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Dispersion relation and unphysical poles of Möbius domain-wall fermions in free field theory at finite L_s

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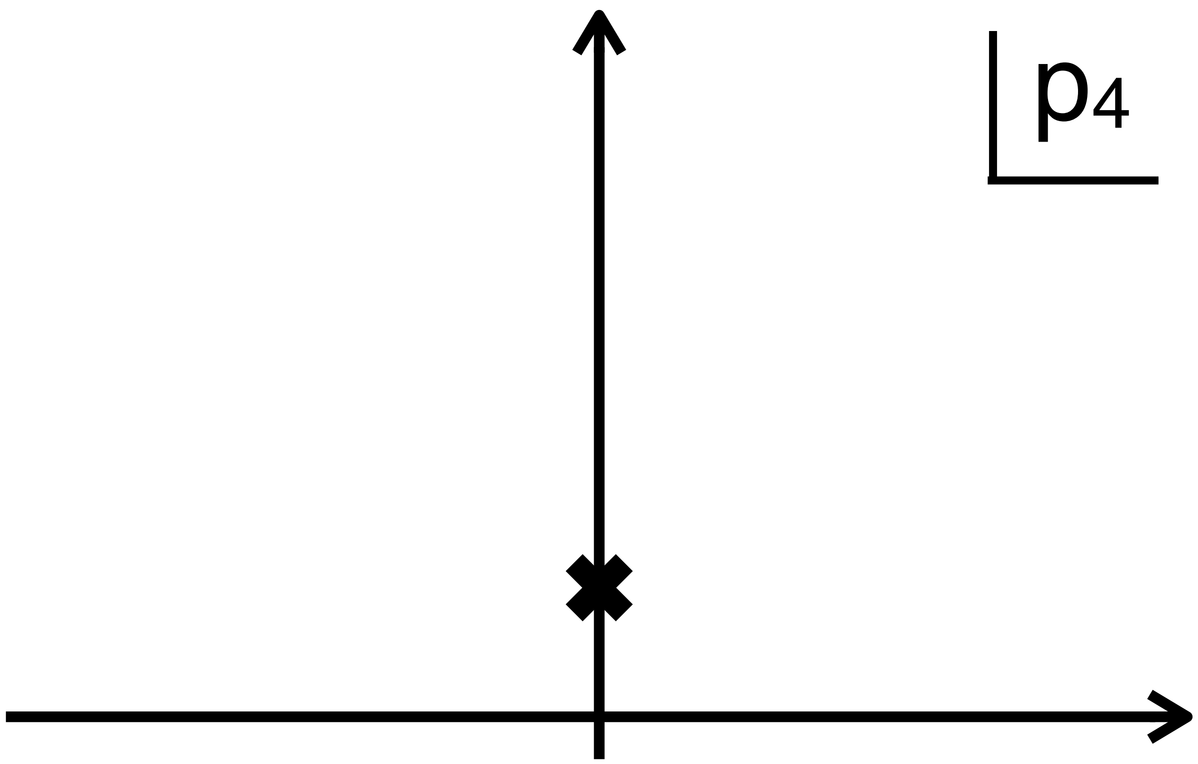
Pole of a free field

- Singular point of propagator in Mom Sp.
  - $p_4 = iE(\vec{p})$
  - FT(prop) $\sim e^{-E(\vec{p})t}$



- Dispersion relation
  - $E(\vec{p})^2 = m^2 + |\vec{p}|^2$

Pole of a free field

- Singular point of propagator in Mom Sp.
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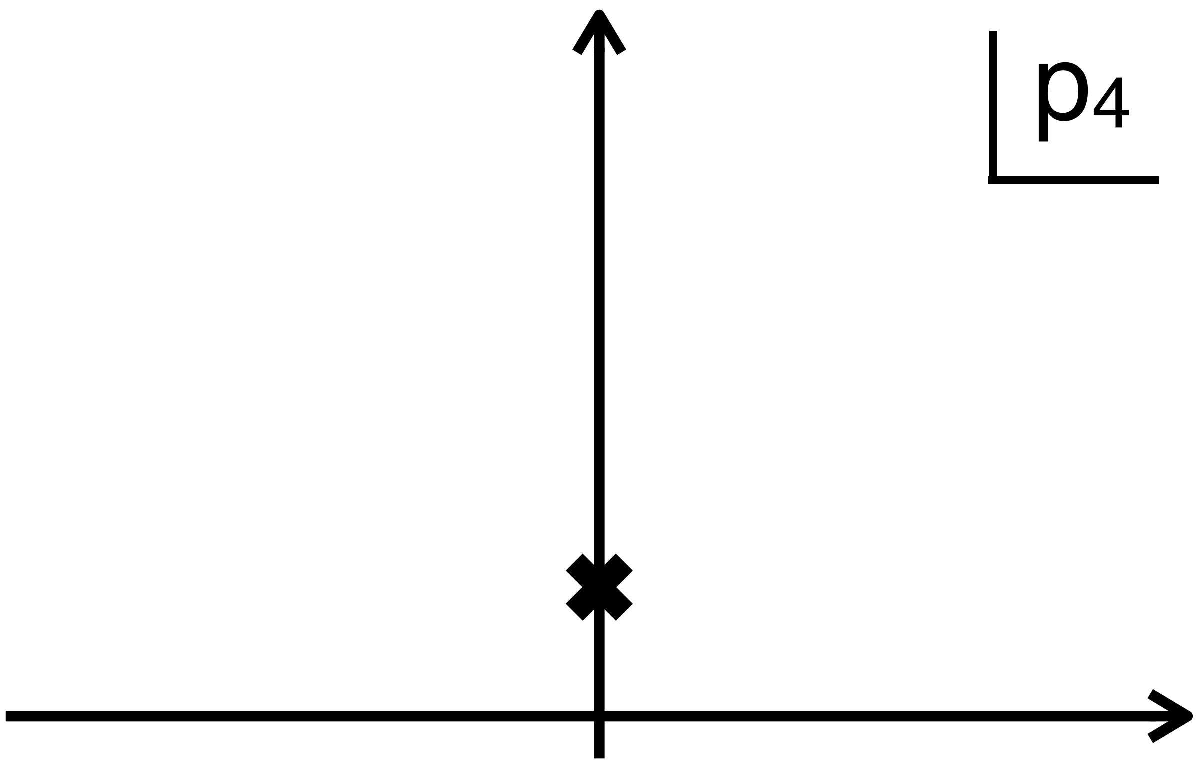
  - FT(prop) ~ $e^{-E(\vec{p})t}$



- Dispersion relation
  - $E(\vec{p})^2 = m^2 + |\vec{p}|^2 + O(a^2(m^4, m^2\,\vec{p}^2, \ \vec{p}^4))$

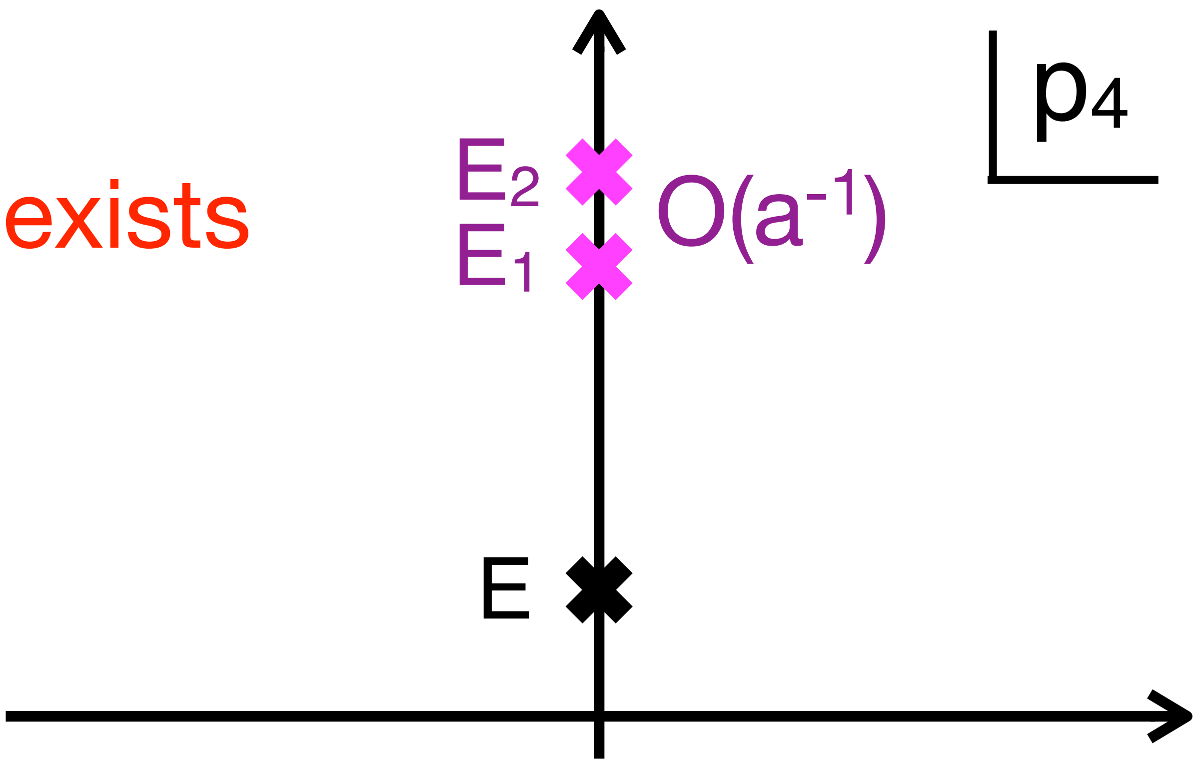
Lattice artifact

Unphysical poles on the lattice

- Other poles ***can*** exist (depending on action)
- Doubler-managed action

  $\Rightarrow E(\vec{0}) = m,$

  $E_1 \sim E_2 \sim \cdots \sim O(a^{-1})$ if exists



Unphysical poles on the lattice

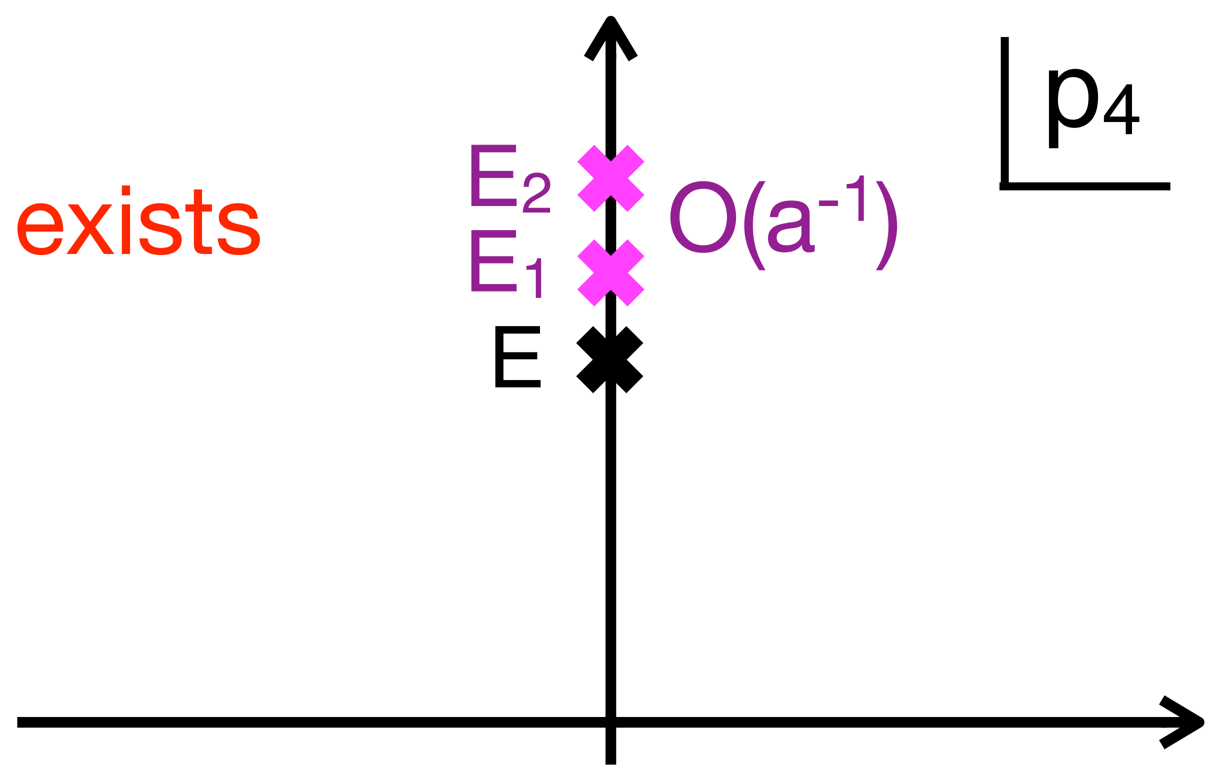
- Other poles ***can*** exist (depending on action)
- Doubler-managed action

