High-order perturbative expansions in massless gauge theories with NSPT

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Motivations

Numerical Stochastic Perturbation Theory (**NSPT**) is a technique which allows to perform perturbative expansions numerically in a quantum (field) theory.

In general, the perturbative series for a generic observable is divergent:

$$R \sim \sum_n r_n \alpha^n$$

$$r_n \sim K\, a^n\, n!\, n^b \qquad n \to +\infty$$

The pattern of divergence (e.g. renormalons) gives informations on non-perturbative physics (e.g. power corrections in the OPE).

There are well-established results in lattice gauge theories only for gluodynamics.

[Bali, Bauer, Pineda '12,'14]

In fact fermions are a handle on the beta function, they affect the running of the coupling and should control and determine the high-order perturbative behaviour.

(Quick) Summary of NSPT

Consider the Euclidean Wilson action $$S_g[U] = -\frac{\beta}{2N_c}\sum_P \mathrm{Tr}(U_P + U_P^\dagger)$$

and set up a Langevin process in a new time τ: $$\frac{\partial U_\tau}{\partial \tau} = -i(\nabla S_g + \eta)U_\tau$$

[Batrouni et al. '85]

gaussian noise

Stochastic quantisation:

[Parisi, Wu '81]

$$\langle O[U_\tau]\rangle_\eta \xrightarrow{\tau\to\infty} \langle O[U]\rangle$$

stochastic average → **expectation value**

If we expand the links in the coupling $$U_n = 1 + \sum_{k=1} \beta^{-k/2} U_n^{(k)}$$

and plug the expansion into a numerical integration scheme $$U_{n+1} = e^{-F[U_n,\eta]}\, U_n$$

→ it is possible to evolve separately each component $U_n^{(k)}$

[Di Renzo, Marchesini, Marenzoni, Onofri '94]

Wilson fermions in NSPT

The fermion determinant gives a new contribution to the action

$$S[U] = S_g[U] - \mathrm{Tr}\ln M[U]$$

Wilson Dirac operator

and leads to an additional drift term

$$\nabla \mathrm{Tr}\ln M[U] = \mathrm{Tr}(\nabla M)M^{-1} = \langle \mathrm{Re}\,\xi^\dagger(\nabla M)M^{-1}\xi\rangle_\xi$$

gaussian noise

Now the stochastic process will average both on η and ξ.

[Di Renzo, Scorzato '01]

<u>There is no need of an actual matrix inversion</u>: the Dirac operator inherits a perturbative expansion from the gauge links and the inversion is done order by order in perturbation theory.

$$(M^{-1})^{(0)} = (M^{(0)})^{-1}$$

$$(M^{-1})^{(k)} = -(M^{(0)})^{-1}\sum_{j=0}^{k-1} M^{(k-j)}(M^{-1})^{(j)}$$

Only $(M^{(0)})^{-1}$ has to be known: the tree level Wilson fermion propagator.

Twisted boundary conditions

When a theory is defined in finite volume, the fields can have boundary conditions compatible with all the symmetries of the action.

We can impose the gauge field to be periodic up to a gauge transformation,
['t Hooft '79]

$$U(x+L\hat{\nu}) = \Omega_\nu U(x) \Omega_\nu^\dagger \qquad \Omega_\nu \Omega_\mu = z_{\mu\nu} \Omega_\mu \Omega_\nu \qquad z_{\mu\nu} \in Z_{N_c}$$

New Fourier transform and finer momentum quantisation:
[Lüscher, Weisz '85]

$$\tilde{A}_\mu(p_\parallel, p_\perp) = \sum_x e^{-i(p_\parallel + p_\perp)x} \, \mathrm{Tr} \Gamma_{p_\perp}^\dagger A_\mu(x)$$

$$p_\parallel = \frac{2\pi}{L}(n_1, n_2, n_3, n_4) \qquad n_\mu = 0, \dots, L-1$$

$$p_\perp = \frac{2\pi}{N_c L}(\tilde{n}_1, \tilde{n}_2, 0, 0) \qquad \tilde{n}_{1,2} = 0, \dots, N_c - 1$$

(for a twist on the 12 plane)

The traceless property of the gauge field forces $\tilde{A}_\mu(p_\parallel,0)=0$ → **no zero mode!**

Fermions with smell

The group property of the twist matrices comes from the consistency condition

$$U(x + L\hat{\mu} + L\hat{\nu}) = \Omega_\mu U(x + L\hat{\nu})\Omega_\mu^\dagger = \Omega_\nu U(x + L\hat{\mu})\Omega_\nu^\dagger$$

The same condition does not work for fermions in the fundamental representation:

$$\psi(x + L\hat{\mu} + L\hat{\nu}) = \Omega_\mu \psi(x + L\hat{\nu}) = \Omega_\mu \Omega_\nu \psi(x)$$
$$\psi(x + L\hat{\mu} + L\hat{\nu}) = \Omega_\nu \psi(x + L\hat{\mu}) = \Omega_\nu \Omega_\mu \psi(x) = z_{\mu\nu} \Omega_\mu \Omega_\nu \psi(x)$$

A solution is to introduce a new degree of freedom: **smell**. *[Parisi '84]*
Smells transform into each other according to the antifundamental representation of SU(N_c):

$$\big(\psi(x + L\hat{\nu})\big)^{c,s} = \sum_{c',s'} \big(\Omega_\nu\big)^{c\,c'} \big(\psi(x)\big)^{c',s'} \big(\Omega_\nu^\dagger\big)^{s',s}$$

