

Progress on Complex Langevin simulations of a finite density matrix model for QCD

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In collaboration with J. Bloch (Regensburg U.), J. Glesaaen (Swansea U.), J. Verbaarschot (Stony Brook U.)

Stochastic quantization as an alternative

- consider the trivial "QFT" given by the partition function
- $\mathcal{Z} = \int e^{-S(x)} dx$
- in the real Langevin formulation
- $x(t + \delta t) = x(t) - \partial_x S(x(t)) \delta t + \delta \xi$
- stochastic variable $\delta \xi$ with zero mean and variance given by $2\delta t$
- generalization to complex actions [Parisi \(1983\)](#), [Klauder \(1983\)](#)
- $x \rightarrow z = x + iy$
- $z(t + \delta t) = z(t) - \partial_z S(z(t)) \delta t + \delta \xi$
- one can study gauge theories with complex actions [Aarts, James, Seiler, Sexty, Stamatescu, ...](#)

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- proof relating Langevin dynamics to the path integral quantization-no longer holds
- simulations are not guaranteed to converge to "the correct solution"
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Validity criteria

- action S and observables \mathcal{O} should be holomorphic in the complexified variables (up to singularities)
- the probability distribution of the complexified variables z along the CL trajectories should be suppressed close to the singularities of the drift term and of the observable
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- same flavor symmetries with QCD which uniquely determine (in the ϵ -regime)
 - mass dependence of the chiral condensate $\langle \bar{\eta}\eta \rangle = \partial_m \log Z$
 - the baryon number density $\langle \eta^\dagger \eta \rangle = \partial_\mu \log Z$

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The Stephanov Model

- $\mathcal{Z} = \int DW e^{-n\Sigma^2 \text{Tr}WW^\dagger} \det^{N_f} \begin{pmatrix} m & iW + \mu \\ iW^\dagger + \mu & m \end{pmatrix}$

Stephanov (1996) and Halasz, Jackson, Verbaarschot(1997)

- solve via bosonization

- $\mathcal{Z}(m, \mu) = \int d\sigma d\sigma^* e^{-n\sigma^2} (\sigma\sigma^* + m(\sigma + \sigma^*) + m^2 - \mu^2)^n$

- where σ is an $N_f \times N_f$ matrix

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The phase transition

- there is a phase transition separating a phase with zero and non-zero baryon density
- in the chiral limit $\mu_c = 0.527$ for $\mu \in \mathbb{R}$
- we can compute $\Sigma(m, \mu)$ and $n_B(m, \mu)$ and compare it with the Complex Langevin simulation
- first attempts in the Osborn model

$$\mathcal{Z} = \int D[W, W'] e^{-n \Sigma^2 \text{Tr}(WW^\dagger + W'W'^\dagger)} \det^{N_f} \begin{pmatrix} m & iW + \mu W' \\ iW^\dagger + \mu W'^\dagger & m \end{pmatrix}$$

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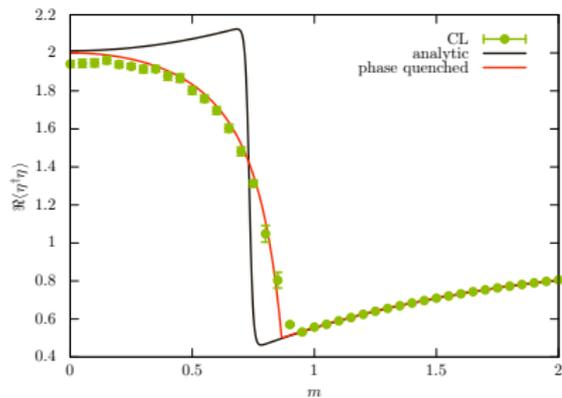
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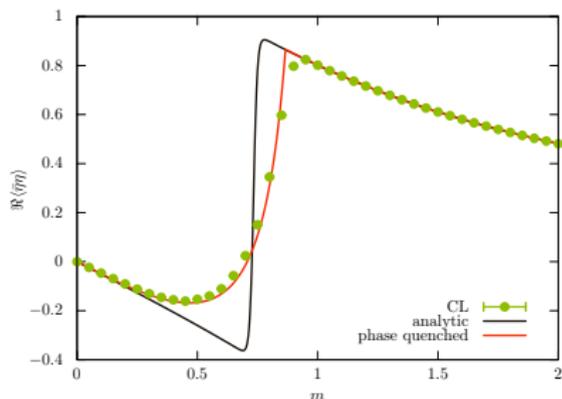
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m -scan for $\mu = 1$