  $\Rightarrow E(\vec{0}) = m$,

  $E_1 \sim E_2 \sim \cdots \sim O(a^{-1})$ if exists

- Contamination by UPPs
  - Significant when $m \sim a^{-1}$

  $e^{-Et} \sim e^{-E_1 t} \sim e^{-E_2 t}$



Ex: UPP of Wilson fermions

$$D_W = \sum_\mu [\mathrm{i}\gamma_\mu \sin p_\mu + r(1 - \cos p_\mu)] + m$$

$$D_W^{-1} = \frac{\sum_\mu [-\mathrm{i}\gamma_\mu \sin p_\mu + r(1 - \cos p_\mu)] + m}{\sum_\mu \sin^2 p_\mu + (r \sum_\mu (1 - \cos p_\mu) + m)^2}$$

Ex: UPP of Wilson fermions

$$D_W = \sum_\mu [\mathrm{i}\gamma_\mu \sin p_\mu + r(1-\cos p_\mu)] + m$$

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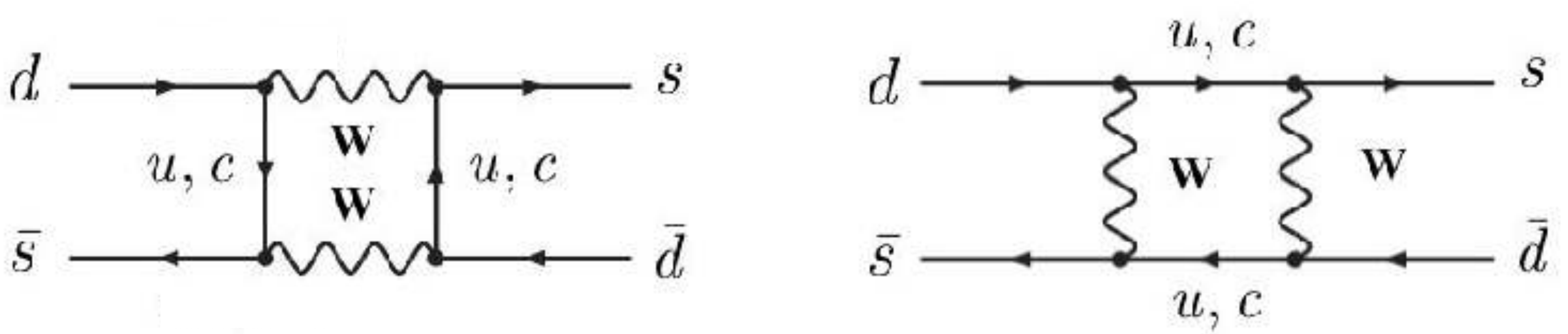
- r = ±1
  - $\cdots$ = 0 : linear equation of cos p_4
  - Only one solution cos p_4 = C (>1) ⇒ p_4 = i $\cosh^{-1}$ C
    physical pole only
- r ≠ ±1
  - $\cdots$ = 0 : quadratic equation of cos p_4
  - Two values of cos p_4 ⇒ one UPP exists

Motivation

- Are there UPPs of DWFs?
- Investigate properties of UPPs
  - Dependence of UPP energies on m, L_s, M_5, b and c
- Search the best choice of these parameters to minimize cutoff effects on heavy quarks

DWFs for heavy quarks

- Charm quark: needed for GIM mechanism
  - Ex: ΔM_K — quadratic divergence cancelled by charm



- GIM mechanism on the lattice
  - Same regularization for up & charm is needed
  - If DW up ⇒ DW charm necessary

Möbius domain-wall fermions

$$D_{\mathrm{MDW}} = \begin{pmatrix} \tilde{D} & -P_- & 0 & \dots & 0 & mP_+ \\ -P_+ & \tilde{D} & -P_- & \ddots & 0 & 0 \\ 0 & -P_+ & \tilde{D} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -P_- & 0 \\ 0 & 0 & \ddots & -P_+ & \tilde{D} & -P_- \\ mP_- & 0 & \dots & 0 & -P_+ & \tilde{D} \end{pmatrix}$$

$$\tilde{D} = D_-^{-1} D_+, \quad D_+ = 1 + bD_W(-M_5), \quad D_- = 1 - cD_W(-M_5)$$

- b = 1, c = 0 ⇒ Shamir DWFs
- Consider only r = 1

Propagator of DWFs

- 5D propagator : D_{MDW}^{-1}
- 4D propagator

$$S_F^{4d}(p) = P_-(D_{\mathrm{MDW}}^{-1})_{0,0}P_+ + P_+(D_{\mathrm{MDW}}^{-1})_{L_s-1,L_s-1}P_- + P_-(D_{\mathrm{MDW}}^{-1})_{0,L_s-1}P_- + P_+(D_{\mathrm{MDW}}^{-1})_{L_s-1,0}P_+$$

$$= \frac{2\sinh(\alpha L_s)}{F_{L_s}} \frac{b+c}{D_-^\dagger D_-} \mathrm{i}\tilde{\not{p}} + \frac{2}{F_{L_s}} \{m[W\sinh(\alpha(L_s-1)) - \sinh(\alpha L_s)] - W\sinh\alpha\}$$

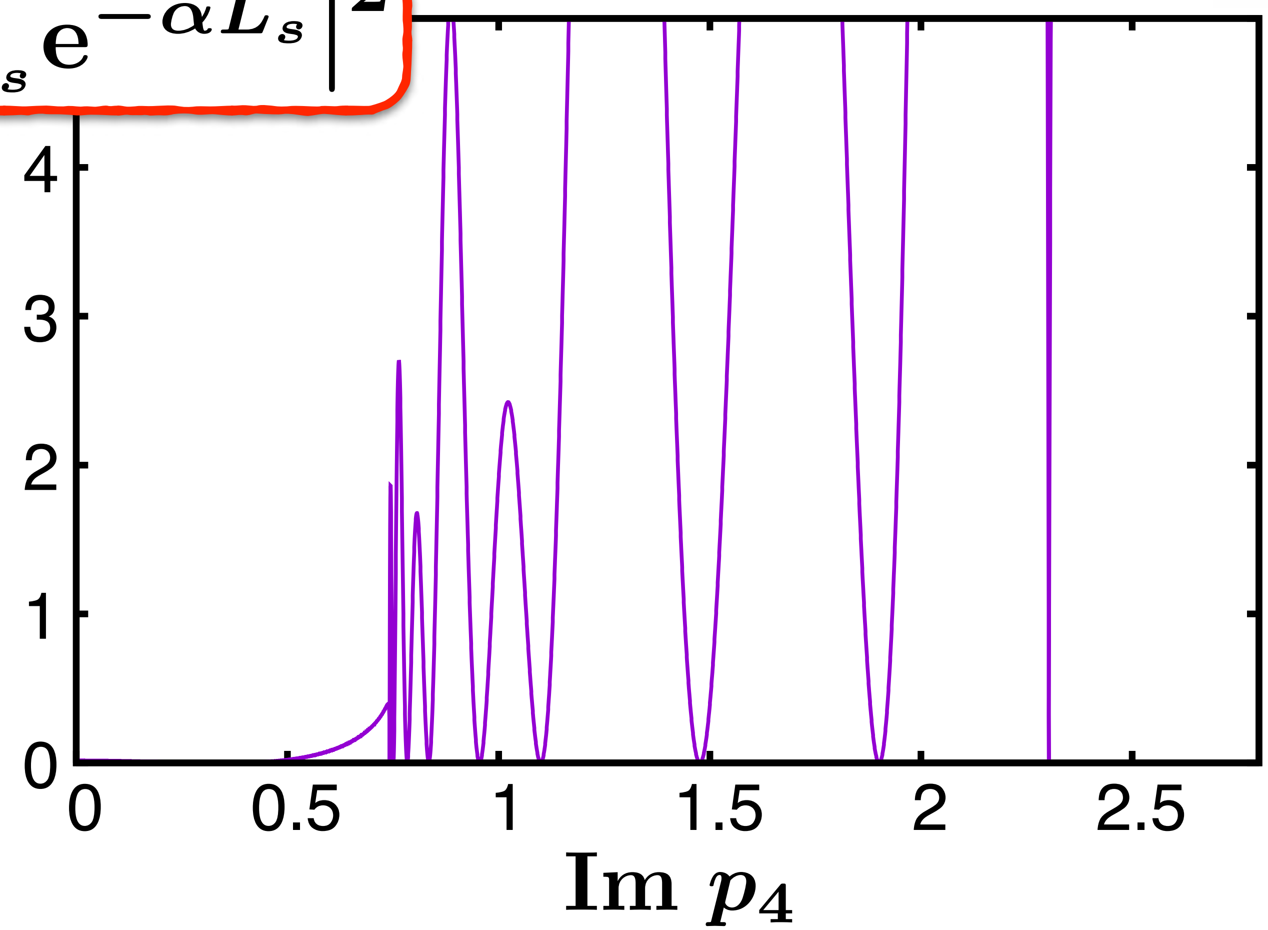
$$\boldsymbol{F_{L_s} = \mathrm{e}^{\alpha L_s}(1 - W\mathrm{e}^{\alpha} + m^2(W\mathrm{e}^{-\alpha} - 1)) - 4mW\sinh\alpha + \mathrm{e}^{-\alpha L_s}(W\mathrm{e}^{-\alpha} - 1 + m^2(1 - W\mathrm{e}^{\alpha}))}$$

$$W = \frac{D_-^\dagger D_+}{D_-^\dagger D_-}, \quad \cosh\alpha = \frac{\left(\frac{b+c}{D_-^\dagger D_-}\right)^2 \tilde{p}^2 + W^2 + 1}{2W}$$

