Fighting topological freezing in the two-dimensional CP^{N-1} model

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The CP^{N-1} model

Square lattice with sites $x = (x_0, x_1)$, where $x_i \in \{0, 1, 2, ..., L_i - 1\}$.
The lattice spacing is set to $a = 1$
Periodic or open boundary conditions in 0-direction
Periodic boundary conditions in 1-direction

$$S = -\beta N \sum_{x,\mu} \left(\bar{z}_{x+\hat{\mu}} z_x \lambda_{x,\mu} + z_{x+\hat{\mu}} \bar{z}_x \bar{\lambda}_{x,\mu} - 2 \right) ,$$

- z_x is a complex N-component vector with $z_x \bar{z}_x = 1$
- $\lambda_{x,\mu}$ is a complex number with $\lambda_{x,\mu} \bar{\lambda}_{x,\mu} = 1$

Toy model of (lattice) QCD

- asymptotically free
- $1/N$-expansion computed to $O(1/N)$
- Stable instanton solutions
- Problem shared with lattice QCD: topological freezing

Topological susceptibility

Topological charge

$$Q_{plaq} = \frac{1}{2\pi} \sum_x \theta_{plaq,x}$$

where

$$\theta_{plaq,x} = \theta_{x,\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x,\nu} - 2n\pi \quad , \qquad \mu \neq \nu$$

where $\theta_{x,\mu} = \arg\{\bar{z}_x z_{x+\hat{\mu}}\}$ and n is such that $-\pi < \theta_{plaq,x} \leq \pi$.

$$\chi_t = \frac{1}{V} \langle Q^2 \rangle$$

Basic algorithm: hybrid of local

- Heatbath/Metropolis (spin/gauge fields) sweeps
- $n_{ov} \propto \xi$ Overrelaxation sweeps

In 2D XY model: dynamical critical exponent $z \approx 1$

CP^{N-1} model, periodic boundary conditions in both directions: autocorrelation times of the topological modes increase very rapidly with the correlation length ξ. Looks like an exponential increase.

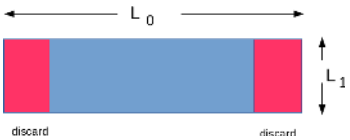
We can go up to $\xi \approx 23$, 6, 2.4 for $N = 10$, 21, and 41

Periodic boundary conditions:

Continuum: disconnected sectors characterized by their charge Q

Finite lattice spacing:
Free energy barriers increase with increasing correlation length.
Markov chain Monte Carlo algorithms become non-ergodic

M. Lüscher and S. Schaefer (2011) Open boundary conditions (lattice QCD)



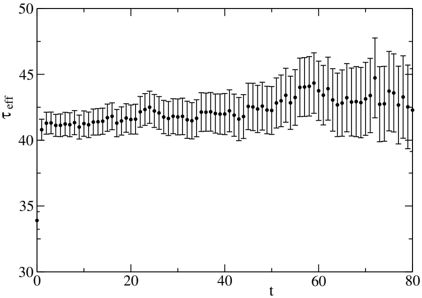
Here $L_0 = 4L_1$, $L_1 \approx 15\xi_{2nd}$, discard $l_0 \approx 10\xi_{2nd}$

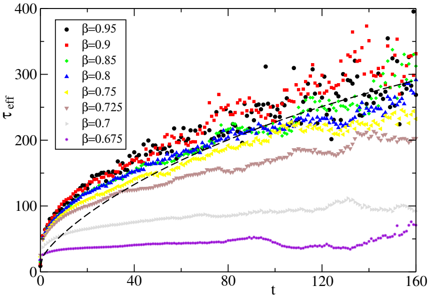
Effective autocorrelation time of the topological susceptibility

$$\tau_{eff,s,A}(t) = -s/\ln[\rho_A(t+s)/\rho_A(t)]$$

periodic b.c. $N = 10$,
$\beta = 0.96$, $\xi_{2nd} = 12.87(1)$

open b.c., N=21,
ξ_{2nd} up to 18.242(2)





G. McGlynn and R. D. Mawhinney (2014) Effective model for diffusion of instantons:

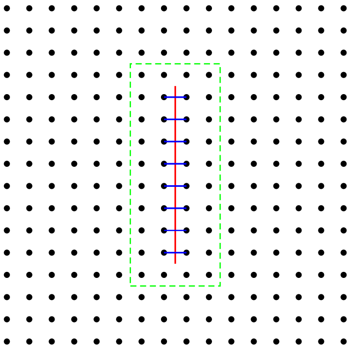
$$\rho(t) \propto \sum_n c_n \exp\left(-\frac{n^2}{\tau_{exp}} t\right) \ ,$$

Dashed line in our plot, ad hoc:
$c_n \propto 1/n$, $\tau_{exp} = 455$

Parallel tempering in a line defect

$$S = -\beta N \sum_{x,\mu} c_{t,x,\mu} \left(\bar{z}_{t,x+\hat{\mu}} z_{t,x} \lambda_{t,x,\mu} + z_{t,x+\hat{\mu}} \bar{z}_{t,x} \bar{\lambda}_{t,x,\mu} - 2\right)$$

where $c_{t,x,\mu} = c_r(t)$ for x, μ on the defect line and $c_{t,x,\mu} = 1$ else.



