# Supermultiplets in $\mathcal{N} = 1$ $SU(2)$ SUSY Yang-Mills Theory

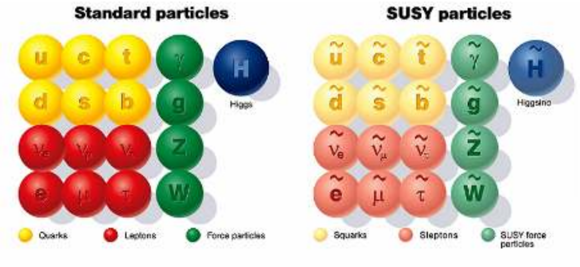
Henning Gerber

Münster-DESY(-Regensburg-Jena) collaboration

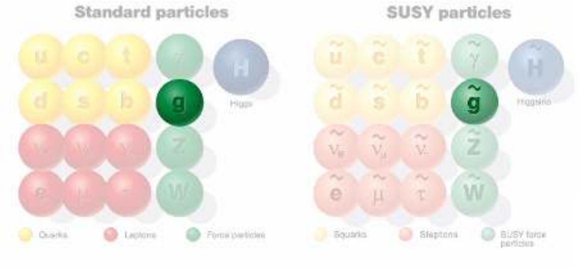
Lattice 2017 - Granada

Westfälische Wilhelms-Universität Münster

Graduiertenkolleg 2149 Research Training Group




- $\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\left(F^a_{\mu\nu}F^{\mu\nu,a}\right) + \frac{\mathrm{i}}{2}\bar{\tilde{g}}\not{D}\tilde{g} + \frac{m_g}{2}\bar{\tilde{g}}\tilde{g}$
- $A^a_\mu(x)$ : gauge fields
- $\tilde{g}(x)$: gluino fields, Majorana fermions in adjoint representation
  - $\left(D_\mu\tilde{g}\right)^a = \partial_\mu\tilde{g}^a + gf_{abc}A^b_\mu\tilde{g}^c$, here: $f_{abc} = \epsilon_{abc}$
- $\frac{m_g}{2}\bar{\tilde{g}}\tilde{g}$: soft SUSY-breaking term

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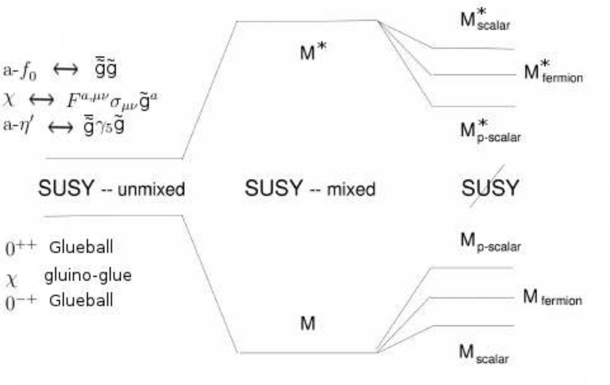


G. R. Farrar, G. Gabadadze, M. Schwetz, ", Phys. Rev. D60 (1999) 035002, arXiv:hep-th/9806204.

- Correlation Matrix $C_{ij}(\Delta t) = \langle O_i(\Delta t) O_j(0) \rangle$
- Generalized Eigenvalue Problem (GEVP):
  $C(t)\vec{v}^{(n)} = \lambda^{(n)}(t, t_0) C(t_0) \vec{v}^{(n)}$
- $\lim_{t\to\infty} \lambda^{(n)}(t, t_0) \propto \mathrm{e}^{-m_n(t-t_0)} \left(1 + \mathcal{O}\left(\mathrm{e}^{-\Delta m_n (t-t_0)}\right)\right)$
  - $\Delta m_n = \min_{l\neq n} |m_l - m_n|$

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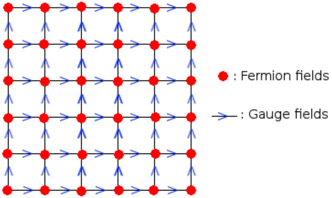
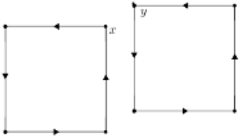