Zero points of F_{L_s}

$L_s = 8$,
b=1, c=0,
$M_5 = 0.9$,
$\vec{p} = 0$,
Re $p_4 = 0$

$|F_{L_s} e^{-\alpha L_s}|^2$



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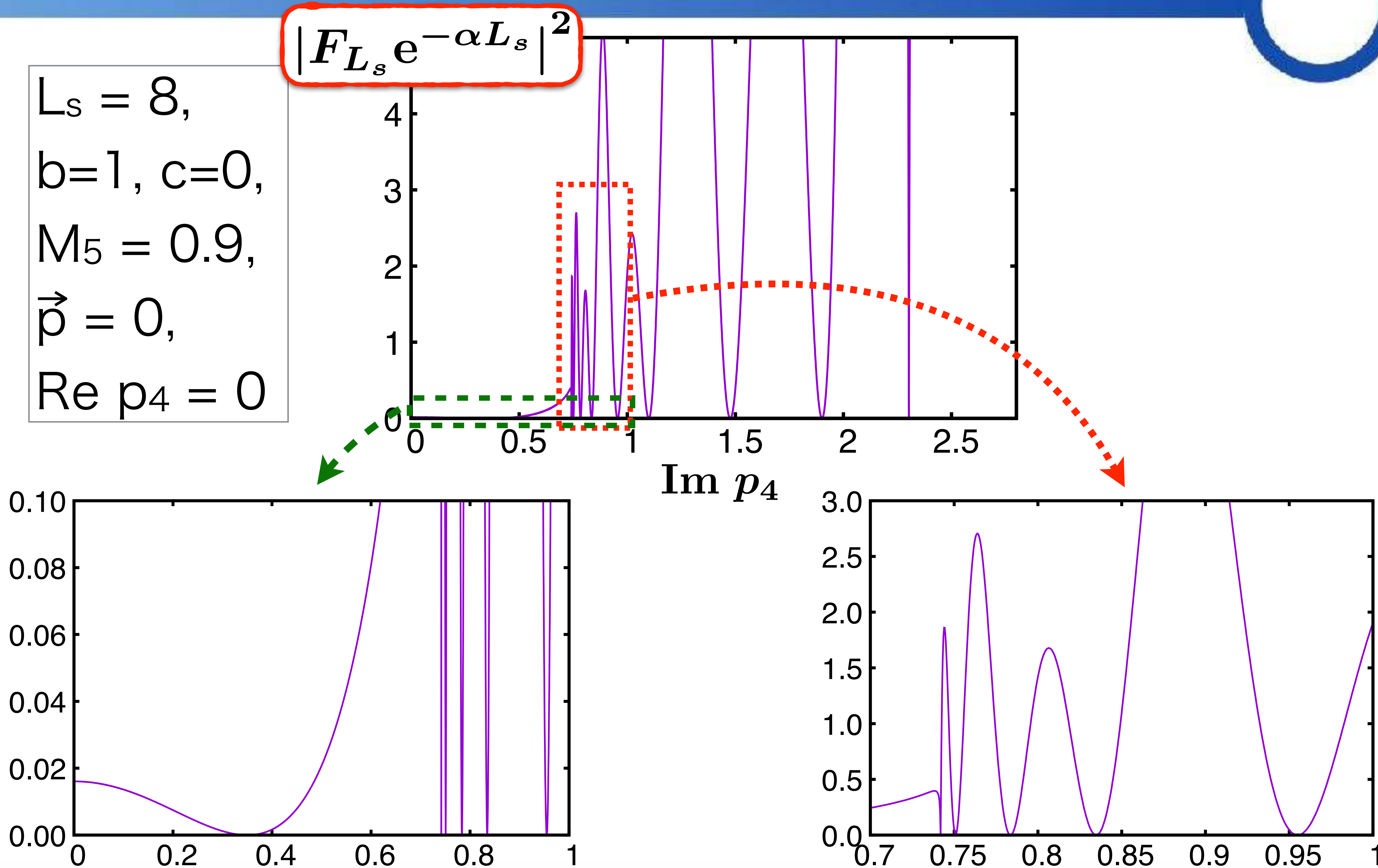

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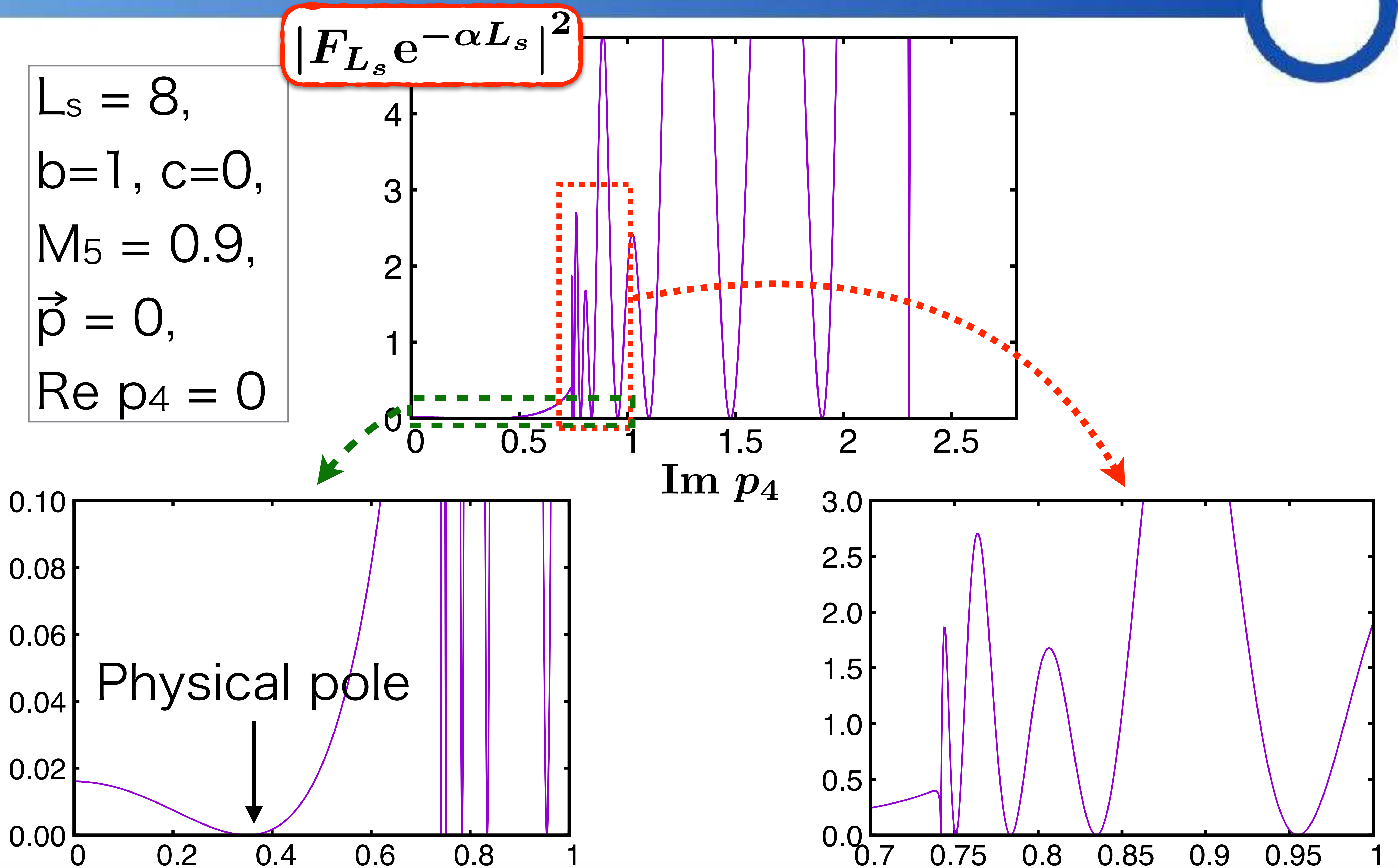

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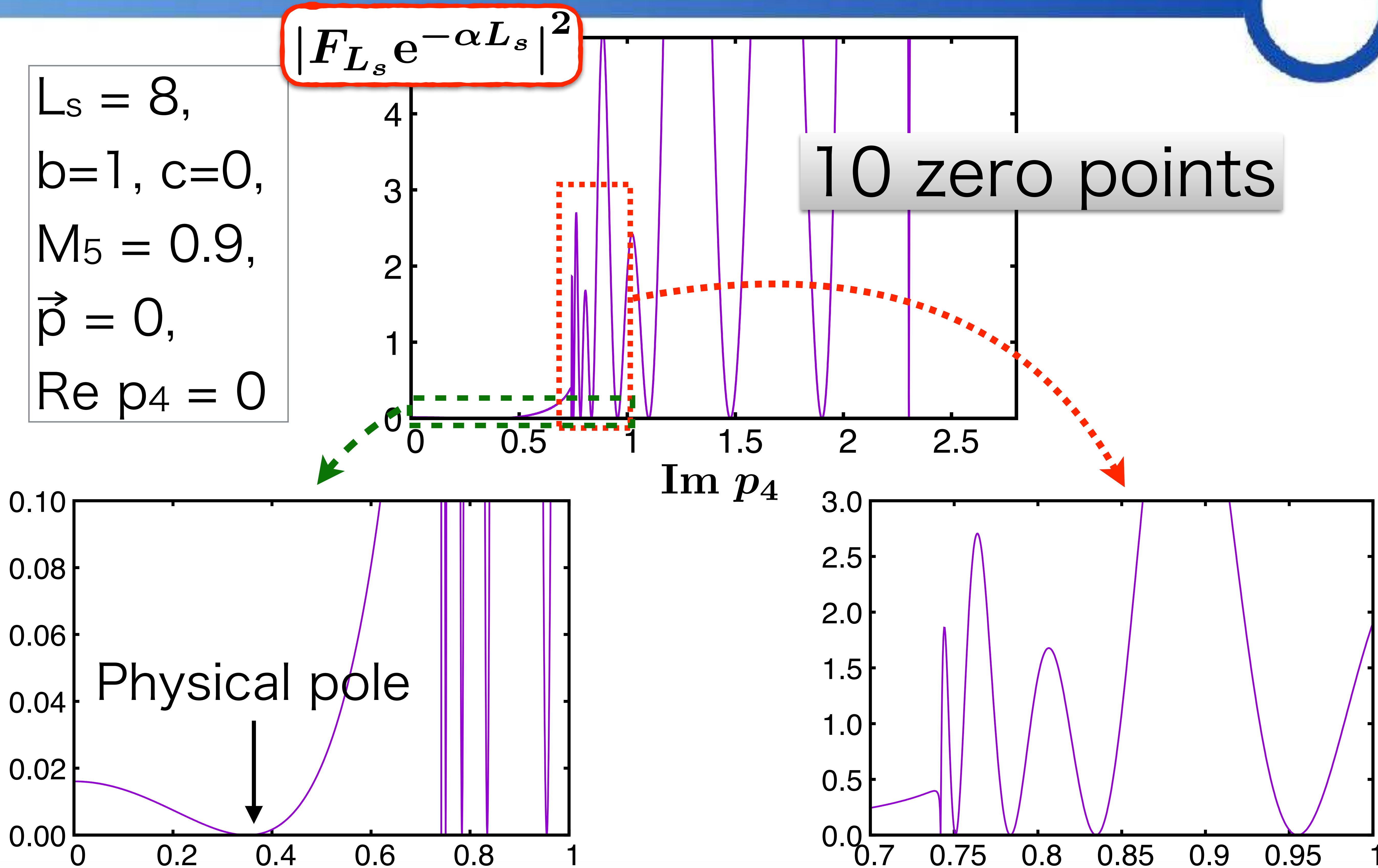
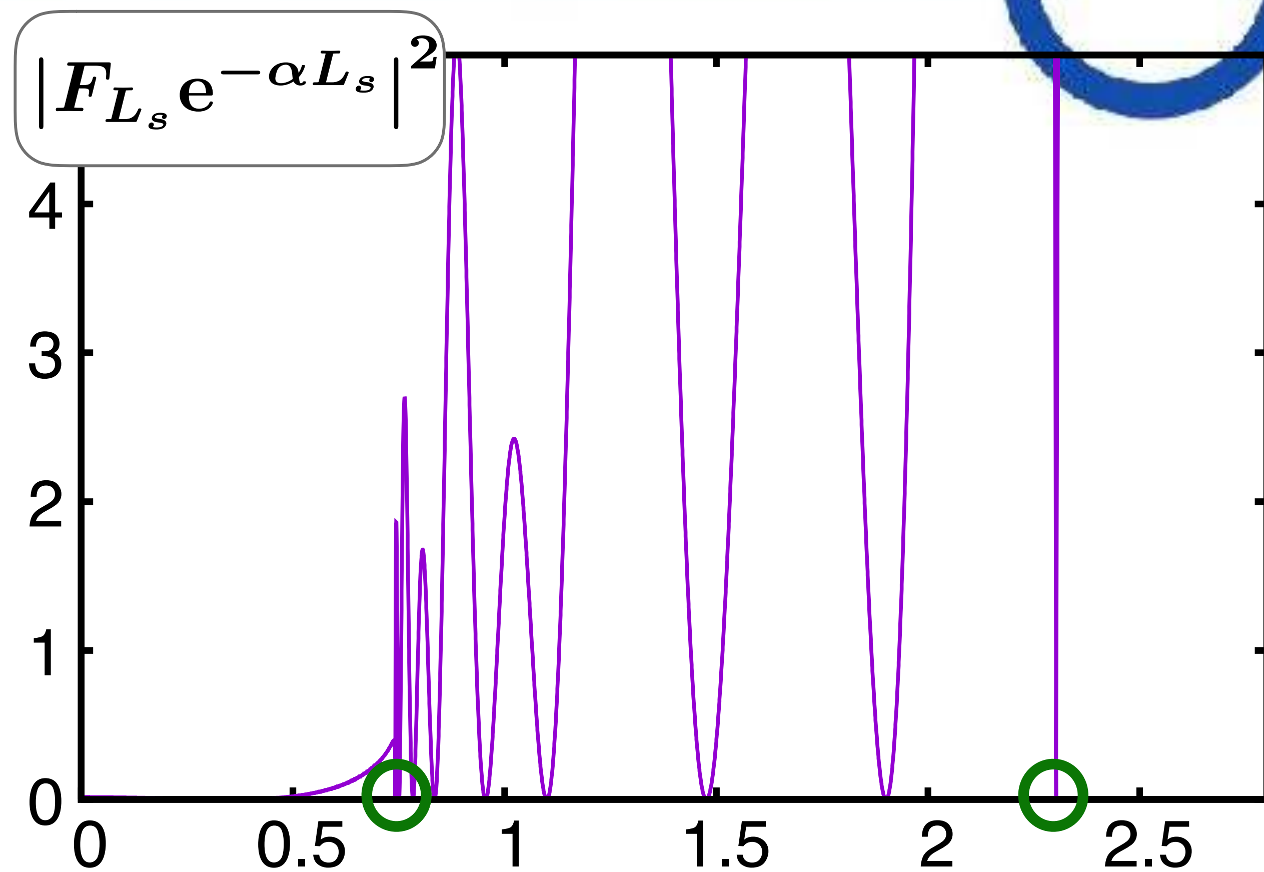