$t \in \{0, 1, ..., N_t - 1\}$ labels the points of the tempering chain. In our simulations we take $c_r(t) = 1 - t/(N_t - 1)$

$A_{swap} = \min[1, \exp(-\beta N$
$[c_r(t_1) - c_r(t_2)] \, [E_r(t_2) - E_r(t_1)])]$

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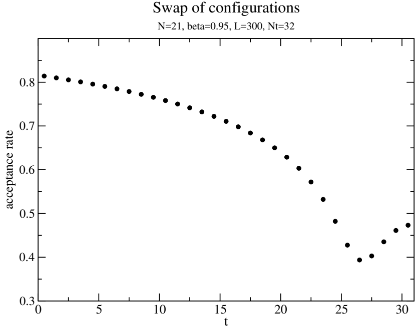
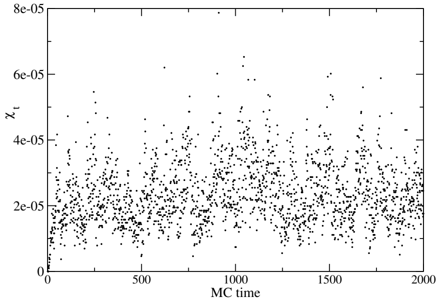
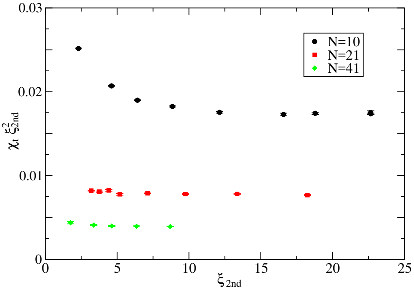
Sweeps over the full lattice; replica ex.; transl.;
  for(i1=0;i1<n1;i1++)
    {
    Sweeps over box(l_1); replica ex.; transl.; measure;
    for(i2=0;i2<n2;i2++)
      {
      Sweeps over box(l_2); replica ex.; transl;
      for(i3=0;i3<n3;i3++)
        {
        Sweep over box(l_3); replica ex.; transl.;
        .
        .
        .  until l_i = 1
    } } ...}
```

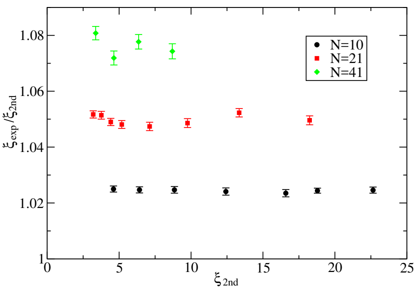
$N = 21$, $\beta = 0.95$, $L = 300$, $l_d = 16$, Largest box: 80×64
Tempering cycle $n_1 = 24$, $n_2 = n_3 = n_4 = 3$, and $n_5 = n_6 = 2$.
The number of overrelaxation updates: 28, 14, 7, 7, 3, and 3 at levels 1, 2, 3, 4, 5, and 6. 4849 config. exchange updates per cycle.
50370 update cycles, 25 days on a CPU with 4 cores.





$\tau_{int,pos} = 1.875(13)/72$, $\chi_t = 0.00002305(16)$, $\tau_{int,\chi_t} = 7.2(5)$,
$\tau_{exp} \approx 16$





$$\chi_t \xi_{2nd}^2 = \frac{1}{2\pi N}\left(1 - \frac{0.38088...}{N}\right) + O\left(N^{-3}\right)$$

$$\frac{\xi_{2nd}}{\xi_{exp}} = \sqrt{\frac{2}{3}} + O\left(N^{-2/3}\right)$$

$$\sqrt{\frac{3}{2}} = 1.2247...$$

Conclusions and outlook

- open boundary conditions avoid topological freezing (as expected)
- Dynamical critical exponent of hybrid overrelaxation $z \approx 1$
- Parallel tempering of line defect looks fine; with respect to the error of χ_t slightly better than standard simulation with open boundary conditions
- Does it work for lattice QCD?

M. H., [arXiv:1706.04443]