Lattice simulation with the Majorana positivity

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Majorana Positivity

fermion sign problem

- QCD at nonzero density
- spin-imbalanced Fermi gas
- repulsive Hubbard model

sign-problem-free class

- QCD at nonzero isospin density
- spin-balanced Fermi gas
- attractive Hubbard model

Majorana Positivity

Dirac (complex) fermion action: $S_\psi = \int d^d x \psi^\dagger D \psi$

$$Z = \int D\psi^\dagger D\psi DA \; e^{-S_\psi - S_A} = \int DA \; \det D \; e^{-S_A}$$

$$\boxed{\det D \geq 0}$$

Majorana Positivity

Majorana (real) fermion action: $$S_\Psi = \int d^d x \frac{1}{2} \Psi^\top P \Psi$$

$$Z = \int D\Psi DA \, e^{-S_\Psi - S_A} = \int DA \, \mathrm{pf}P \, e^{-S_A}$$

$$\boxed{\mathrm{pf}P \geq 0}$$

Majorana Positivity

Li et al. (2015) Wei et al. (2016)

"Majorana positivity"

$$\mathrm{pf}P \geq 0$$

Majorana (real) fermion

$$\Psi$$

Dirac (complex) fermion

$$\psi = \frac{1}{\sqrt{2}}(\Psi_1 + i\Psi_2)$$

Majorana Positivity

Wei et al. (2016) AY & Hayata (2017)

sufficient (not necessary) condition

$$P = \begin{pmatrix} P_1 & iP_2 \\ -iP_2^\top & P_3 \end{pmatrix}$$

condition 1: P_2 is semi-positive

condition 2: $P_3 = -P_1^\dagger$ and $P_2 = P_2^\dagger$

Majorana Positivity

if $\mathrm{pf}P \geq 0$,

$$Z = \int DA \; \mathrm{pf}P \; e^{-S_A}$$
$$= \int DA \; (\det P)^{\frac{1}{2}} \; e^{-S_A}$$
$$= \int DA \; (\det PP^{\dagger})^{\frac{1}{4}} \; e^{-S_A}$$

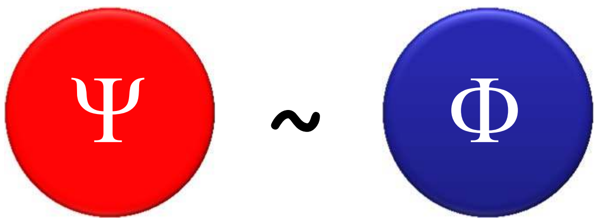
$\longrightarrow$ standard simulation by Hybrid Monte Carlo

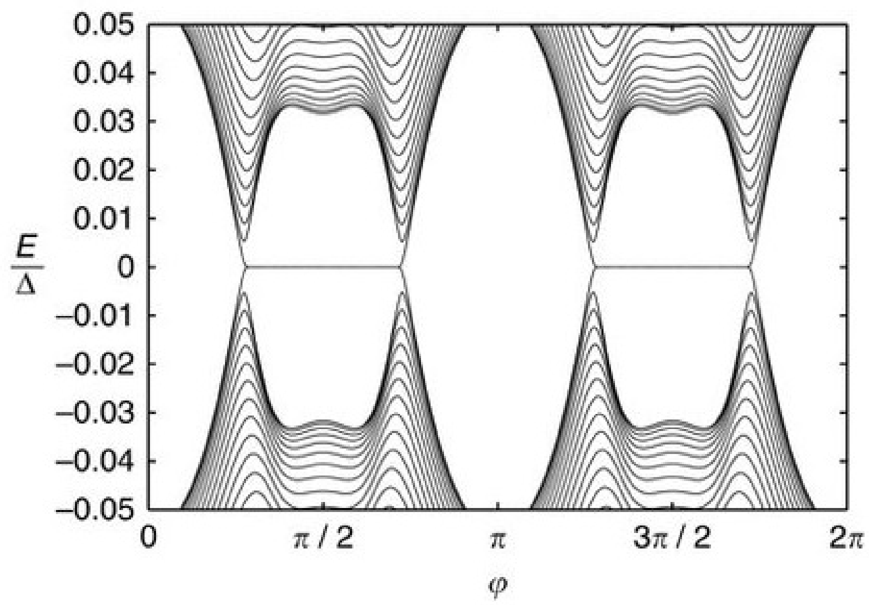
Application to Majorana Fermions

Li et al. (2016)

supersymmetry

Majorana zero mode





Application to Majorana Fermions

Majorana $$\Psi = \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix}$$

$$S = \int d^2x \left[\frac{1}{2} \Psi^\top (\partial_\tau + i\sigma_3 \partial_x + m\sigma_2) \Psi - g(\Psi_\uparrow \partial_x \Psi_\uparrow)(\Psi_\downarrow \partial_x \Psi_\downarrow) \right]$$