Classification of the zero points

- One physical pole
- Two solvable zero points (○)
  - $\cosh \alpha = \pm 1$
  - Not poles

$$S_F^{4d}(p) = \frac{2\sinh(\alpha L_s)}{F_{L_s}} \frac{b+c}{D_-^\dagger D_-} \mathrm{i}\not{p} + \frac{2}{F_{L_s}} \{m[W \sinh(\alpha(L_s - 1)) - \sinh(\alpha L_s)] - W \sinh \alpha\}$$

! Numerator also vanish at $\cosh \alpha = \pm 1$



- Seven (L_s–1) UPPs
  - $-1 < \cosh \alpha < +1$

Region of UPPs

- $-1 < \cosh\alpha < +1$
  - α: pure imaginary
  - F_{L_s}: oscillatory w/ varying p_4

$$F_{L_s} = e^{\alpha L_s}(1 - We^{\alpha} + m^2(We^{-\alpha} - 1)) - 4mW \sinh\alpha$$
$$+ e^{-\alpha L_s}(We^{-\alpha} - 1 + m^2(1 - We^{\alpha}))$$

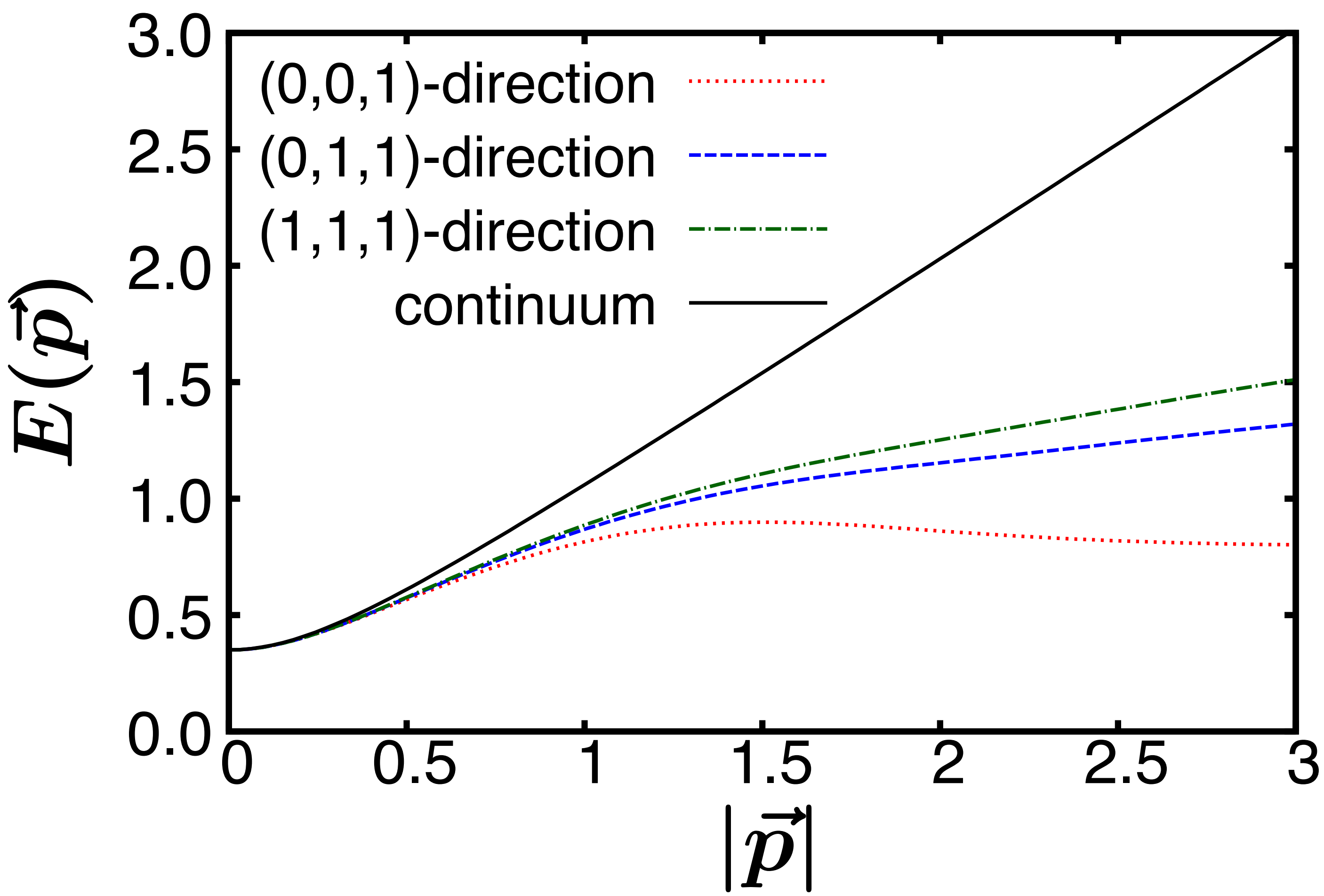
⭐ Any terms not suppressed at large L_s

  - Frequency $\propto L_s + 1 \Rightarrow$ zero points increased by L_s

Dispersion relation for physical pole