- Glueballs: $O(x) = \sum_{i<j} P_{ij}(x)$




- Gluino-glue: $O^a(x) = \sum_{i<j} \sigma_{ij}^{\alpha\beta} Tr_c \left[P_{ij}(x)\tilde{g}^{\beta}(x)\right]$

- Mesons: $O(x) = \bar{\tilde{g}}(x)\Gamma\tilde{g}(x)$

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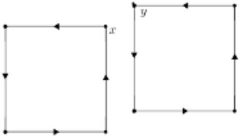


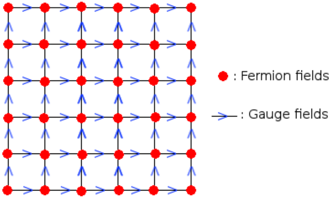


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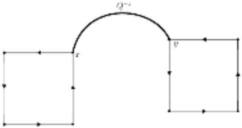

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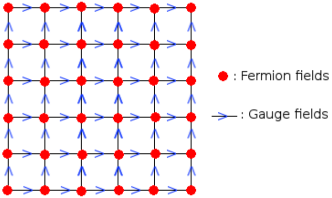


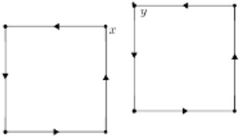


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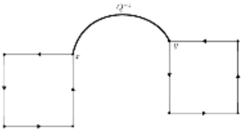

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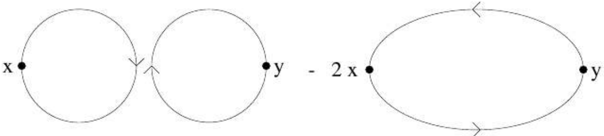

- Build new operators by applying smearing techniques:
- For gluonic operators: APE Smearing

$$U'_\mu(x) = U_\mu + \epsilon_{\text{APE}} \sum_{\nu=\pm1,\nu\neq\mu}^{\pm3} U^\dagger_\nu(x+\mu)U_\mu(x+\hat{\nu})U_\nu(x)$$

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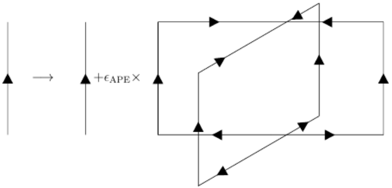


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- For mesonic operators: Jacobi Smearing
  $\tilde{g}'(\vec{x},t) = \sum_{\vec{y}} F(\vec{x},\vec{y})\tilde{g}(\vec{y},t)$

- $F(\vec{x},\vec{y}) =$Iterative solution of 3d Klein-Gordon equation for source and sinks :

$$F_{\beta b,\alpha a}(\vec{x},\vec{y}) = \delta_{\vec{x},\vec{y}}\delta_{\beta\alpha} + \delta_{\beta\alpha}\sum_{i=1}^{N_J}\left(\kappa_J \sum_{\mu=1}^{3}\left[\delta_{\vec{y},\vec{x}+\mu}U_{\mu,ba}(x) + \delta_{\vec{y}+\hat{\mu}}U^{\dagger}_{\mu,ba}(x)\right]\right)^i$$

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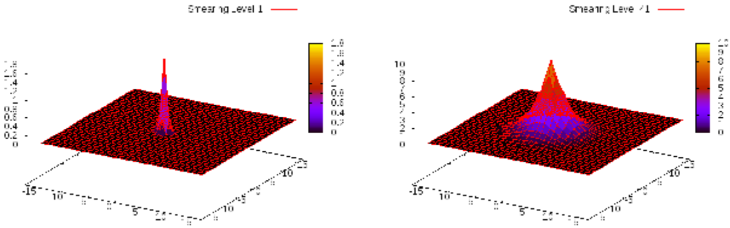

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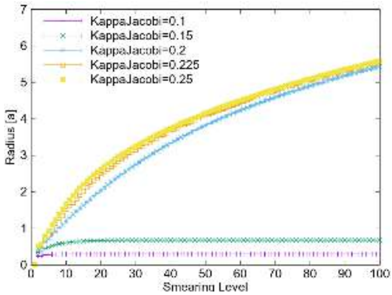
## Jacobi Smearing

- Smearing Radius:

$$R_J = \frac{\sum_{\vec{x}} |\vec{x}|^2 \, |F(\vec{x},0)|^2}{\sum_{\vec{x}} |F(\vec{x},0)|^2}$$