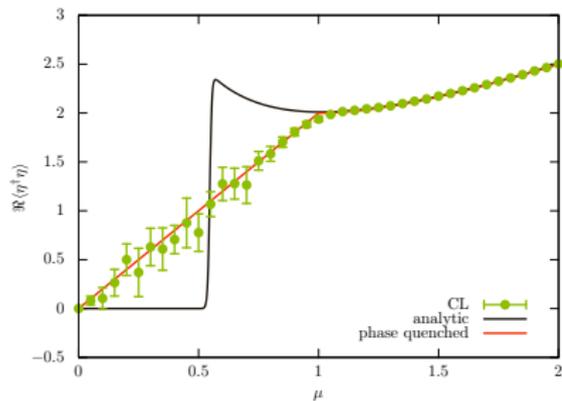
$\langle \eta^\dagger \eta \rangle$ for $\mu = 1$



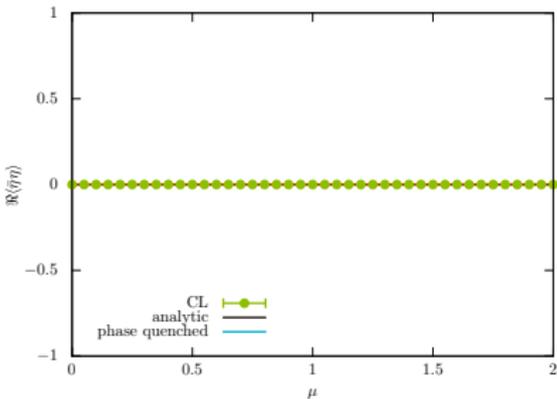
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analytical pq results by Glesaaen, Verbaarschot and SZ (2016)

μ -scan for $m = 0$

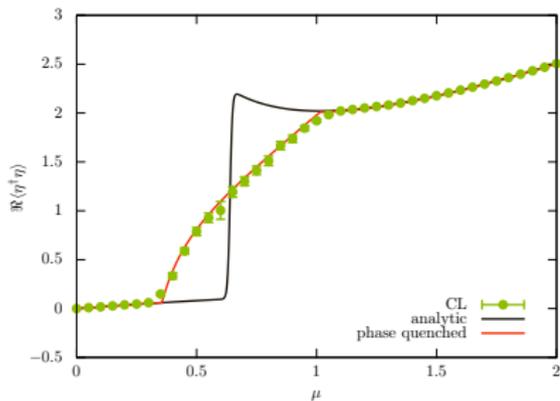


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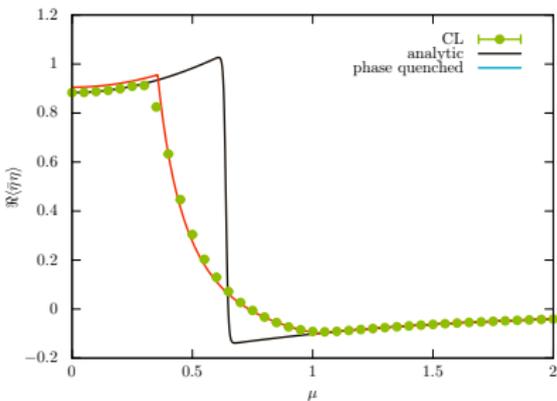


