{had}(<)$	42.34 (17)	45.86 (19)	46.50 (20)	46.58 (20)
$a_\mu^{had}(>)$	4.27 (18)	0.75 (5)	0.10 (1)	0.02 (1)
a_μ^{had}	46.61 (24)	46.61 (20)	46.60 (20)	46.60 (20)

$$a_\mu^{HVP} = a_\mu^{HVP}(<) + a_\mu^{HVP}(>)$$

$$a_\mu^{HVP}(<) = 4\alpha_{em}^2 q_f^2 \sum_{\bar{t}=0}^{\bar{T}_{data}} \bar{f}(\bar{t})\bar{V}(\bar{t})$$

$$a_\mu^{HVP}(>) = 4\alpha_{em}^2 q_f^2 \sum_{\bar{t}=\bar{T}_{data}+1}^{\infty} \bar{f}(\bar{t}) \frac{\bar{G}_V}{2\bar{M}_V} e^{-\bar{M}_V \bar{t}}$$

$(t_{\min}, t_{\max}) =$ ground-state dominance

four choices:

$$\bar{T}_{data} = \begin{Bmatrix} (\bar{t}_{\min}+2), (\bar{t}_{\min}+\bar{t}_{\max})/2, \\ (\bar{t}_{\max}-2), (\bar{T}/2-4) \end{Bmatrix}$$

* the sum is independent on T_{data}

in what follows $\bar{T}_{data} = \bar{T}/2 - 4$

$$a_\mu^{HVP}(>)/a_\mu^{HVP} < 1\%$$

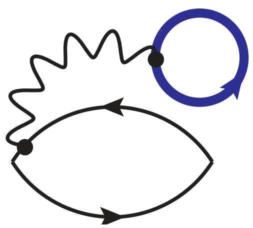
and within the statistical errors

quenched QED

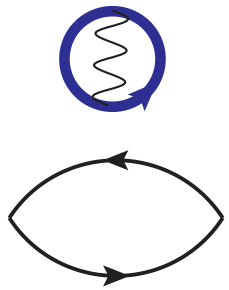
* neglect of the electric sea-quark charges: $q_f^{sea} = 0$

- in the RM123 approach $\Longrightarrow$ neglect of (noisy) fermionic disconnected diagrams