$\Rightarrow$Use $\kappa_{\text{Jacobi}} = 0.2$

$\Rightarrow$Use smearing levels up to 80

$\Rightarrow$Optimizing the signal lead to choosing smearing levels 0, 40 and 80

APE smearing:

- similar analyis as for Jacobi smearing
- $\epsilon_{\text{APE}} \sim 0.4$
- smearing levels up to 95 for $32^3 \times 64$ lattice
- i.e. {5,15,...,95} for gluino-glue

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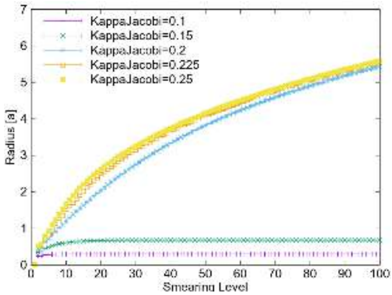


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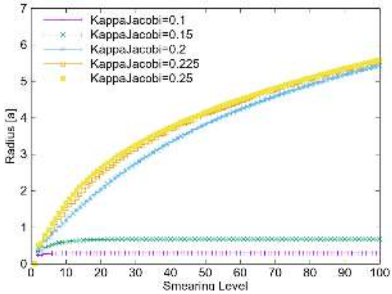


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  - e.g. a-$f_0$ and $0^{++}$-glueball
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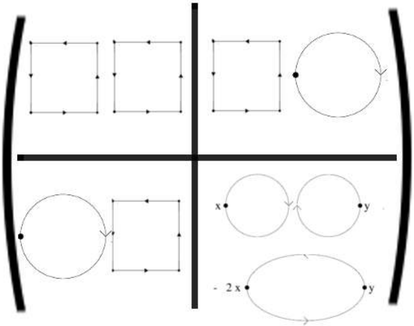
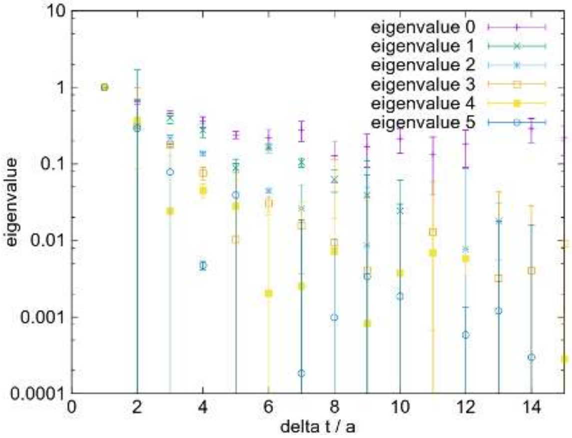
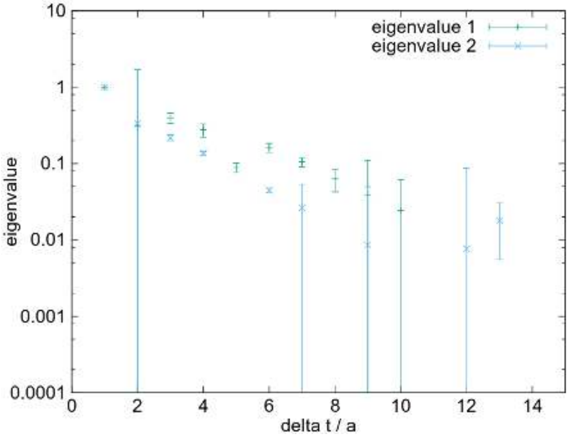


## Ordering the eigenvalues




Generalized eigenvalues of $0^{++}$-channel at $\beta = 1.9$, $\kappa = 0.1433$ sorted by value

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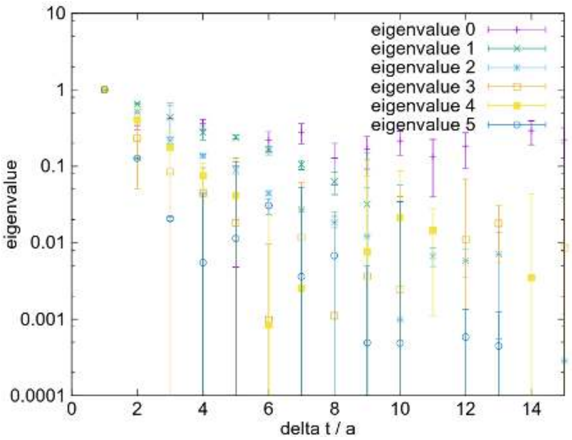