$\langle\bar{\eta}\eta\rangle$ for $m = 0$

μ -scan for $m = 0.2$

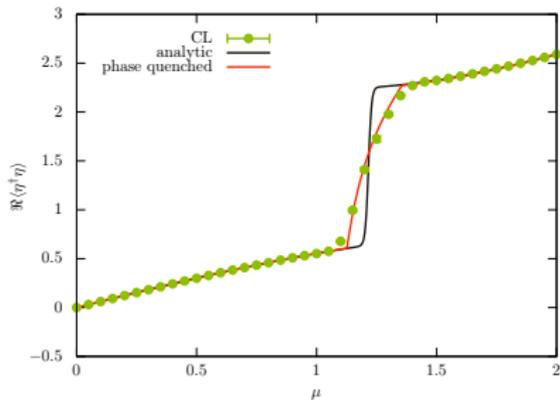


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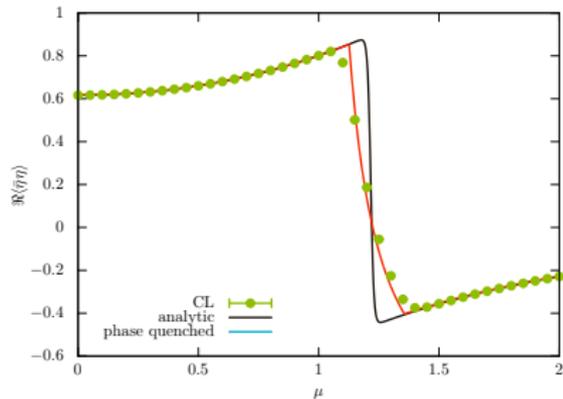


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Gauge Cooling

Originally suggested for QCD by Seiler, Sexty and Stamatescu(2012)

implemented for RMT by Nagata, Nishimura, Shimasaki (2015) in the Osborn model

- complexified action invariant under an enhanced gauge trafo
- steer the evolution in Langevin time towards more physical solutions
- successful application to the Osborn model
- complexified action is invariant $GL(N, \mathbb{C})$
- $A' = \frac{1}{2} \left(hAh^{-1} + (hA^T h^{-1})^T + i(hBh^{-1} - (hB^T h^{-1})^T) \right)$
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The Cooling norms

- $\mathcal{N}_H = \frac{1}{N} \text{tr} \left[(X - Y^\dagger)^\dagger (X - Y^\dagger) \right]$

(N.B. $X(0) = W$ and $Y(0) = W^\dagger$)

- $\mathcal{N}_{AH} = \frac{1}{N} \text{tr} \left[\left((\phi + \psi^\dagger)^\dagger (\phi + \psi^\dagger) \right)^2 \right]$

- ψ and ϕ are the off-diagonal elements of D : $\psi = iY + \mu$,
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- $\mathcal{N}_{\text{ev}} = \sum_{i=1}^{n_{\text{ev}}} e^{-\xi \gamma_i}$

- $\mathcal{N}_{\text{agg}} = (1 - s)N_{AH/\text{ev}} + sN_H$, where $s \in [0, 1]$

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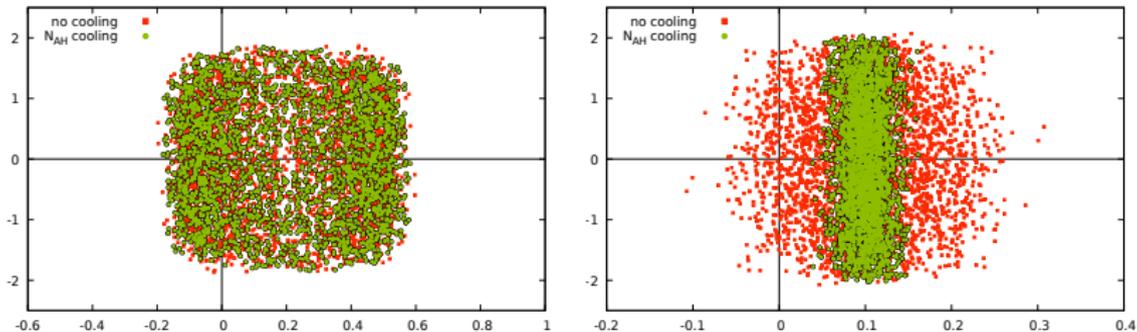
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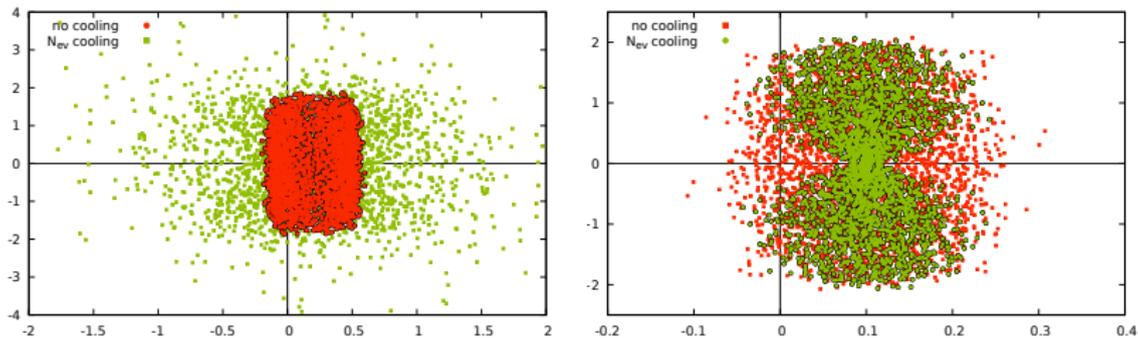
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Cooling and the Dirac Spectrum



