DUALS OF $U(N)$ LGT WITH STAGGERED FERMIONS

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I. Duals of lattice gauge models

- Dual representations based on the plaquette formulation. Dual variables are introduced as variables conjugate to local Bianchi identities. The dual model is non-local due to the presence of connectors
- Dual representations based on 1) the character expansion of the Boltzmann weight and 2) the integration over link variables using Clebsch-Gordan expansion

$$Z = \sum_{r_p, r_l} \prod_p C_{r_p}(\beta_{\mu\nu}) \prod_x (6j \text{ links}) \prod_{\text{c}} (\text{6j cubes})$$

- In the strong coupling limit the model can be mapped onto monomer-dimer-closed baryon loop model for $SU(N)$
- Other approaches: n-link action, abelian colour cycles, $\cdots$
  Remark: abelian *vs* non-abelian dual representations

II. $U(N)$ group integrals

Consider the Taylor expansion of the Boltzmann weight

$$\exp\beta\,\mathrm{Re}\,\chi_{\mathrm{f}}(\mathrm{U_p}) \;=\; \sum_{n_1,n_2=0}^{\infty} \frac{(\beta/2)^{n_1+n_2}}{n_1!n_2!}\,(\chi_f(U_p))^{n_1}(\chi_f(U_p^{\dagger}))^{n_1}$$

One link integral becomes

$$\mathcal{I}_N(r_1,r_2) \;=\; \int dU\,\prod_{k=1}^{r_1} U^{i_k j_k}\,\prod_{n=1}^{r_2} U^{m_n l_n *}$$

Only matrices in fundamental representation appear in the integrand.

(B. Collins, 2003, B.Collins et.al. 2003-2017)

$$\mathcal{I}_N(r_1,r_2) = \delta_{r_1,r_2}\sum_{\tau,\sigma\in S_r} Wg^N(\tau^{-1}\sigma)\prod_{k=1}^{r}\delta_{i_k,m_{\sigma(k)}}\delta_{j_k,l_{\tau(k)}}.$$

S_r - group of permutations of r elements. $Wg^N(\sigma)$ - Weingarten function which depends only on the length of cycles of σ.

$$Wg^N(\sigma) = \frac{1}{(r!)^2} \sum_\lambda \frac{d^2(\lambda)}{s_\lambda(1^N)} \chi_\lambda(\sigma) \; , \; \lambda_1 \geq \cdots \geq \lambda_N \geq 0$$

$d(\lambda), \chi_\lambda(\sigma)$ - dimension and character of S_r, $s_\lambda(1^N)$ - the Schur function.

Example: Simple integral

$$\int dU \; (TrU)^s \; (TrU^*)^q \; = \; \delta_{s,q} \sum_\lambda d^2(\lambda) \; .$$

Example: $2D$ model in the large volume limit

$$Z = [z_0]^{N_p} \; , \; z_0 = \det I_{i-j}(\beta), \; 1 \leq i,j \leq N \; .$$

$$z_0 = \sum_{k=0}^{\infty} \frac{(\frac{\beta}{2})^{2k}}{(k!)^2} \sum_\lambda d^2(\lambda), \; \lambda_1 \geq \cdots \geq \lambda_N, \; \sum_{i=1}^{N} \lambda_i = k$$

Example: One-link integral in the strong coupling limit

$$\sum_{k=0}^{N} (\eta_\nu/2)^{2k} \frac{1}{k!} (\sigma_x \sigma_{x+e_\nu})^k \sum_{\tau \in S_k} (-1)^{|\tau|} Wg^N(\tau)$$

1. Properties of $Wg^N(\sigma)$ - general form, recurrence relations, large-N asymptotic expansion, bounds - are known.

2. In abelian case one recovers the conventional dual model.

3. The constraint $r_1 = r_2$ is essentialy abelian one. One solves the constraint by introducing genuine dual variables, like in $U(1)$ model. No other constraints are generated.

4. For $U(N)$, in any dimension, summation over group (matrix) indices is factorized in every lattice site and can be done locally. Dual theory is a theory with only local interaction.

5. Last property holds also in the presence of fermions.

III. Pure gauge models

$3D$ model

Summation over matrix indices and duality transformations lead to the following representation on the dual lattice

$$Z = \sum_{\{r_x, k_l\}} \prod_l \frac{\beta^{2k_l + |r_x - r_{x+e_n}|}}{k_l!(k_l + |r_x - r_{x+e_n}|)!}$$
$$\sum_{\{\tau_p, \sigma_p\}} \prod_p Wg^N(\tau_p^{-1}\sigma_p) \prod_c (\text{group factor})$$

$$\text{group factor} = \text{symmetric function } P_\mu(1) = N^{|\mu|}$$

$\sigma_p, \tau_p \in S_L$, $L = \sum_{l\in p}(k_l + |r_x - r_{x+e_n}|/2)$. $|\mu|$ - the number of cycles in combined permutations, $\mu \in S_C$, $C = \sum_{l\in c}(2k_l + |r_x - r_{x+e_n}|)$.