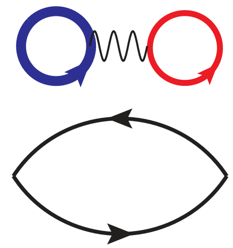
$q_f \sum_{f'} q_{f'}^{sea}$



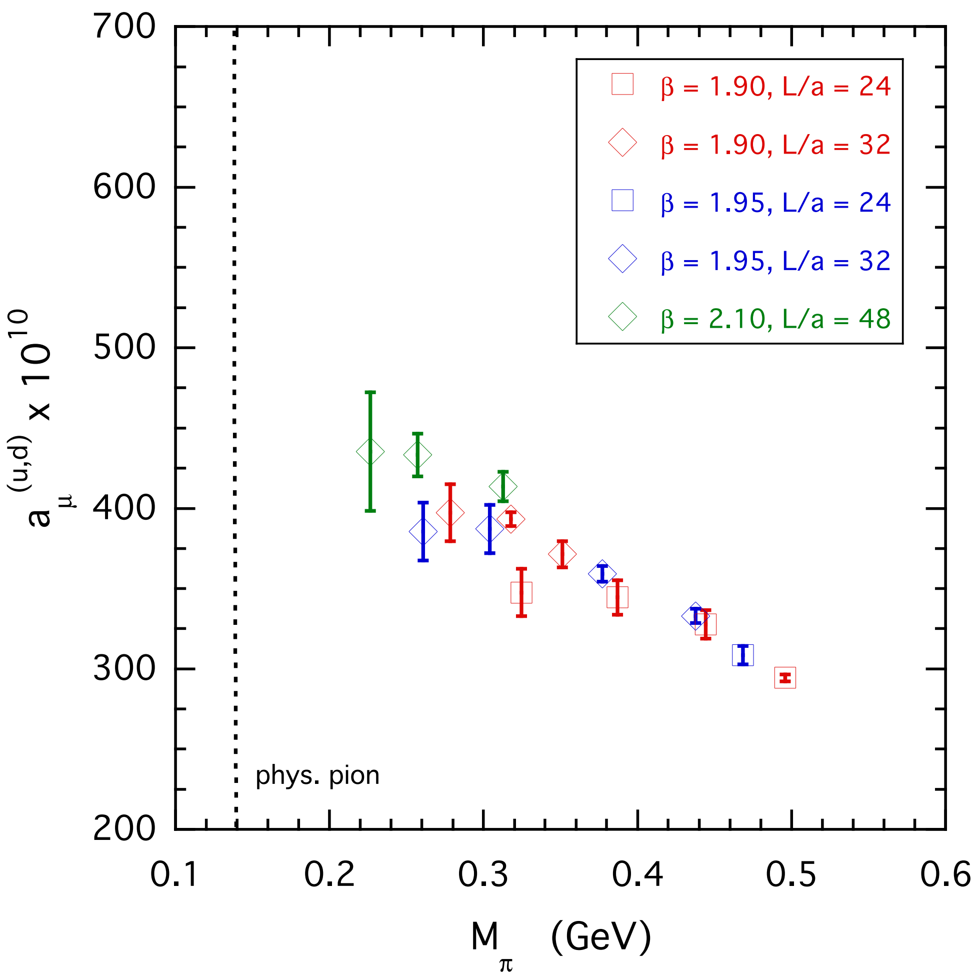
$\sum_{f'} q_{f'}^{sea} q_{f'}^{sea}$



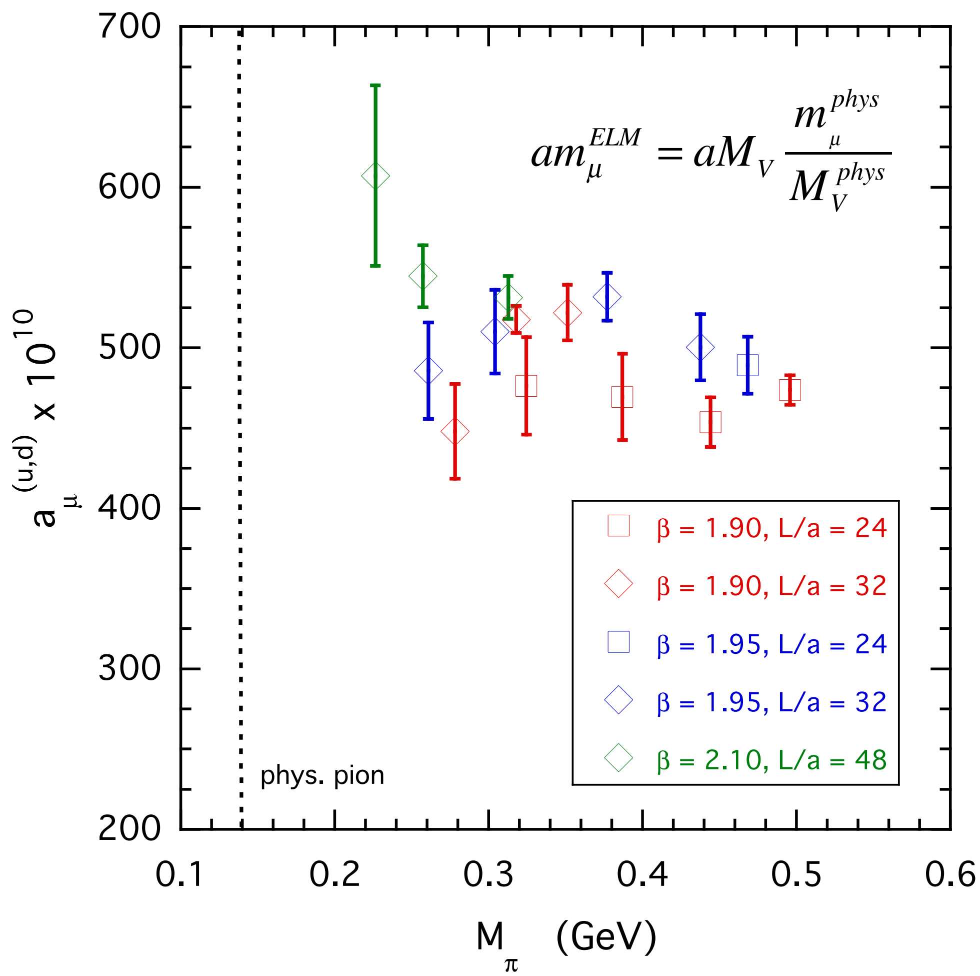
$\sum_{f_1 f_2} q_{f_1}^{sea} q_{f_2}^{sea}$



with **physical** lepton mass



with **effective** lepton mass



- FSEs are clearly visible

- the ELM procedure makes the pion mass dependence milder, but increases the statistical uncertainty