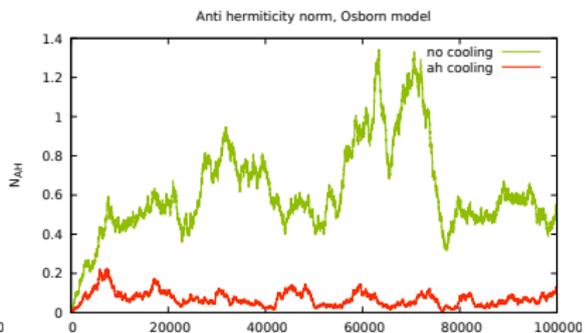
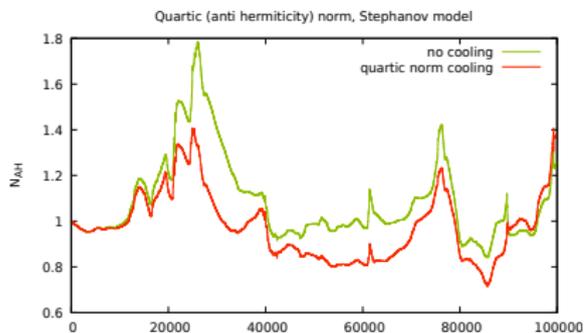
Scatter plots of the eigenvalues of the fermion matrix for a standard CL run together with the ones from a run cooled with the \mathcal{N}_{AH} cooling norm. The plots show the eigenvalues from the last 60 trajectories, separated by 100 updates. The left hand plot shows the Stephanov model, while the Osborn model is shown to the right.

Cooling and the Dirac Spectrum

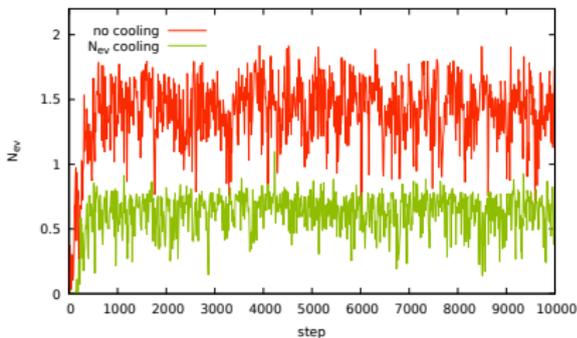
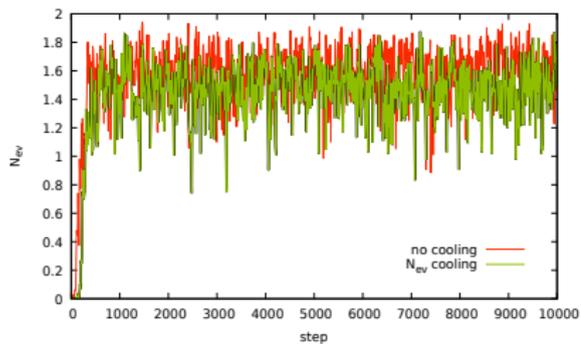


Scatter plots of the eigenvalues of the fermion matrix for a standard CL run together with the ones from a gauge cooled run. We chose the parameters $\{\xi = 100, n_{\text{EV}} = 2\}$ for \mathcal{N}_{EV} . The plots show the eigenvalues from the last 60 trajectories, separated by 100 updates. The left hand plot shows the Stephanov model, while the Osborn model is shown to the right