- $E(\vec{p})^2 = m^2 + |\vec{p}|^2 + O(m^4, m^2\vec{p}^2, \vec{p}^4)$

Lattice artifact



$L_s = 8$,
b=1, c=0,
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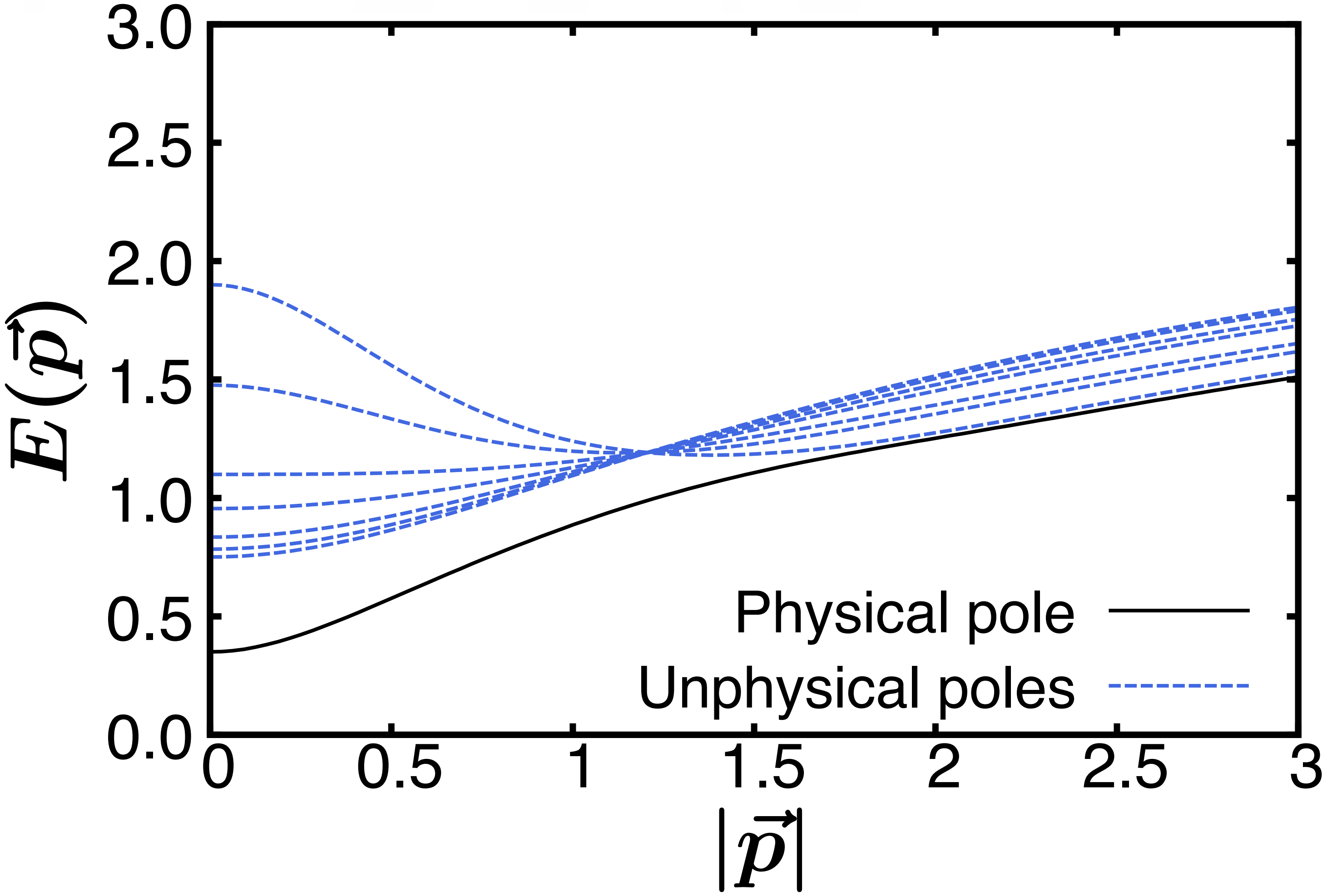
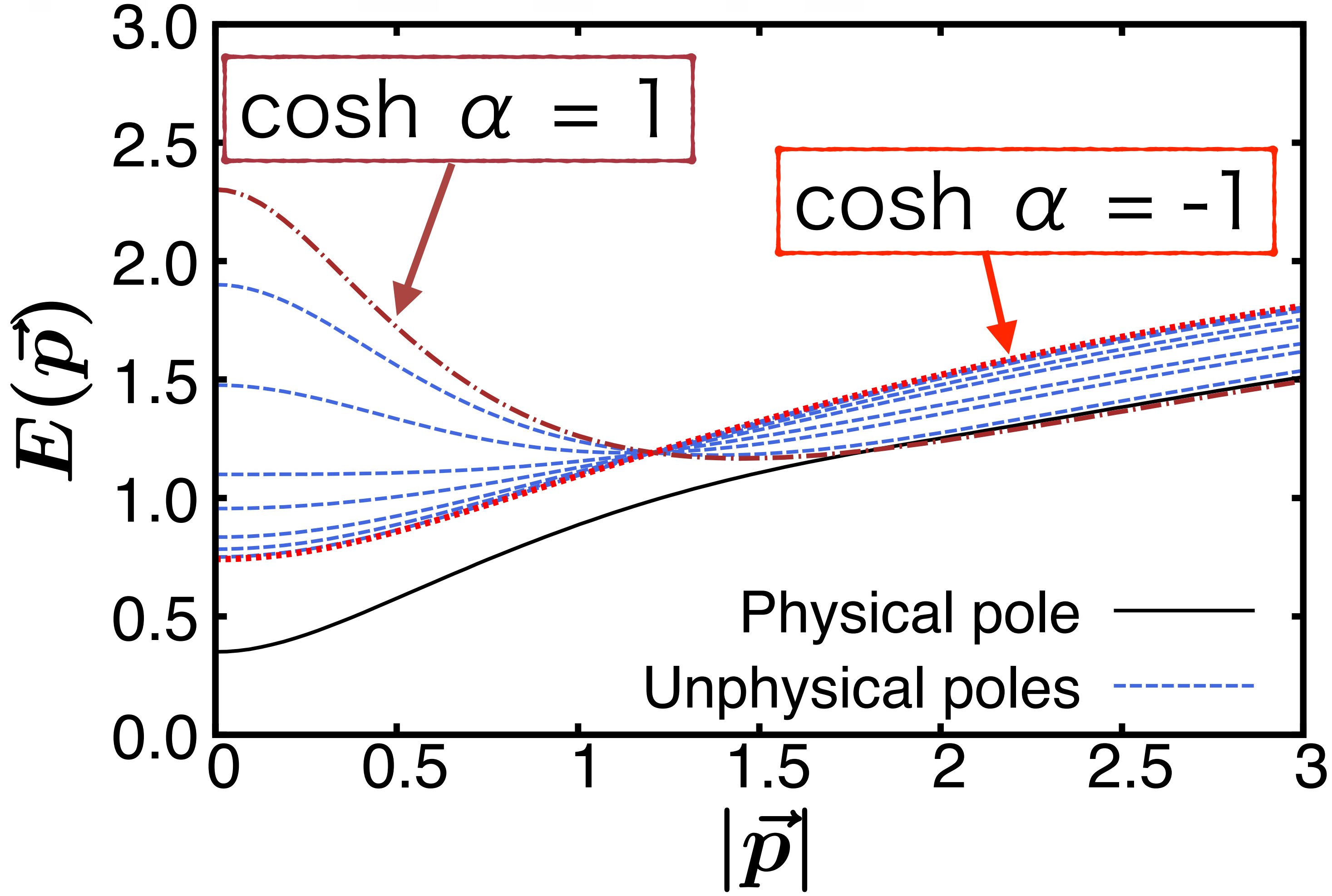
Dispersion relation for UPPs



$L_s = 8$,
$b=1$, $c=0$,
$M_5 = 0.9$,
$\mathrm{Re}\, p_4 = 0$
$\vec{p} = \frac{1}{\sqrt{3}}(|\vec{p}|, |\vec{p}|, |\vec{p}|)$

Dispersion relation for UPPs



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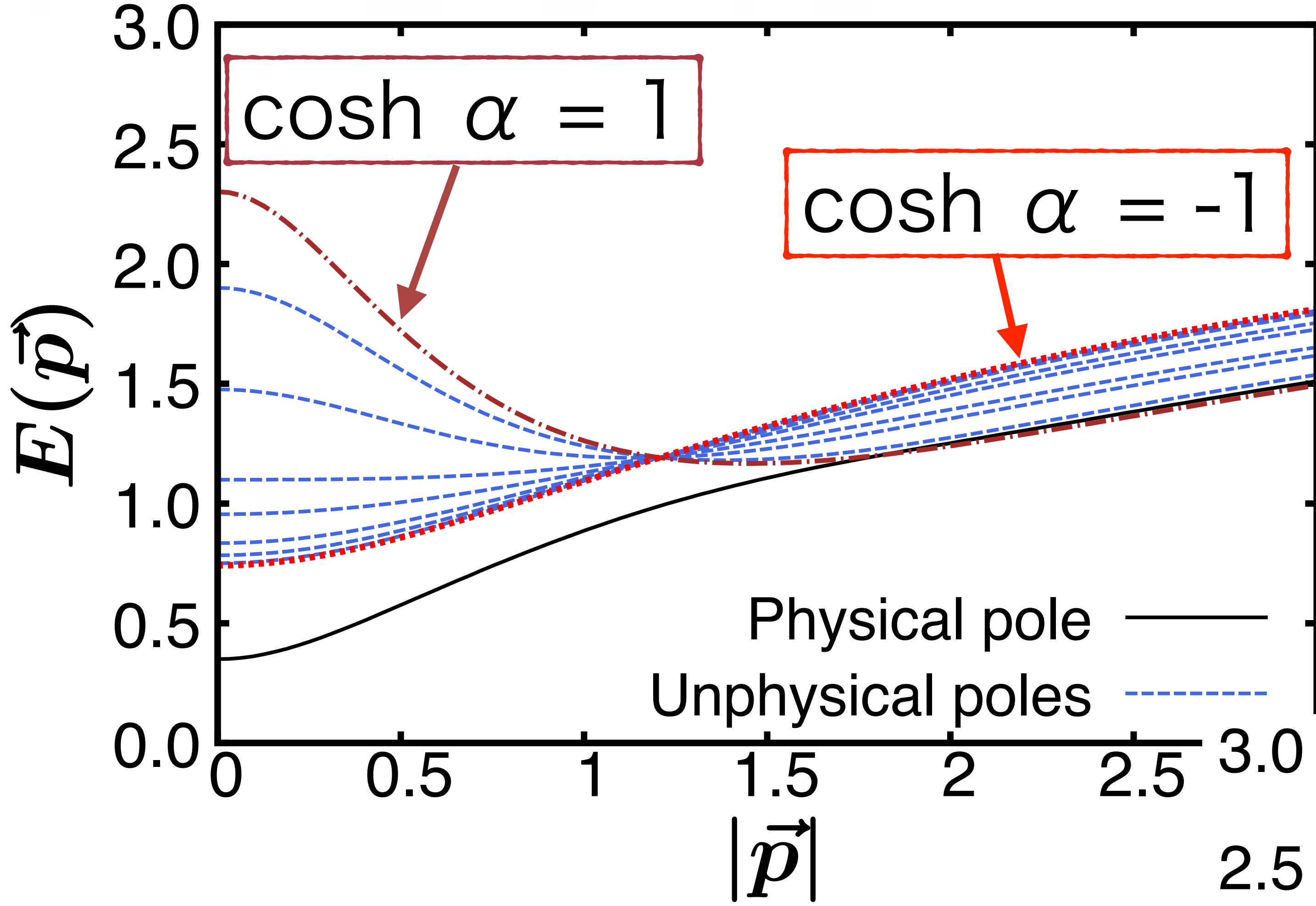
- UPPs
  - $-1 < \cosh\alpha < +1$

Dispersion relation for UPPs



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b=1, c=0,
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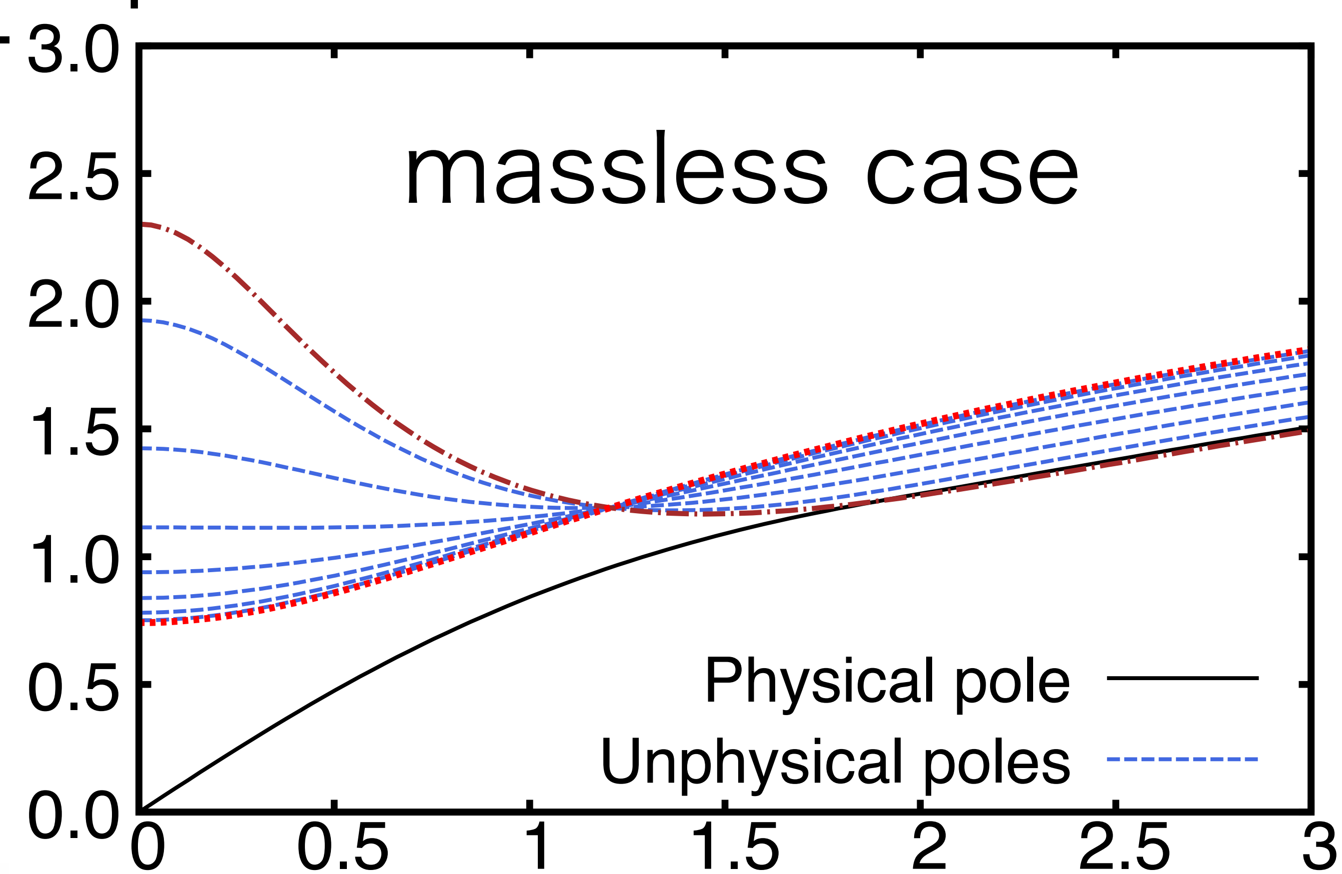
Re $p_4 = 0$
$\vec{p} = \frac{1}{\sqrt{3}}(|\vec{p}|, |\vec{p}|, |\vec{p}|)$