Generalized eigenvalues of $0^{++}$-channel at $\beta = 1.9$, $\kappa = 0.1433$ sorted by new method

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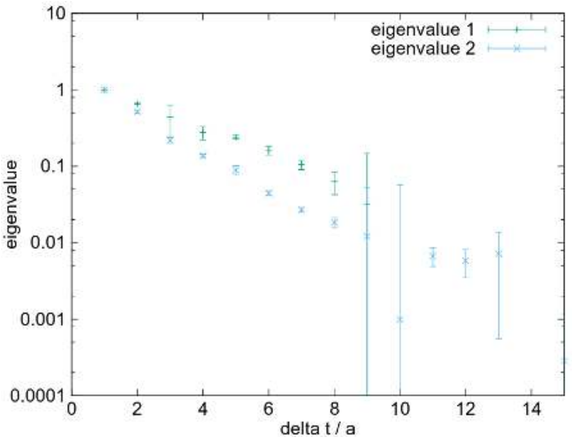


Generalized eigenvalues of $0^{++}$-channel at $\beta = 1.9$, $\kappa = 0.1433$ sorted by new method

1. Solve GEVP at some fixed $t$:
   - $C(t)\vec{w}^{(m)} = \lambda^{(m)}(t, t_0)C(t_0)\vec{w}^{(m)}$
2. Solve GEVP at all other $t$:
   - $C(t)\vec{v}^{(n)} = \lambda^{(n)}(t, t_0)C(t_0)\vec{v}^{(n)}$
3. Sort $\vec{v}^{(n)}$ by their scalar products with $\vec{w}^{(n)}$:
   - $M_{mn} = \vec{w}^{(m)} \cdot \vec{v}^{(n)}$
   - scan for largest entry in $M$, call the indices $m_{\text{max}}, n_{\text{max}}$
   - identify $\vec{v}_{\text{sorted}}^{(n_{\text{max}})} = \vec{v}^{(m_{\text{max}})}$
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- Tree-level Symanzik improved gauge action
- 1 level of stout smearing
- Configurations produced on JUQUEEN - Juelich - Germany
- Measurements performed on
  - JURECA - Juelich - Germany
  - local HPC clusters in Münster, standard CPU-clusters and Xeon-Phi Knights Corner cluster

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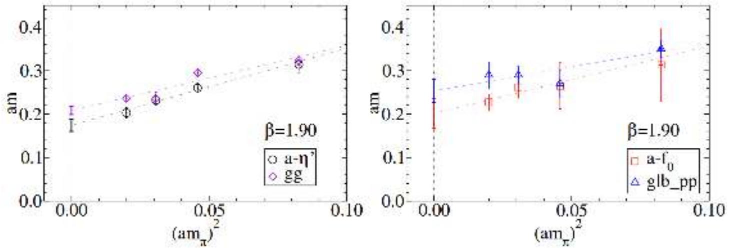
## Groundstate multiplet




Extrapolation of the lightest supermultiplet to the chiral limit. Desy-Münster collaboration arXiv:1512.07014

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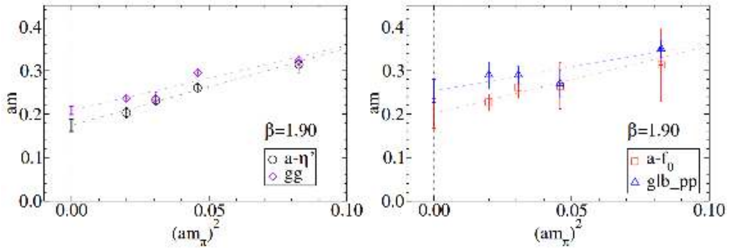