The anti-hermiticity norm



The eigenvalue norm



Value of N_{ev} as a function of Langevin time. Stephanov model to the left, Osborn model to the right

The shifted representation

- shift μ away from the fermionic term by a COV

- initially $D = \begin{pmatrix} m & iA - B + \mu \\ iA^T + B^T + \mu & m \end{pmatrix}$

- Absorb μ into A with a COV $A' = A - i\mu$. The action in terms of A' and B is

$$S = N \text{tr} (A'^T A' - 2i\mu A' + \mu^2 + B^T B) - N_f \text{tr} \log (m^2 + X' Y')$$

$X' = A' + iB$ and $Y' = A'^T - iB^T$. Now, the μ dependence has shifted from the fermionic to the bosonic term.

- Computing again the CL force term ...
- Advantage of the shifted representation is that it starts in an anti-Hermitian state, and since CL is non-deterministic, the configurations could evolve to a different minimum.

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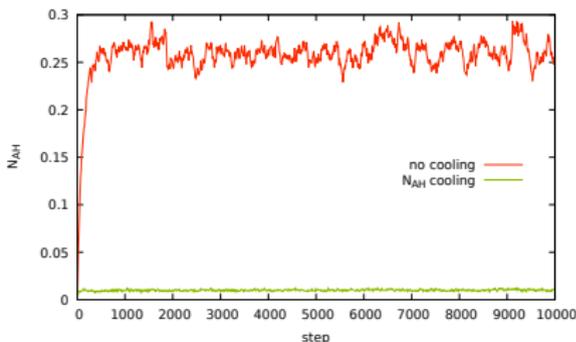
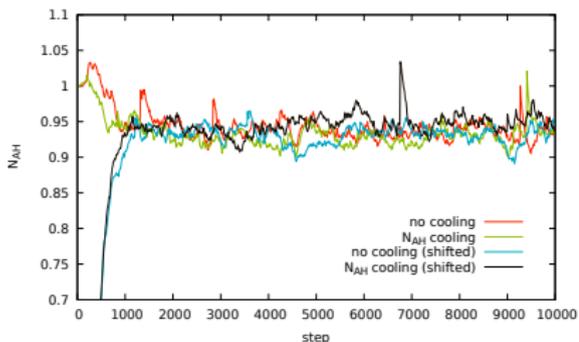
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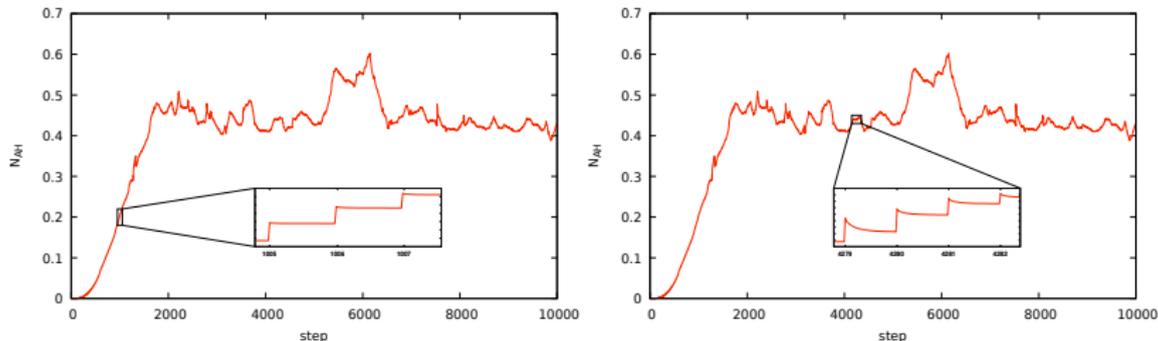
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Shifted representation and cooling



\mathcal{N}_{AH} as a function of Langevin time. Stephanov model (RHS), Osborn model (LHS). The Stephanov plot also includes the values from the shifted representation. These start at 0 for $t = 0$, but quickly shoot up to meet the unshifted curves. Although A and A' start out very different, they coincide after thermalization. This means that $\langle A' \rangle_{\text{CL, shifted}} = \langle A \rangle_{\text{CL, standard}} - i\mu$, and thus they converge to the same solution. **The advantage of the shifted representation is that it starts in an anti-Hermitian state, and due to the fact that CL is non-deterministic, the configurations could evolve to a different minimum.**

Shifted representation and cooling



\mathcal{N}_{AH} for the shifted representation of the Stephanov model as a function of Langevin time for 8×8 block matrices. The zoomed in plots show the evolution of \mathcal{N}_{AH} with the application of the cooling step. There are 50 gauge cooling transformations between each Langevin step.