Application to Majorana Fermions

Majorana $\Psi = \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix}$

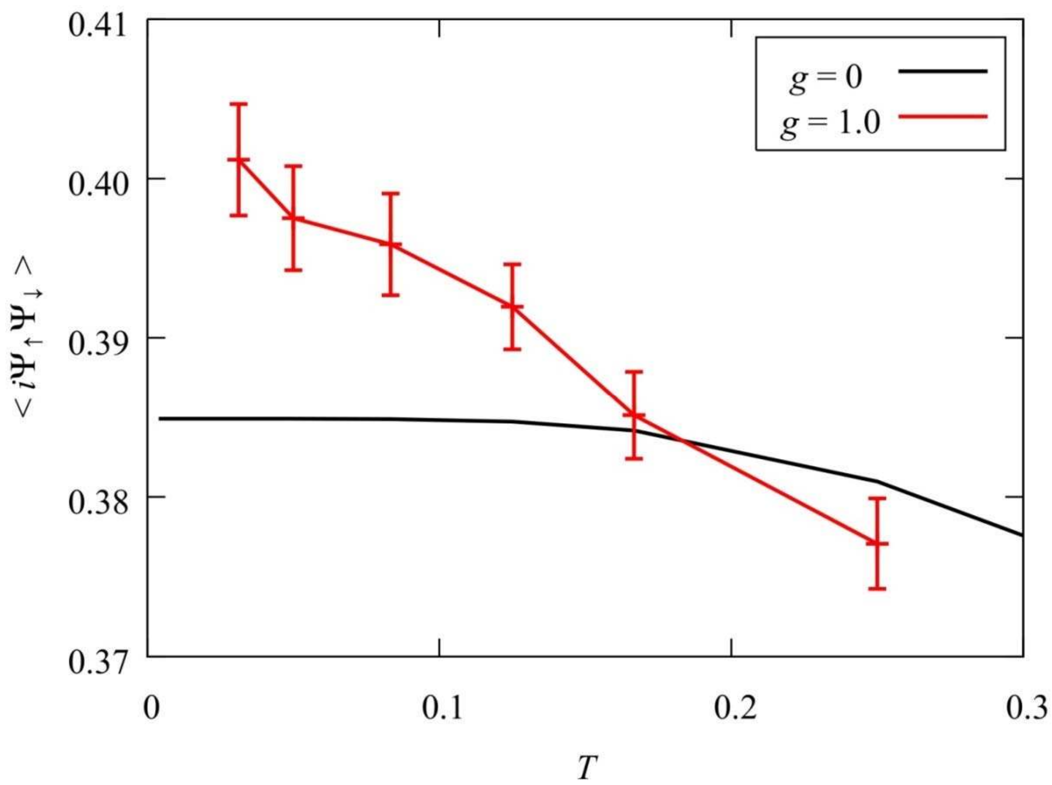
$$S = \int d^2x \left[\frac{1}{2} \Psi^\top (\partial_\tau + i\sigma_3 \partial_x + m\sigma_2) \Psi - \underline{g(\Psi_\uparrow \partial_x \Psi_\uparrow)(\Psi_\downarrow \partial_x \Psi_\downarrow)} \right]$$

Hubbard-Stratonovich

$$S_\Psi = \int d^2x \frac{1}{2} \Psi^\top \left\{ \partial_\tau + (i\sigma_3 + A) \partial_x + m\sigma_2 \right\} \Psi$$

Application to Majorana Fermions

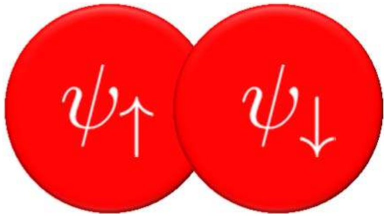
Majorana condensate $\langle i\Psi_{\uparrow}\Psi_{\downarrow}\rangle$