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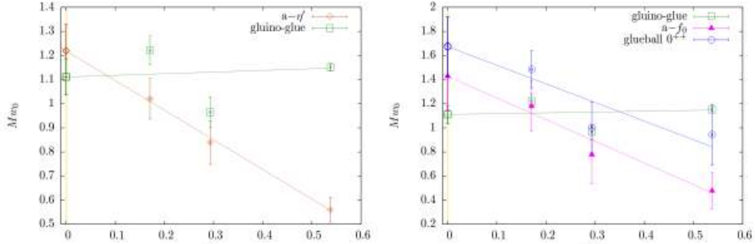
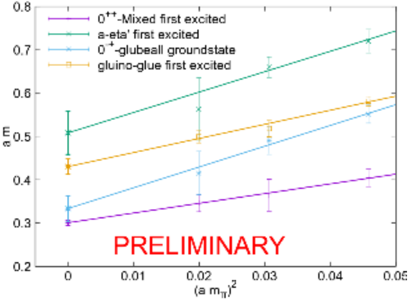



Extrapolation of the lightest supermultiplet to the continuum limit. Desy-Münster collaboration arXiv:1512.07014

Excited Multiplet @ $\beta = 1.9$

# PRELIMINARY!!!




Linear chiral extrapolation at $\beta = 1.9$

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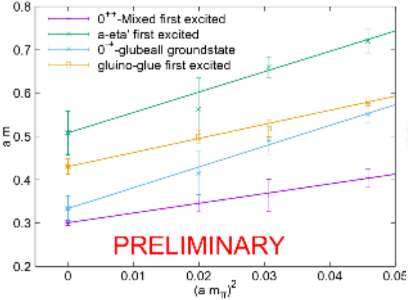


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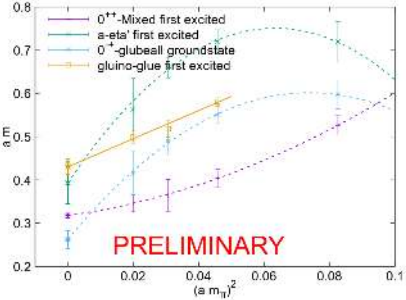


Quadratic chiral extrapolation at $\beta = 1.9$

- $\beta = 1.9$ (0.036fm)

| $\kappa$ | $N_{\text{confs}}^{\text{measured}}$ (mesos) | $N_{\text{confs}}^{\text{available}}$ |
|---|---|---|
| 0.1433 | 1151 | 10374 |
| 0.14387 | 1276 | 10237 |
| 0.14415 | 2814 | 21090 |
| 0.14435 | 1343 | 10680 |

- $\beta = 1.75$ (0.054fm)

| $\kappa$ | $N_{\text{confs}}^{\text{measured}}$ (mesons) | $N_{\text{confs}}^{\text{available}}$ |
|---|---|---|
| 0.1490 | 1163 | 9350 |
| 0.1492 | 1070 | 13017 |
| 0.14925 | 879 | 7224 |
| 0.1493 | 612 | 10280 |

- still too noisy for extraction of first excited states

- Second order polynomial fit fits the data too well
  - errorbars overestimated?
- Within errors there seem so be no mixing in the $0^{-+}$-channel
  - off-diagonal entries of the correlation matrix are zero within errors
- $0^{-+}$-glueball and $a-\eta'$ masses differ

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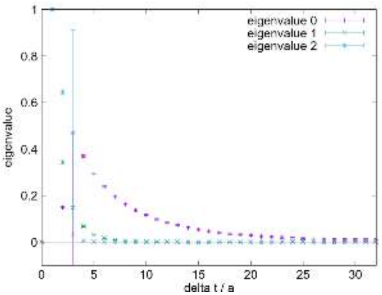


a-$\eta'$ @$\beta = 1.9,\ \kappa = 0.14435$

- Large autocorrelation
- Talk C. Urbach, J. Simeth, Tuesday 18.10, 1830
  - Derivative trick $\Rightarrow$ Use $\tilde{C}(t) = C(t) - C(t+1)$

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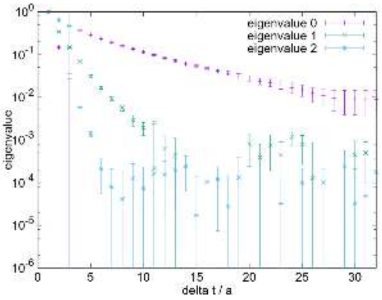