- UPPs
  - $-1 < \cosh \alpha < +1$
  - largely independent of input mass



Boundaries $\cosh\alpha = \pm 1$

$$\cos p_4\big|_{\cosh\alpha=1} = \frac{\sum_{i=1}^{3}\sin^2 p_i + B^2 + 1}{2B},$$

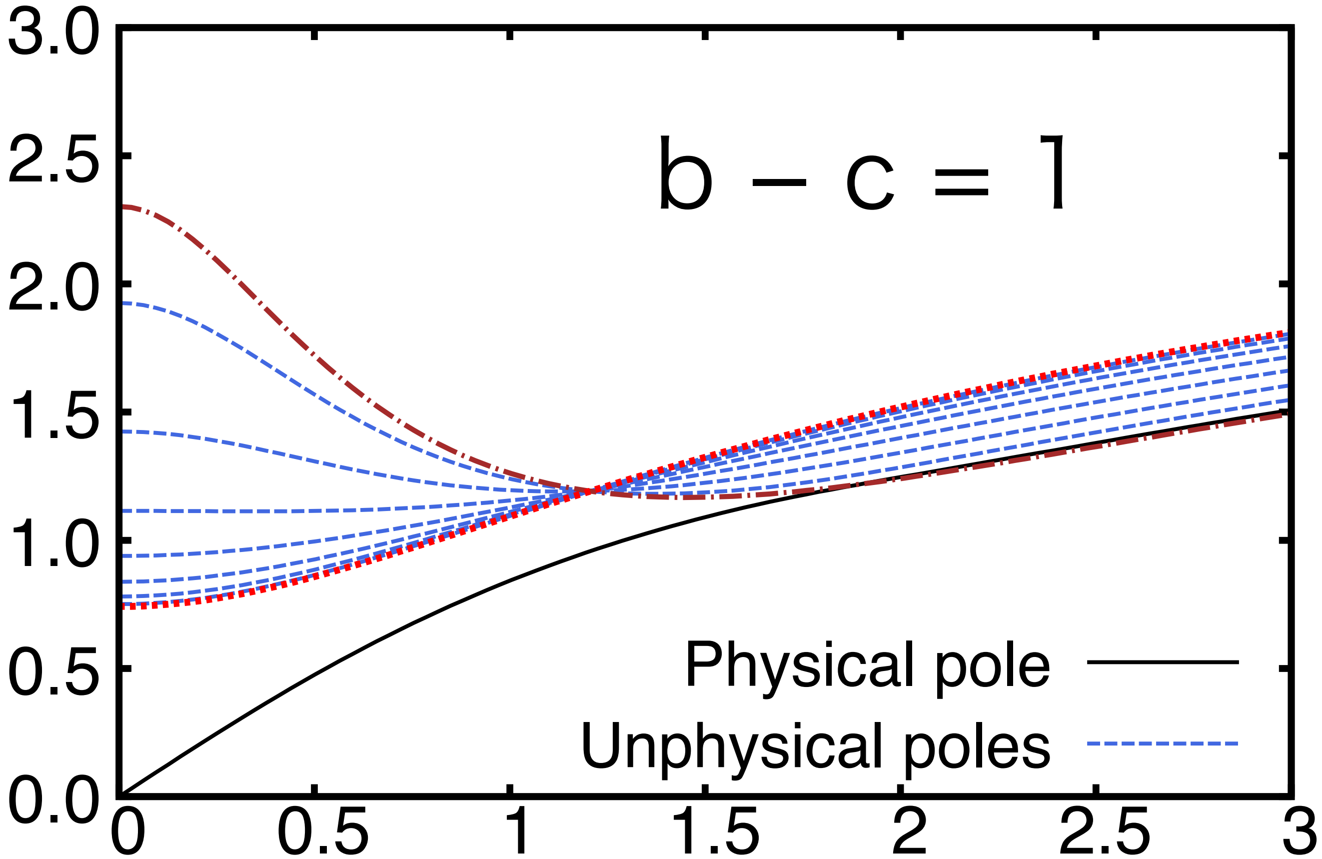
$$\cos p_4\big|_{\cosh\alpha=-1} = \frac{4 + 4(b-c)B + (b-c)^2(\sum_{i=1}^{3}\sin^2 p_i + B^2 + 1)}{4(b-c) + 2(b-c)^2 B},$$

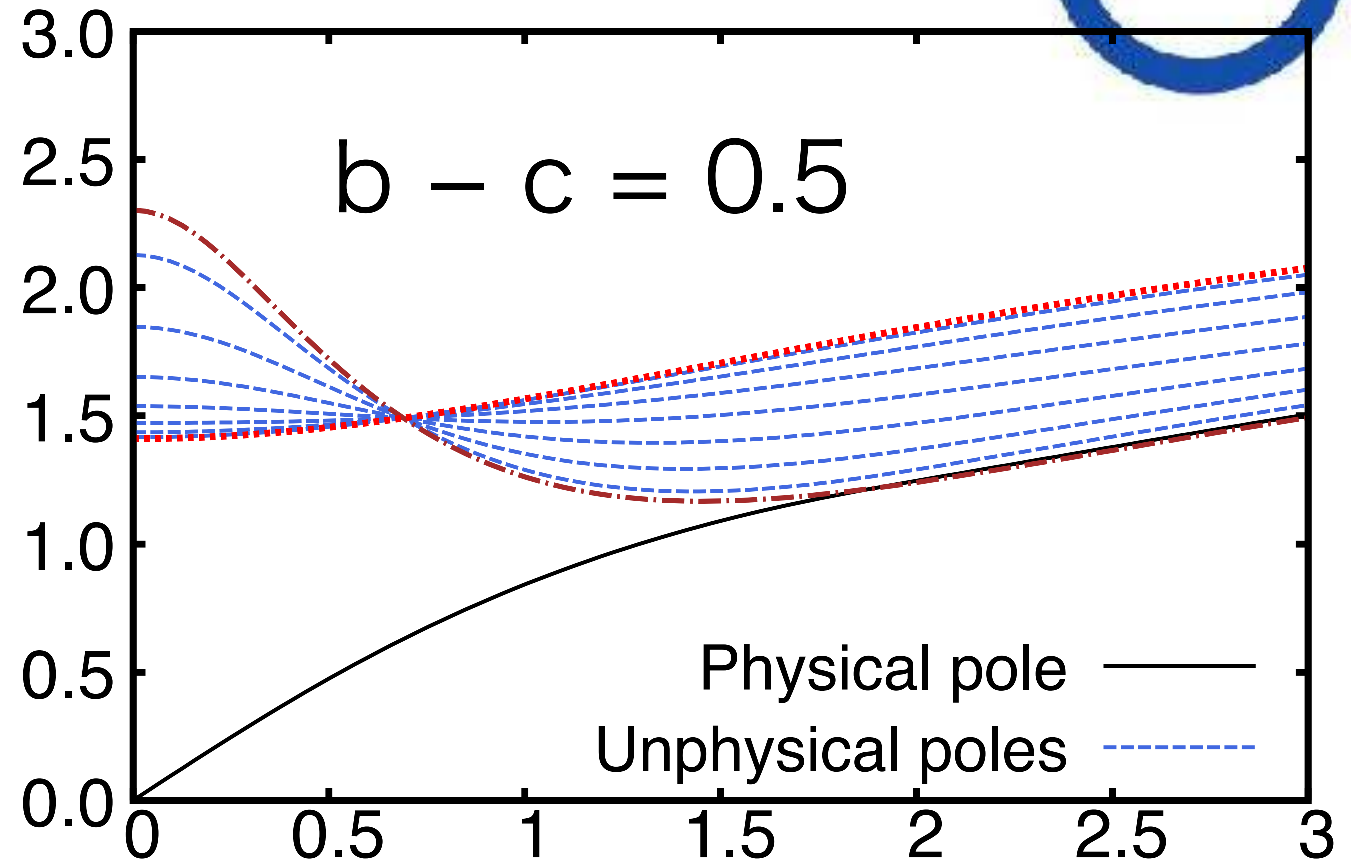
$$B = 4 - M_5 - \sum_{i=1}^{3}\cos p_i$$

- Independent of m, L_s, b+c
- $\cosh\alpha = 1$: depends on M_5
- $\cosh\alpha = -1$: M_5 & b – c
  - b – c: usually fixed to 1

    ⇒ other b-c reduces the effects of UPPs?

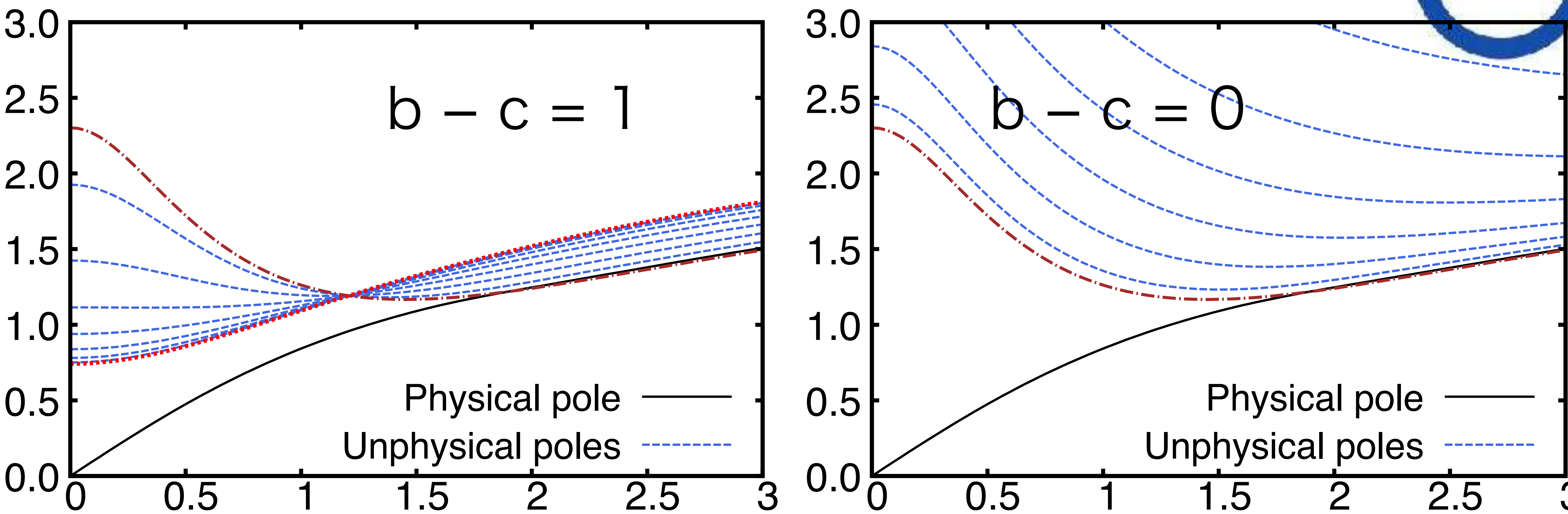
Dependence on b − c





- One boundary (cosh α = -1) increased by decreasing b – c
- Unphysical effects likely small at small b – c