Reweighted Complex Langevin

- generate a CL trajectory for parameter values where complex Langevin is correct
- perform a reweighting of the trajectory to compute observables in an extended range of the parameters where CL used to fail

Reweighting

- **target** ensemble with parameters $\xi = (m, \mu, \beta)$ (for the general QCD case-drop β for RMT)
- **auxiliary** ensemble with parameters $\xi_0 = (m_0, \mu_0, \beta_0)$
- Reweighting from **auxiliary** to **target**

$$\begin{aligned}\langle O \rangle_\xi &= \frac{\int dx w(x; \xi) \mathcal{O}(x; \xi)}{\int dx w(x; \xi)} = \frac{\int dx w(x; \xi_0) \left[\frac{w(x; \xi)}{w(x; \xi_0)} \mathcal{O}(x; \xi) \right]}{\int dx w(x; \xi_0) \left[\frac{w(x; \xi)}{w(x; \xi_0)} \right]} \\ &= \frac{\langle \frac{w(x; \xi)}{w(x; \xi_0)} \mathcal{O}(x; \xi) \rangle_{\xi_0}}{\langle \frac{w(x; \xi)}{w(x; \xi_0)} \rangle_{\xi_0}}\end{aligned}$$

- but now we have a **complex weight** $w(x; \xi_0) = e^{-S(x; \xi_0)}$ so we need CL for this too!

Reweighting

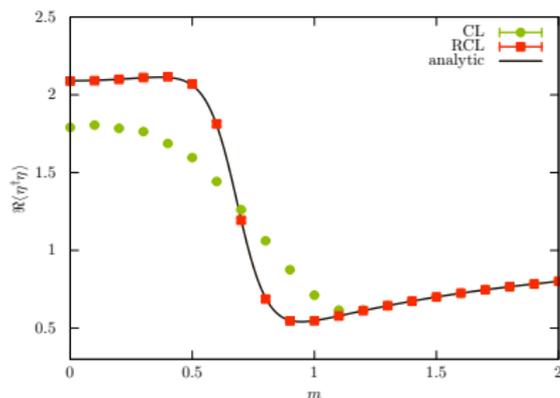
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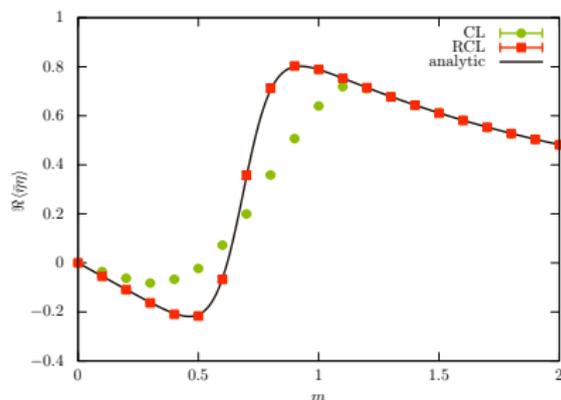
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m -scan for $\mu = 1$

reweighted from an auxiliary ensemble with $m_0 = 4$ and $\mu_0 = 1$
using $\mathcal{O}(15000)$ confs



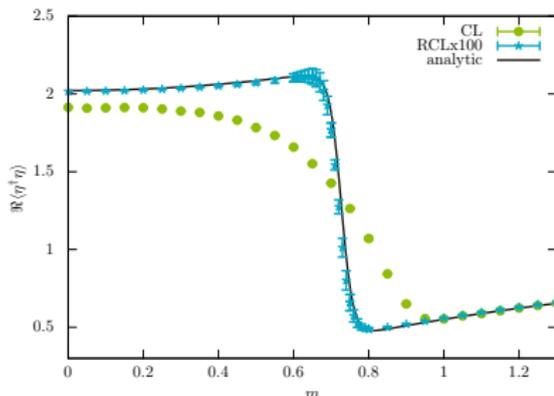
$\langle \eta^\dagger \eta \rangle$ for $\mu = 1$ and $n = 6$