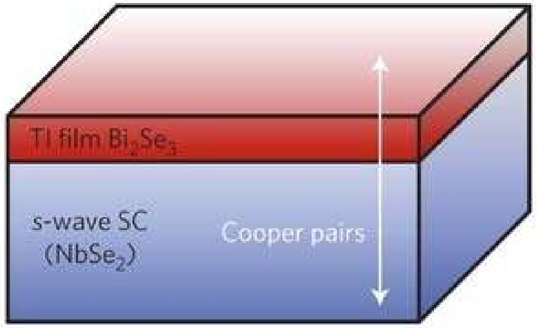
Application to Dirac Fermions

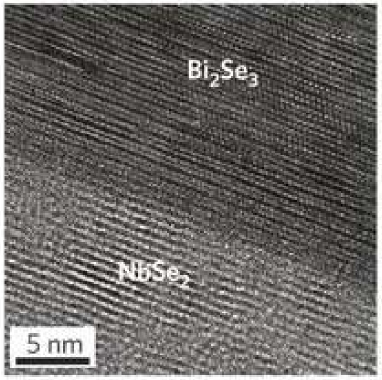
Xu et al. (2014)

diquark source



proximity effect





Application to Dirac Fermions

Dirac ψ & ψ_c $\longrightarrow$ Majorana $\begin{pmatrix} \Psi_{\uparrow 1} \\ \Psi_{\downarrow 2} \\ \Psi_{\uparrow 1} \\ \Psi_{\downarrow 2} \end{pmatrix}$

$$S = \int d^2x \left[\overline{\psi}(\sigma_2 \partial_\tau + \sigma_1 \partial_x + m)\psi - \frac{\Delta}{2}(\overline{\psi}_c \psi + \overline{\psi}\psi_c) + \frac{g}{2}(\psi^\dagger \psi)^2 \right]$$

Application to Dirac Fermions

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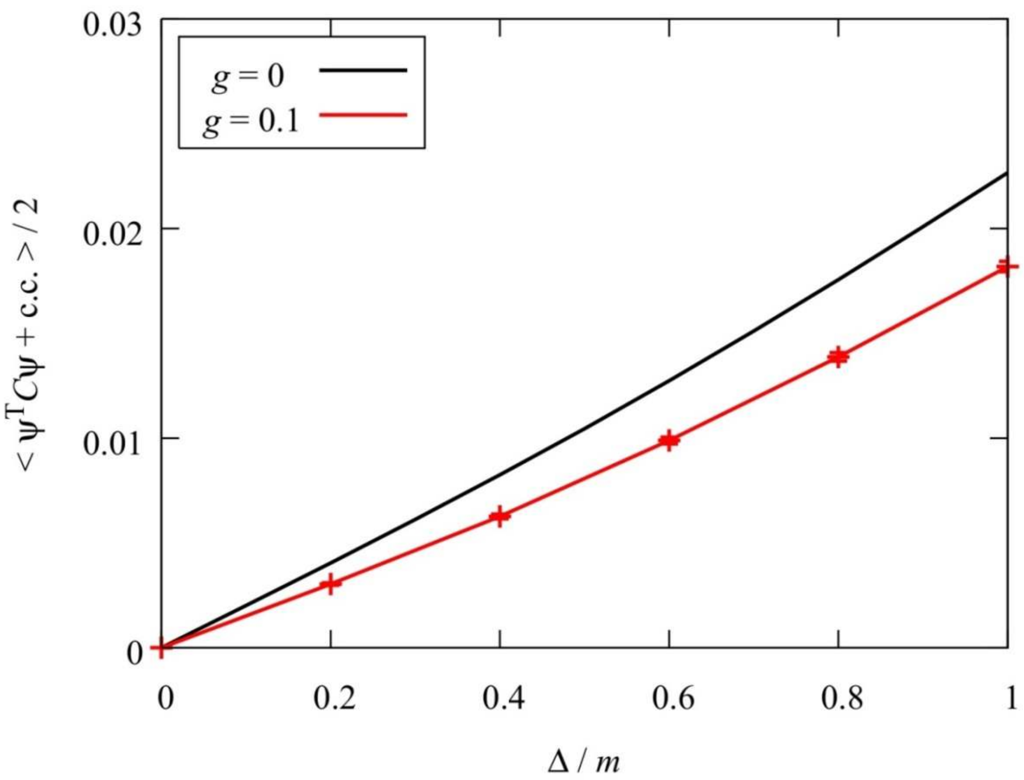
Hubbard-Stratonovich

$$S_\Psi = \int d^2x \left[\overline{\psi}\{\sigma_2(\partial_\tau + iA) + \sigma_1 \partial_x + m\}\psi - \frac{\Delta}{2}(\overline{\psi}_c \psi + \overline{\psi}\psi_c) \right]$$