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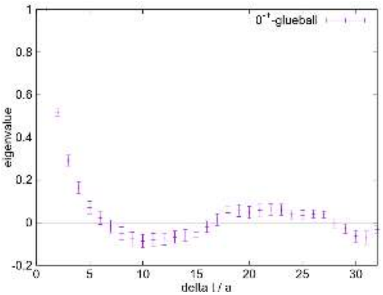
  - off-diagonal entries of the correlation matrix are zero within errors
- $0^{-+}$-glueball and $a-\eta'$ masses differ




$0^{-+}$- glueball
$@\beta = 1.9,\ \kappa = 0.14435$

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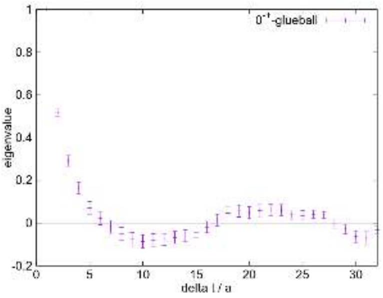


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- $0^{-+}$-glueball groundstate heavier than groundstate multiplet
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- Work to be done:
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  - Continuum extrapolation of the excited multiplet
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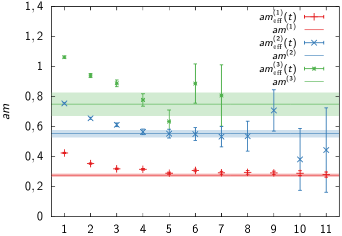
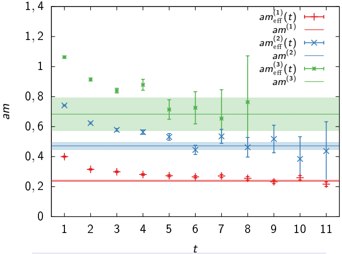
Thank you for your attention!!

Westfälische Wilhelms-Universität Münster

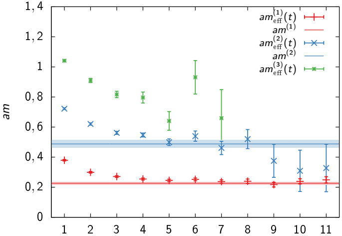
Graduiertenkolleg 2149
Research Training Group




Effective gluino-glue masses at $\kappa = 0.14387$



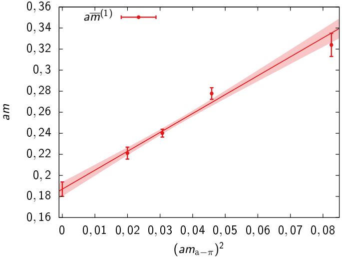

Effective gluino-glue masses at $\kappa = 0.14415$



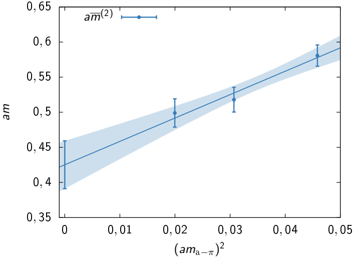

Effective gluino-glue masses at $\kappa = 0.14435$

*S. Kuberski, Masterthesis, Uni Münster*

- $\kappa = \frac{1}{2am+8}$
- $\beta = \frac{2N}{g^2}$
- $m_{\text{eff}} = \ln \frac{C(\Delta t)}{C(\Delta t+1)}$




Extrapolation to the chiral limit (groundstate)




Extrapolation to the chiral limit (first excited state)

*S. Kuberski, Masterthesis, Uni Münster*

- Calculate the lowest eigenvalues of $Q$ and corresponding eigenvectors
  - using Arnoldi (ARPACK)
  - Chebyshev Polynomials of order 11
  - Even/Odd-Preconditioning
- Stochastic estimator technique for space orthogonal to the previously calculated eigenvectors:
  - $\frac{1}{N_S}\sum_i^{N_S} \left|\eta^i\right\rangle \left\langle\eta^i\right| = \mathbb{1} + \mathcal{O}\left(\sqrt{N_S}\right)$
    - use $\mathbb{Z}_4$-noise
  - $Q\left|s^i\right\rangle = \left|\eta^i\right\rangle$
  - $Q^{-1} = \frac{1}{N_S}\sum_i^{N_S} \left|s^i\right\rangle \left\langle\eta^i\right|$
  - Conjugate gradient
  - $N_S = 40$ for $\beta = 1.9$, $32^3 \times 64$