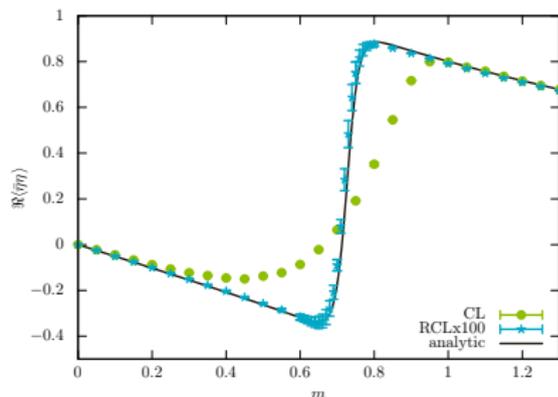
$\langle \bar{\eta} \eta \rangle$ for $\mu = 1$ and $n = 6$

m -scan for $\mu = 1$

Auxiliary ensemble $m_0 = 1.3, \mu_0 = \mu = 1.0$ using $\mathcal{O}(560000)$ confs



$\langle \eta^\dagger \eta \rangle$ for $\mu = 1$ and $n = 24$

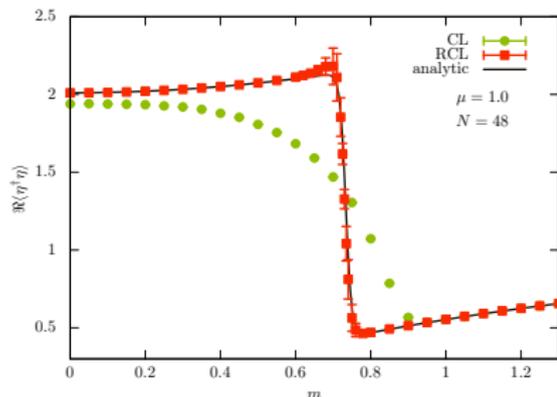


$\langle \bar{\eta} \eta \rangle$ for $\mu = 1$ and $n = 24$

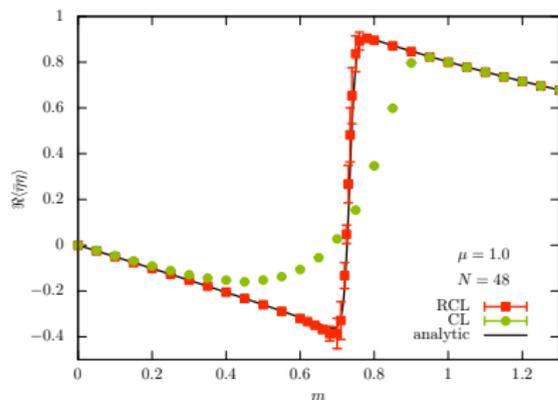
The number of confs needed, builds up very rapidly. So one has to "just" to generate an auxiliary trajectory long enough to overcome the sign problem.

m -scan for $\mu = 1$

Auxiliary ensemble $m_0 = 1.3, \mu_0 = \mu = 1.0$ using $\mathcal{O}(122000000)$ confs



$\langle \eta^\dagger \eta \rangle$ for $\mu = 1$ and $n = 48$



$\langle \bar{\eta} \eta \rangle$ for $\mu = 1$ and $n = 48$

if you (or the computer) work(s) hard enough you can also deal with the matrices of the original size that we were dealing with $n = 48$

Conclusions and outlook

- studied the CL algorithm for an RMT model for QCD/
comparing numerical with exact analytical results for all the
range of parameters (m, μ)
- compared to previous similar studies this model possesses a
phase transition to a phase with non-zero baryon density
- fails to converge to QCD and it converges to $|QCD|$
- partial attempt to fix the issues via RCL procedure \rightarrow correct
results, even when CL itself does not work for the target
ensemble for 2d QCD cf. Bloch 1701.00986
- other standard ways to fix it \rightarrow gauge cooling Seiler, Sexty and
Stamatescu(2012), Nagata, Nishimura, Shimasaki (2015) seem unable to overcome
the pathologies of this model
- *Thanks a lot for your attention!*

Stay Tuned!



for upcoming results ...