Dependence on b − c



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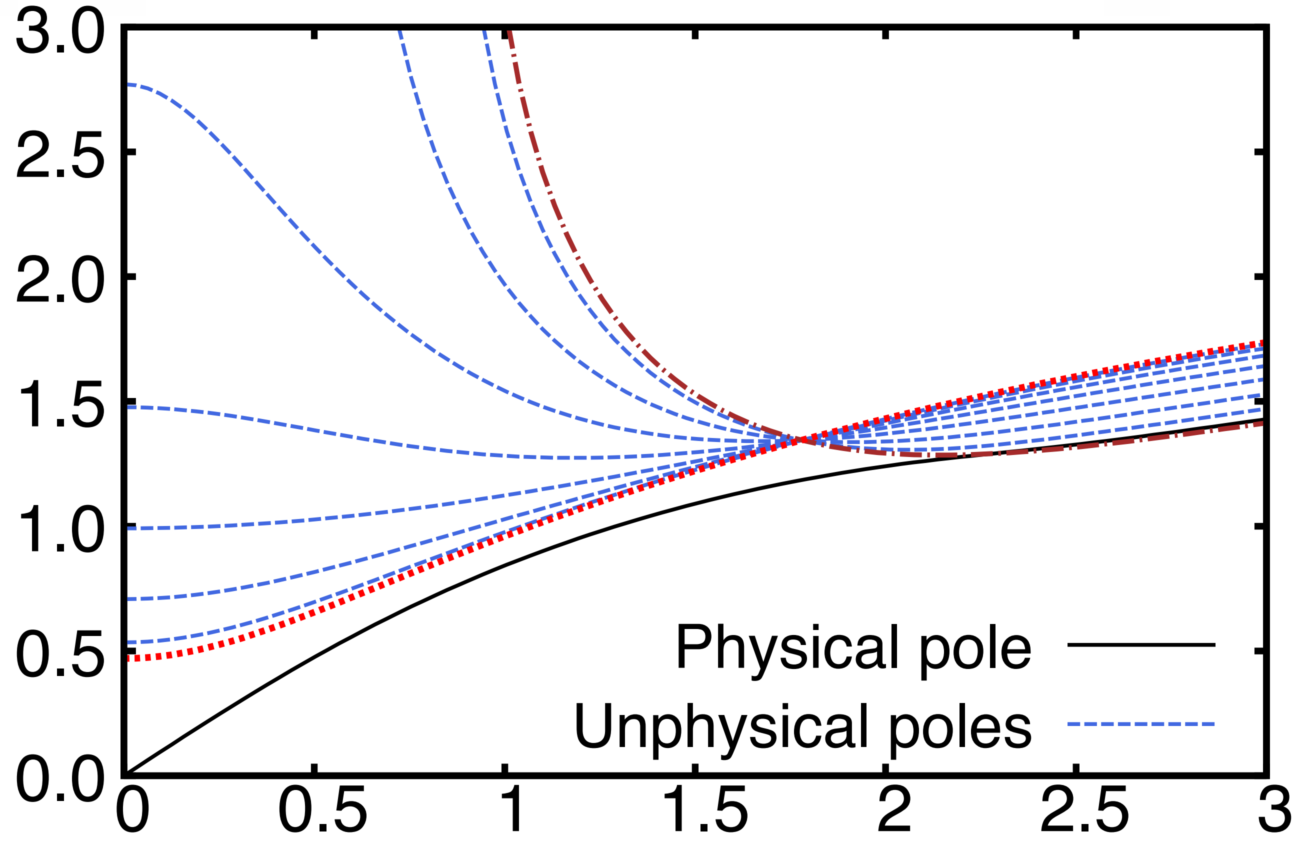
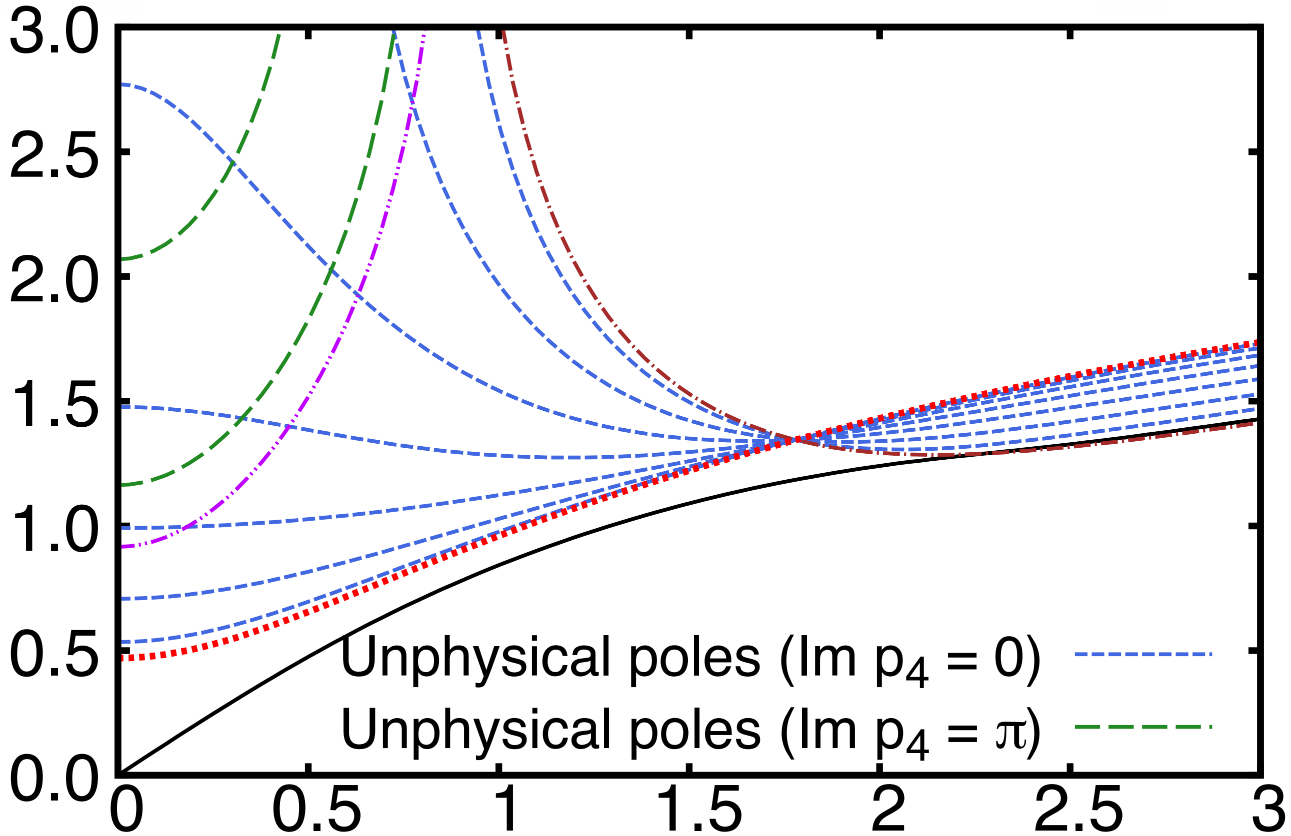
UPPs in $M_5 > 1$