Application to Dirac Fermions

Cooper pair condensate $\frac{1}{2}\langle \psi^{\top} C \psi + \psi_c^{\top} C \psi_c \rangle$



Application to Dirac Fermions

one-flavor Dirac fermion in 4 dim.

term	condition 1	condition 2	examples
$\bar{\psi} M \psi$	✓	$M = M^\dagger$	Dirac mass
$\bar{\psi}\gamma_5 M_5 \psi$	✓	$M_5 = M_5^\dagger$	
$\bar{\psi}\gamma_\mu D_\mu \psi$		$D_\mu = -D_\mu^\dagger$	gauge field imaginary chemical potential imaginary orbit-rotation coupling
$\bar{\psi}\gamma_\mu\gamma_5 D_{5\mu}\psi$		$D_{5\mu} = D_{5\mu}^\dagger$	imaginary axial gauge field chiral chemical potential imaginary spin-rotation coupling
$\bar{\psi}\gamma_\mu\gamma_\nu\Sigma_{\mu\nu}\psi$	✓	$\Sigma_{\mu\nu} = -\Sigma_{\mu\nu}^\dagger$	
$\bar{\psi}_c m^{\mathrm{S}}\psi + \bar{\psi} m'^{\mathrm{S}}\psi_c$		$m^{\mathrm{S}} = -m'^{\mathrm{S}*}$	Majorana mass
$\bar{\psi}_c\gamma_5 m_5^{\mathrm{S}}\psi + \bar{\psi}\gamma_5 m_5'^{\mathrm{S}}\psi_c$		$m_5^{\mathrm{S}} = -m_5'^{\mathrm{S}*}$	
$\bar{\psi}_c\gamma_\mu d_\mu^{\mathrm{AS}}\psi + \bar{\psi}\gamma_\mu d_\mu'^{\mathrm{AS}}\psi_c$	✓	$d_\mu^{\mathrm{AS}} = -d_\mu'^{\mathrm{AS}*}$	
$\bar{\psi}_c\gamma_\mu\gamma_5 d_{5\mu}^{\mathrm{S}}\psi + \bar{\psi}\gamma_\mu\gamma_5 d_{5\mu}'^{\mathrm{S}}\psi_c$	✓	$d_{5\mu}^{\mathrm{S}} = -d_{5\mu}'^{\mathrm{S}*}$	
$\bar{\psi}_c\gamma_\mu\gamma_\nu\sigma_{\mu\nu}^{\mathrm{AS}}\psi + \bar{\psi}\gamma_\mu\gamma_\nu\sigma_{\mu\nu}'^{\mathrm{AS}}\psi_c$		$\sigma_{\mu\nu}^{\mathrm{AS}} = -\sigma_{\mu\nu}'^{\mathrm{AS}*}$	

Application to Dirac Fermions

two-flavor Dirac fermion in 4 dim.