For $U(1)$ model one recovers the conventional form

$$Z = \sum_{\{r_x\}} \prod_l I_{r_x - r_{x+e_n}}(\beta)$$

Dual representation can be simplified by using orthogonality relation

$$\sum_{\omega \in S_r} \chi_\mu(\omega\tau)\chi_\lambda(\omega\sigma) \;=\; r!\,\delta_{\mu,\lambda}\,\frac{\chi_\lambda(\tau^{-1}\sigma)}{d(\lambda)}$$

$$Z = \sum_{\{r_x, k_l\}} \sum_{\{\omega_p\}} \prod_l \frac{\beta^{2k_l + |r_x - r_{x+e_n}|}}{k_l!(k_l + |r_x - r_{x+e_n}|)!} \prod_c Q(c)$$

$$Q(c) = (\prod_{p \in c} \sum_{\sigma_p} B_p \,)\, N^{|\mu|}\,,\; \mu = \mu(\sigma_p)$$

$$B_p = \sum_\lambda (\,\frac{d(\lambda)}{r!}\,)^{3/2} \frac{1}{(s_\lambda(1^N))^{1/2}}\, \chi_\lambda(\omega_p\sigma_p)$$

IV. Strong coupling QCD

A. $U(N)$ spin model

Effective Polyakov loop model in the presence of heavy quarks and finite chemical potential

$$Z = \int \prod_x dU(x)$$
$$\times \ \exp\left[\beta \sum_{x,\mu} \mathrm{ReTr}U(x)\mathrm{Tr}U^\dagger(x+e_\mu) + \sum_x \Big(h_r \mathrm{Tr}U(x) + h_i \mathrm{Tr}U^\dagger(x)\Big)\right]$$

h_r and h_i are functions of quark mass and chemical potential. The effective action is complex if $\mu \neq 0$.

The dual form of $U(N)$ spin model

$$Z = \prod_l \left[\sum_{p(l)=-\infty}^{\infty} \sum_{q(l)=0}^{\infty} \left(\frac{\beta}{2}\right)^{|p(l)|+2q(l)} \frac{1}{(q(l)+|p(l)|)!\,q(l)!} \right]$$

$$\prod_x \left[\sum_{k(x)=0}^{\infty} \frac{(h_r h_i)^{k(x)}}{k(x)!\left(k(x)+\sum_{n=1}^{d}(p_n(x)-p_n(x-e_n))\right)!} \, Q_N(s(x)) \right]$$

$$s(x) = \sum_{i=1}^{2d} \left(q(l_i) + \frac{1}{2}\,|p(l_i)| \right) + k(x) + \frac{1}{2} \sum_{n=1}^{d} \left(p_n(x) - p_n(x-e_n) \right) \, .$$

$$Q_N(s(x)) = \sum_{\lambda \in S_{s(x)}} d^2(\lambda) \, , \; \lambda_1 \geq \cdots \geq \lambda_N \geq 0, \; \sum_{i=1}^{N} \lambda_i = s(x)$$

The dual Boltzmann weight is strictly positive if $h_r h_i \geq 0$.

B. One link integral
for N_f flavours of staggered fermions, $\beta = 0$

$$Z = \int \prod_x \left(d\psi_x d\bar{\psi}_x \right) \prod_{x,\mu} \left[\sum_{n=0}^{NN_f} \sum_{\sigma \in S_n} \frac{(-1)^\sigma}{n!} Wg^N(\sigma) \sum_{f_i, v_i = 1}^{N_f} \prod_{i=1}^{n} r_x^{f_i, v_i} r_{x+\mu}^{v_i, f_{\sigma(i)}} \right]$$

$$r_x^{f,v} = \sum_{i=1}^{N} \bar{\psi}_f^i(x) \psi_v^i(x) \ .$$

V. $2D\ U(N)$ QCD

Partition function of two-dimensional $U(N)$ LGT with one flavour of the staggered fermions

$$Z = \sum_{\{r(p)\}=-\infty}^{\infty} \sum_{\{t(p)\}=0}^{\infty} \sum_{\{k(l)\}=0}^{N_c} \sum_{\{n(l)\}=0}^{N_c} \sum_{\{s(x)\}=0}^{N_c} \sum_{\{\tau_l,\sigma_l\}\in S_z(l)}$$

$$\prod_p \frac{(\beta/2)^{2t(p)+|r(p)|}}{t_p!(t_p+|r(p)|)!} \prod_l \left[\frac{\eta_\nu(x)}{2}\right]^{k(l)+n(l)} [y_\nu]^{k(l)-n(l)}\ Wg^N(\tau_l^{-1}\sigma_l)$$

$$\prod_x m^{s(x)} \times (\text{constraints}) \times \prod_x (\text{group factor}) \times (\text{sign factor})$$

- Gauge (fermion) integration produces constraint on every link (site)

$$\prod_l \delta\left(r(p) - r(p') + k(l) - n(l)\right) \ p, p' \text{ have common link } l,$$

$$\prod_x \delta\left(s(x) + k(x) - N\right) \ \delta\left(s(x) + n(x) - N\right) \ ,$$

$$k(x) = \sum_{\nu=1}^{d} \left[k_\nu(x) + n_\nu(x - e_\nu)\right] , n(x) = \sum_{\nu=1}^{d} \left[n_\nu(x) + k_\nu(x - e_\nu)\right] .$$

- Permutation group $S_z(l)$ is fixed by

$$z(l) = t(p) + t(p') + \frac{1}{2} \left(|r(p)| + r(p) + |r(p')| - r(p')\right) + k(l) \ .$$

- Group factor arises after summation over all matrix (colour) indices

$$\text{Group factor} = N^{|\mu|}$$

with $|\mu|$ - the number of cycles in combined permutations $\sigma_l, \tau_l, l \in x$.

Sign factor is still missing.

VI. Conclusion and perspectives

- LGT at large N: asymptotic expansion of the Weingarten function is known
- Confinement: bounds on the Weingarten function are also known
- Sign factor in the theory with dynamical fermions
- Dual weight with non-zero chemical potential
- Extension to the Wilson fermions
- Extension to $SU(N)$ models. One link integrals include terms proportional to unit anti-symmetric tensor on the group (= baryon states). (J.-B. Zuber, 2016)
- Possibility of numerical simulations.