The analytical solution

Halasz, Jackson, Verbaarschot(1997)

$$\int \mathcal{D}[\dots] \exp \left[-i \sum_{k=1}^{N_f} \psi^{k*} \begin{pmatrix} m & iW + \mu \\ iW^\dagger + \mu & m \end{pmatrix} \psi^k \right] e^{-n\Sigma^2 \text{Tr} W W^\dagger}$$

perform the W integration

$$\int \mathcal{D}[\dots] e \left[-\frac{1}{n\Sigma^2} \psi_{Lk}^{f*} \psi_{Ri}^f \psi_{Ri}^{g*} \psi_{Lk}^g + m (\psi_{Ri}^{f*} \psi_{Ri}^f + \psi_{Lk}^{f*} \psi_{Lk}^g) + \mu (\psi_{Ri}^{f*} \psi_{Li}^f + \psi_{Lk}^{f*} \psi_{Rk}^g) \right]$$

write the four-fermion term as a difference of two squares and

linearize via a **Hubbard-Stratonovich trafo**

$$\exp(-AQ^2) \sim \int d\sigma \exp\left(-\frac{\sigma^2}{4A} - iQ\sigma\right)$$

The analytical solution

then one carries out the Grassmann integrals

$$Z(m, \mu) = \int \mathcal{D}\sigma \exp[-n\Sigma^2 \text{Tr}\sigma\sigma^\dagger] \det^n \begin{pmatrix} \sigma + m & \mu \\ \mu & \sigma^\dagger + m \end{pmatrix}$$

which for **one-flavor** and $\Sigma = 1$ becomes

$$\mathcal{Z}^{N_f=1}(m, \mu) = \int d\sigma d\sigma^* e^{-n\sigma^2} (\sigma\sigma^* + m(\sigma + \sigma^*) + m^2 - \mu^2)^n$$

The analytical solution

in polar coordinates after the angular integration

$\mathcal{Z}^{N_f=1}(m, \mu) = \pi e^{-nm^2} \int_0^\infty du (u - \mu^2)^n I_0(2mn\sqrt{u}) e^{-nu}$ in the thermodynamic limit one can perform a **saddle point analysis**

$$I_0(z) \sim e^z / \sqrt{2\pi z}$$

and the saddle point equation takes the form

$$\frac{1}{u - \mu^2} = 1 - \frac{m}{\sqrt{u}}$$

The analytical solution

A 1st order phase transition takes place at the points where $|Z_{u=u_b}| = |Z_{u=u_r}|$, with u_b and u_r two different solutions of the saddle-point equation give the same free-energy. This condition can be rewritten as $|(u_b - \mu^2)e^{2m\sqrt{u_b}-u_b}| = |(\mu^2 - u_r)e^{2m\sqrt{u_r}-u_r}|$. A general solution is quite cumbersome, but for $m \rightarrow 0$ we find that $u_r = 0$ and $u_b = 1 + \mu^2$. This leads to the critical curve $\text{Re} [1 + \mu^2 + \log \mu^2] = 0$ which for **real** μ , $\mu_c = 0.527\dots$

Phase Quenched QCD

- ignore the complex phase of the fermion determinant
- $\mathcal{Z}_{pq} = \mathcal{Z}_{iso} = \int dA |\det(D(\mu))|^2 e^{-S_g}$
- $|\det(D(\mu))|^2 = \det D(\mu) (\det(D(\mu)))^* = \det D(\mu) \det D(-\mu)$
- since $\gamma_5(\gamma_\mu D^\mu + m - \mu\gamma_0)\gamma_5 = (\gamma_\mu D^\mu + m + \mu\gamma_0)^\dagger$
- $\langle O \rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int dA O |\det D(\mu)|^2 e^{-S_g}$
- this theory has a different phase diagram and different properties than QCD
- for $T \ll$ and $\mu \gg \rightarrow$ Bose condensation of charged pions

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