term	condition 1	condition 2	examples
$\bar{\psi}\begin{pmatrix} M & \widetilde{M} \\ \widetilde{M}' & M^\dagger \end{pmatrix}\psi$	✓	$\widetilde{M} = -\widetilde{M}^\dagger$ $\widetilde{M}' = -\widetilde{M}'^\dagger$	degenerate Dirac mass Wilson term
$\bar{\psi}\gamma_5\begin{pmatrix} M_5 & \widetilde{M}_5 \\ \widetilde{M}'_5 & M_5^\dagger \end{pmatrix}\psi$	✓	$\widetilde{M}_5 = -\widetilde{M}_5^\dagger$ $\widetilde{M}'_5 = -\widetilde{M}_5'^\dagger$	chirally twisted mass
$\bar{\psi}\gamma_\mu\begin{pmatrix} D_\mu & \widetilde{D}_\mu \\ \widetilde{D}'_\mu & -D_\mu^\dagger \end{pmatrix}\psi$	✓	$\widetilde{D}_\mu = \widetilde{D}_\mu^\dagger$ $\widetilde{D}'_\mu = \widetilde{D}_\mu'^\dagger$	gauge field isospin chemical potential isospin electric field
$\bar{\psi}\gamma_\mu\gamma_5\begin{pmatrix} D_{5\mu} & \widetilde{D}_{5\mu} \\ \widetilde{D}'_{5\mu} & D_{5\mu}^\dagger \end{pmatrix}\psi$	✓	$\widetilde{D}_{5\mu} = -\widetilde{D}_{5\mu}^\dagger$ $\widetilde{D}'_{5\mu} = -\widetilde{D}_{5\mu}'^\dagger$	isospin axial gauge field chiral chemical potential
$\bar{\psi}\gamma_\mu\gamma_\nu\begin{pmatrix} \Sigma_{\mu\nu} & \widetilde{\Sigma}_{\mu\nu} \\ \widetilde{\Sigma}'_{\mu\nu} & -\Sigma_{\mu\nu}^\dagger \end{pmatrix}\psi$	✓	$\widetilde{\Sigma}_{\mu\nu} = \widetilde{\Sigma}_{\mu\nu}^\dagger$ $\widetilde{\Sigma}'_{\mu\nu} = \widetilde{\Sigma}_{\mu\nu}'^\dagger$	
$\bar{\psi}_c\begin{pmatrix} m^{\mathrm{S}} & \widetilde{m}^{\mathrm{S}} \\ \widetilde{m}^{\mathrm{S}} & m'^{\mathrm{S}} \end{pmatrix}\psi + \bar{\psi}\begin{pmatrix} m'^{\mathrm{S}\dagger} & -\widetilde{m}^{\mathrm{S}\dagger} \\ -\widetilde{m}^{\mathrm{S}\dagger} & m^{\mathrm{S}\dagger} \end{pmatrix}\psi_c$	✓		
$\bar{\psi}_c\begin{pmatrix} m_5^{\mathrm{S}} & \widetilde{m}_5^{\mathrm{S}} \\ \widetilde{m}_5^{\mathrm{S}} & m_5'^{\mathrm{S}} \end{pmatrix}\psi + \bar{\psi}\begin{pmatrix} m_5'^{\mathrm{S}\dagger} & -\widetilde{m}_5^{\mathrm{S}\dagger} \\ -\widetilde{m}_5^{\mathrm{S}\dagger} & m_5^{\mathrm{S}\dagger} \end{pmatrix}\psi_c$	✓		
$\bar{\psi}_c\gamma_\mu\begin{pmatrix} d_\mu^{\mathrm{AS}} & \widetilde{d}_\mu^{\mathrm{AS}} \\ -\widetilde{d}_\mu^{\mathrm{AS}} & d_\mu'^{\mathrm{AS}} \end{pmatrix}\psi + \bar{\psi}\gamma_\mu\begin{pmatrix} -d_\mu'^{\mathrm{AS}\dagger} & \widetilde{d}_\mu^{\mathrm{AS}\dagger} \\ -\widetilde{d}_\mu^{\mathrm{AS}\dagger} & -d_\mu^{\mathrm{AS}\dagger} \end{pmatrix}\psi_c$	✓		
$\bar{\psi}_c\gamma_\mu\gamma_5\begin{pmatrix} d_{5\mu}^{\mathrm{AS}} & \widetilde{d}_{5\mu}^{\mathrm{AS}} \\ -\widetilde{d}_{5\mu}^{\mathrm{AS}} & d_{5\mu}'^{\mathrm{AS}} \end{pmatrix}\psi + \bar{\psi}\gamma_\mu\gamma_5\begin{pmatrix} d_{5\mu}'^{\mathrm{AS}\dagger} & -\widetilde{d}_{5\mu}^{\mathrm{AS}\dagger} \\ \widetilde{d}_{5\mu}^{\mathrm{AS}\dagger} & d_{5\mu}^{\mathrm{AS}\dagger} \end{pmatrix}\psi_c$	✓		
$\bar{\psi}_c\gamma_\mu\gamma_\nu\begin{pmatrix} \sigma_{\mu\nu}^{\mathrm{AS}} & \widetilde{\sigma}_{\mu\nu}^{\mathrm{AS}} \\ -\widetilde{\sigma}_{\mu\nu}^{\mathrm{AS}} & \sigma_{\mu\nu}'^{\mathrm{AS}} \end{pmatrix}\psi + \bar{\psi}\gamma_\mu\gamma_\nu\begin{pmatrix} -\sigma_{\mu\nu}'^{\mathrm{AS}\dagger} & \widetilde{\sigma}_{\mu\nu}^{\mathrm{AS}\dagger} \\ -\widetilde{\sigma}_{\mu\nu}^{\mathrm{AS}\dagger} & -\sigma_{\mu\nu}^{\mathrm{AS}\dagger} \end{pmatrix}\psi_c$	✓		

Summary

"Majorana positivity" : sign-problem-free condition of Pfaffian

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- ✓ Majorana (real) fermion

  supersymmetry, Majorana zero mode

- ✓ Dirac (complex) fermion

  diquark source, proximity effect