$L_s = 8$,
b=1, c=0,
$M_5 = 1.4$,
Re $p_4 = 0$



UPPs in $M_5 > 1$

$L_s = 8$,
b=1, c=0,
$M_5 = 1.4$,
Re $p_4 = 0,\ \pi$

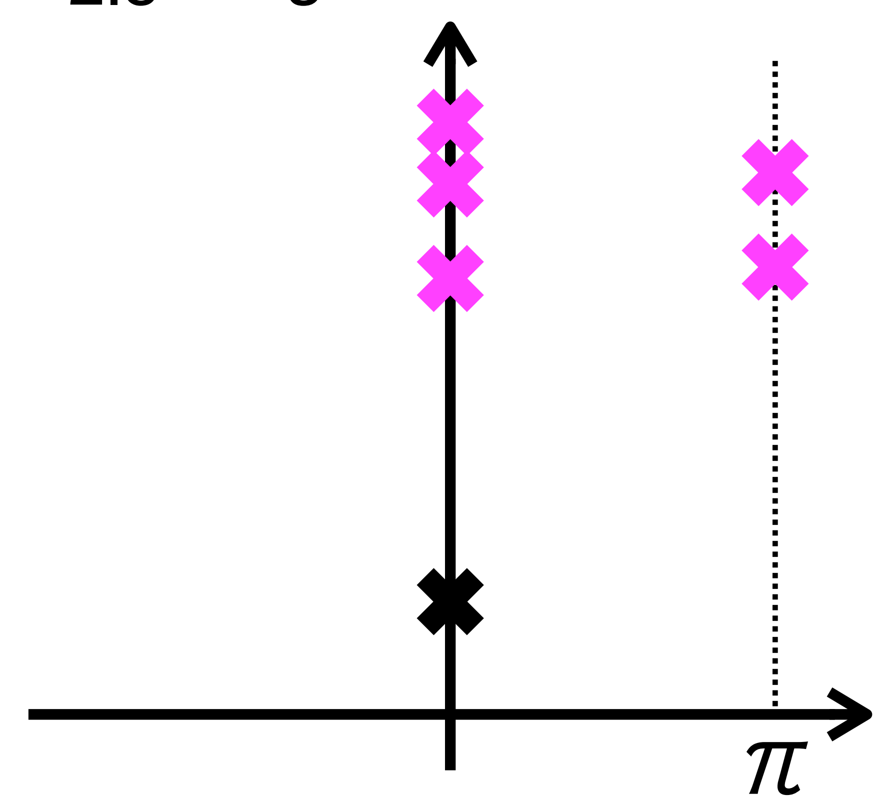


- $\cos p_4\big|_{\cosh\alpha=1} < -1$ at small $|\vec{p}|$

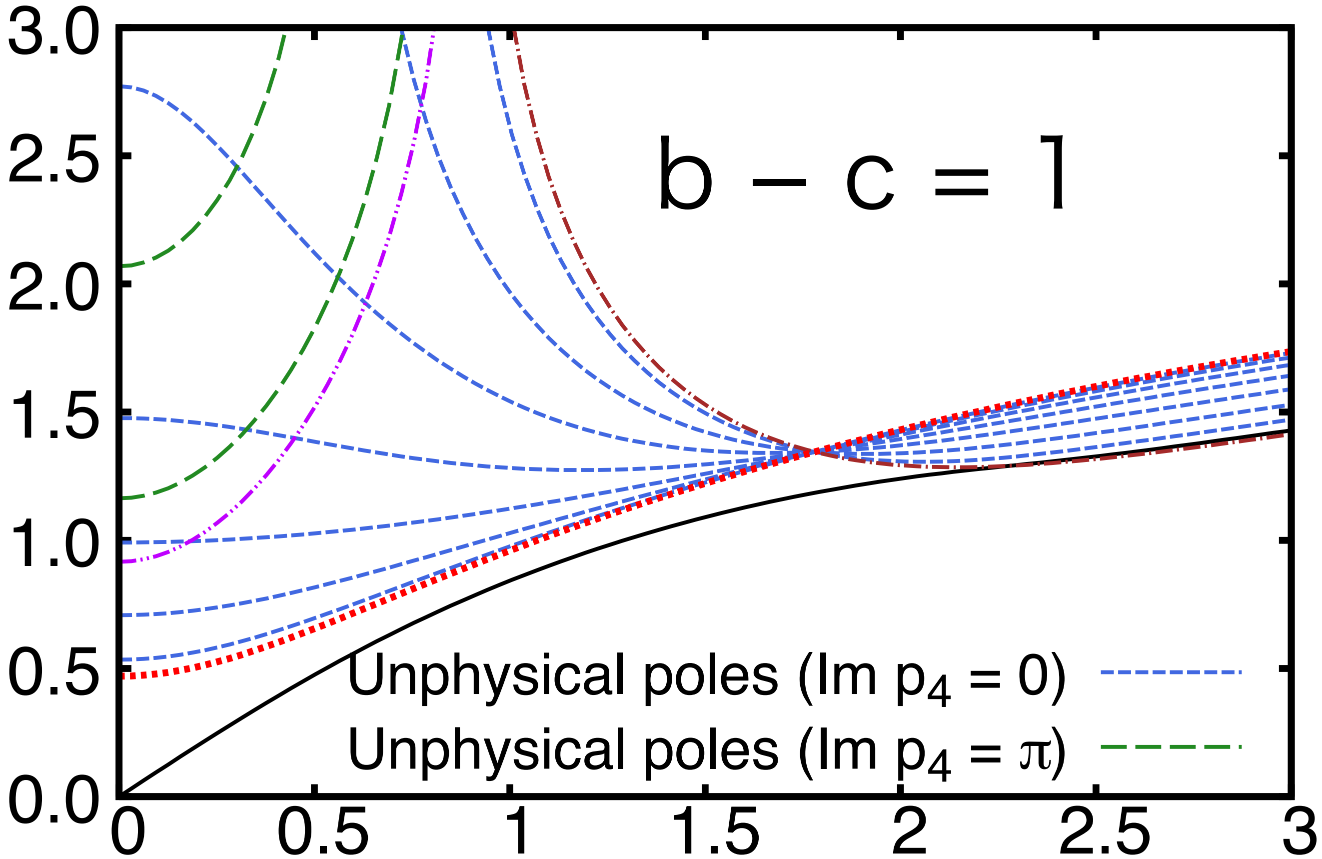
  $\Rightarrow \mathrm{Re}\, p_4\big|_{\cosh\alpha=1} = \pi$

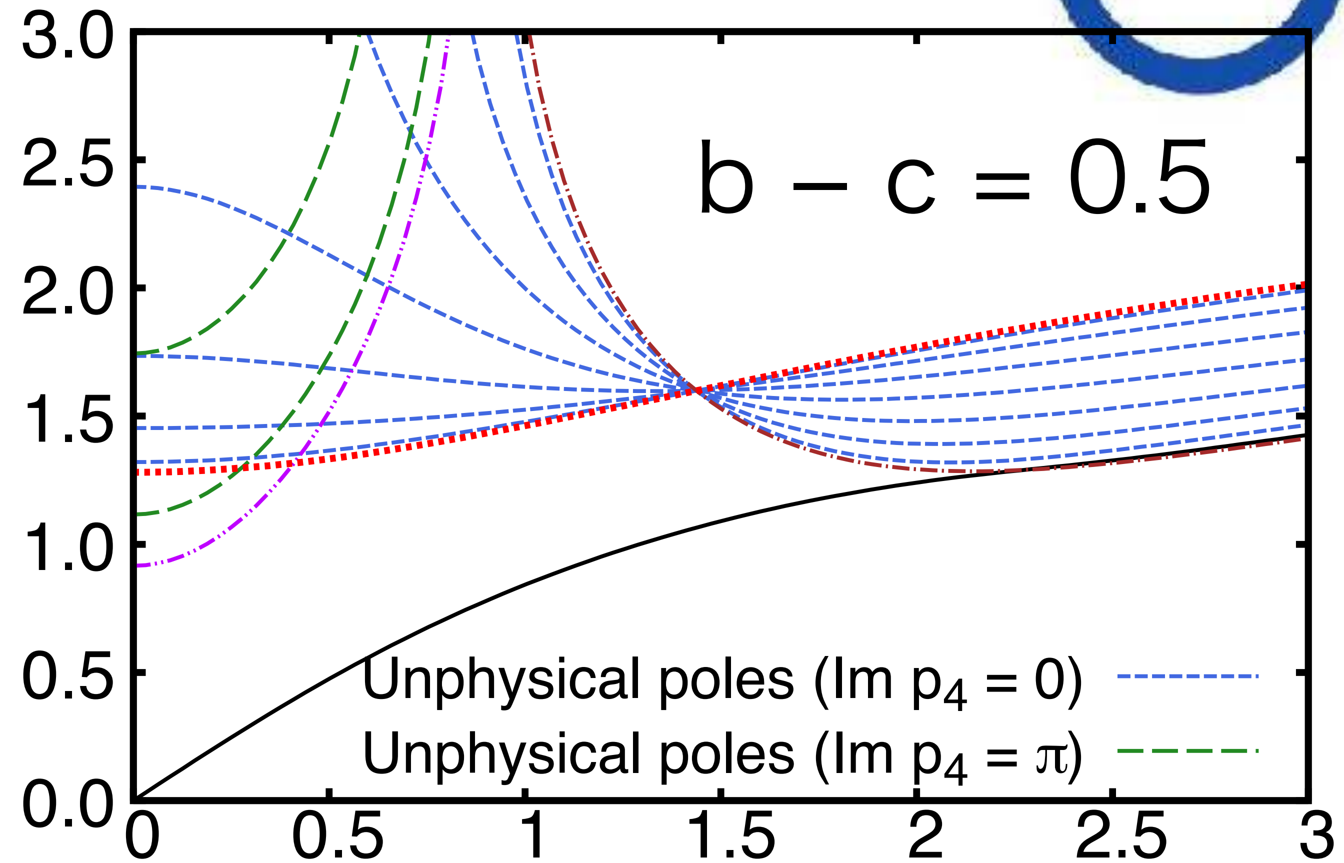
  Non-zero real part

- $-1 < \cosh\alpha < 1 \Rightarrow$ Re $p_4 = 0$ or π



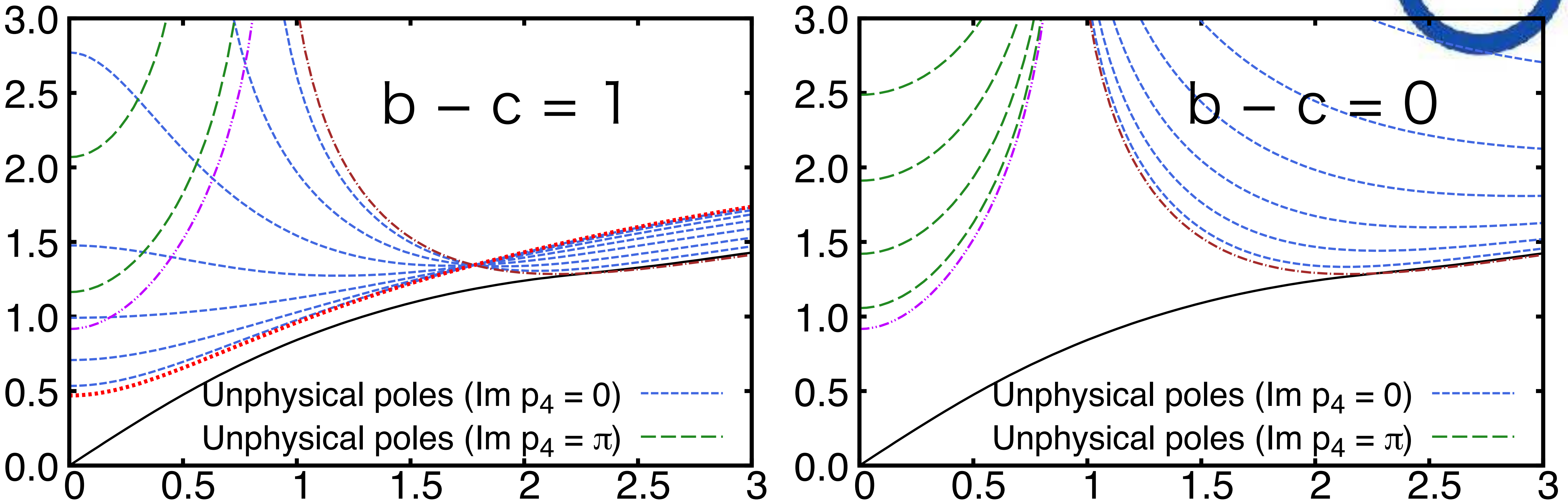
Dependence on b − c





- One bound ($\cosh \alpha = -1$) increased by decreasing b – c
- Unphysical effects likely small at small b – c

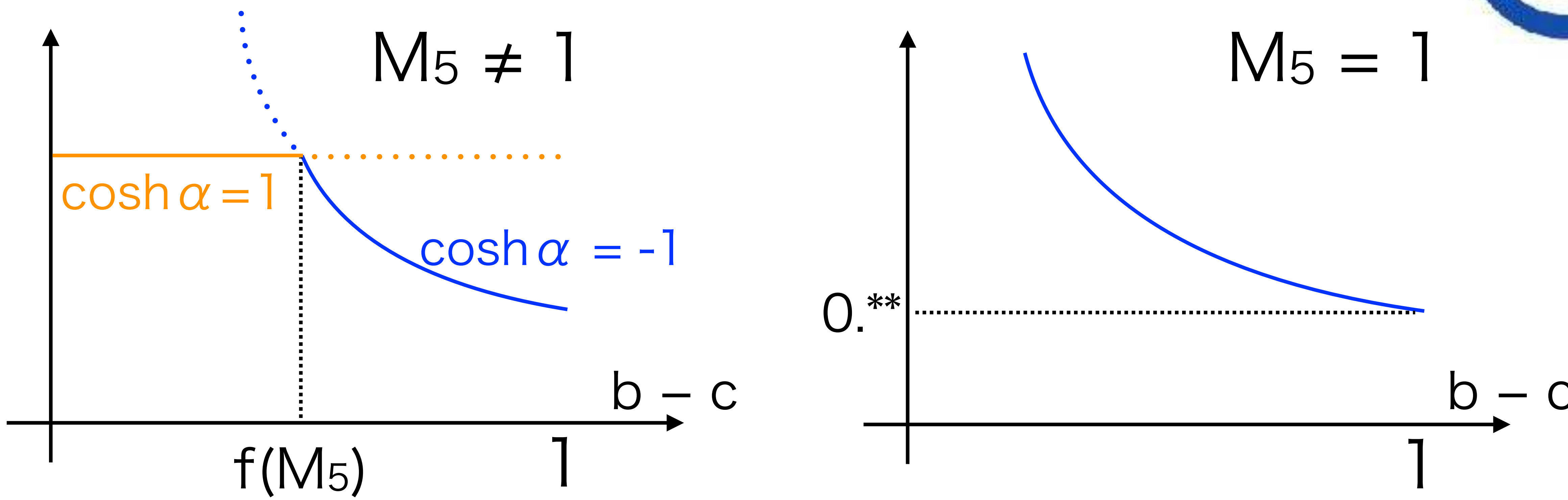
Dependence on b − c



- One bound (cosh α = -1) increased by decreasing b – c

- Unphysical effects likely small at small b – c

Lower bound on UPP masses V.S. b − c



- M_5 also important
- Mean-field approx:

$$M_5 = 1 \rightarrow M_5 = 4 - 3u_0, \quad b \rightarrow u_0 b,\ c \rightarrow u_0 c$$

Summary & comment

- DWFs have L_s – 1 UPPs
- UPP masses basically depend on b – c & M_5
  - So far b – c has been usually fixed to 1
  - Discretization error of large mass could be reduced by using smaller b – c
  - Non-perturbative test on-going

- Possible weak point of small b – c — chiral symmetry
  - Möbius kernel

$$H_M = \gamma_5 \frac{(b+c)D_W}{2+(b-c)D_W}$$

  - Small b – c sets a large upper limit on λ_M
    ⇒ sign function may become worse