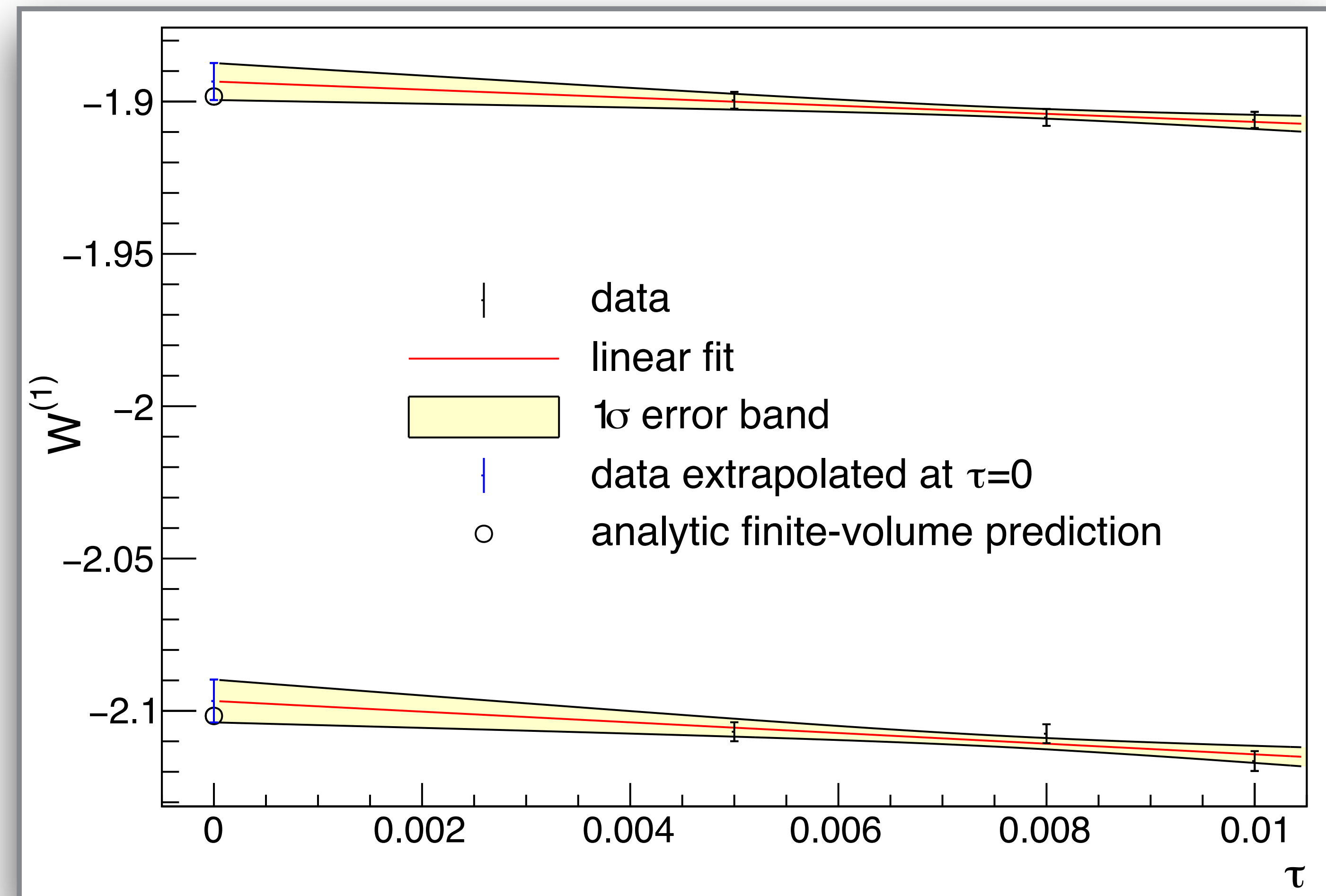
All physical observables are singlet under the smell group.

(Smell is not required for fermions in the adjoint representation)

Plaquette with TBC

Let's study the plaquette: $$W = \frac{1}{N_c}\langle \mathrm{Tr}\, U_P \rangle = 1 + \frac{1}{\beta} W^{(1)} + \frac{1}{\beta^2} W^{(2)} + \ldots$$

Triple twisted boundary conditions

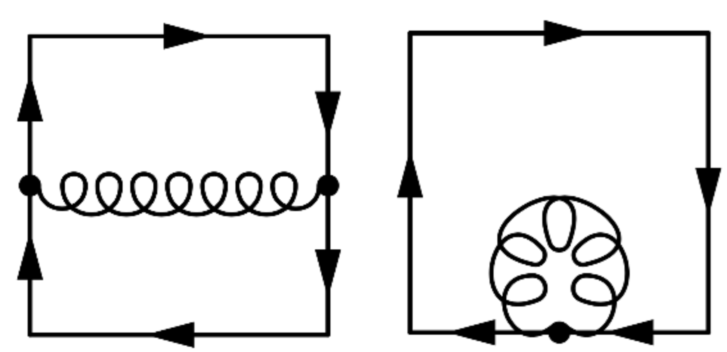


planes identified by one twisted direction and one periodic direction

$L^4=2^4$, $N_c=3$

planes identified by two twisted directions

Finite-volume predictions from twisted lattice perturbation theory



Second order integration scheme

The Langevin process is discretised according to $U_{n+1} = e^{-F[U_n,\eta]}\,U_n$

So far a first order **Euler integrator** was used

$$F[U_n,\eta] = \tau\nabla S[U_n] + \sqrt{\tau}\,\eta$$

→ results extrapolate linearly in τ.

[Bali, Bauer, Pineda, Torrero '13]

We have also implemented a second-order **Runge-Kutta integrator**

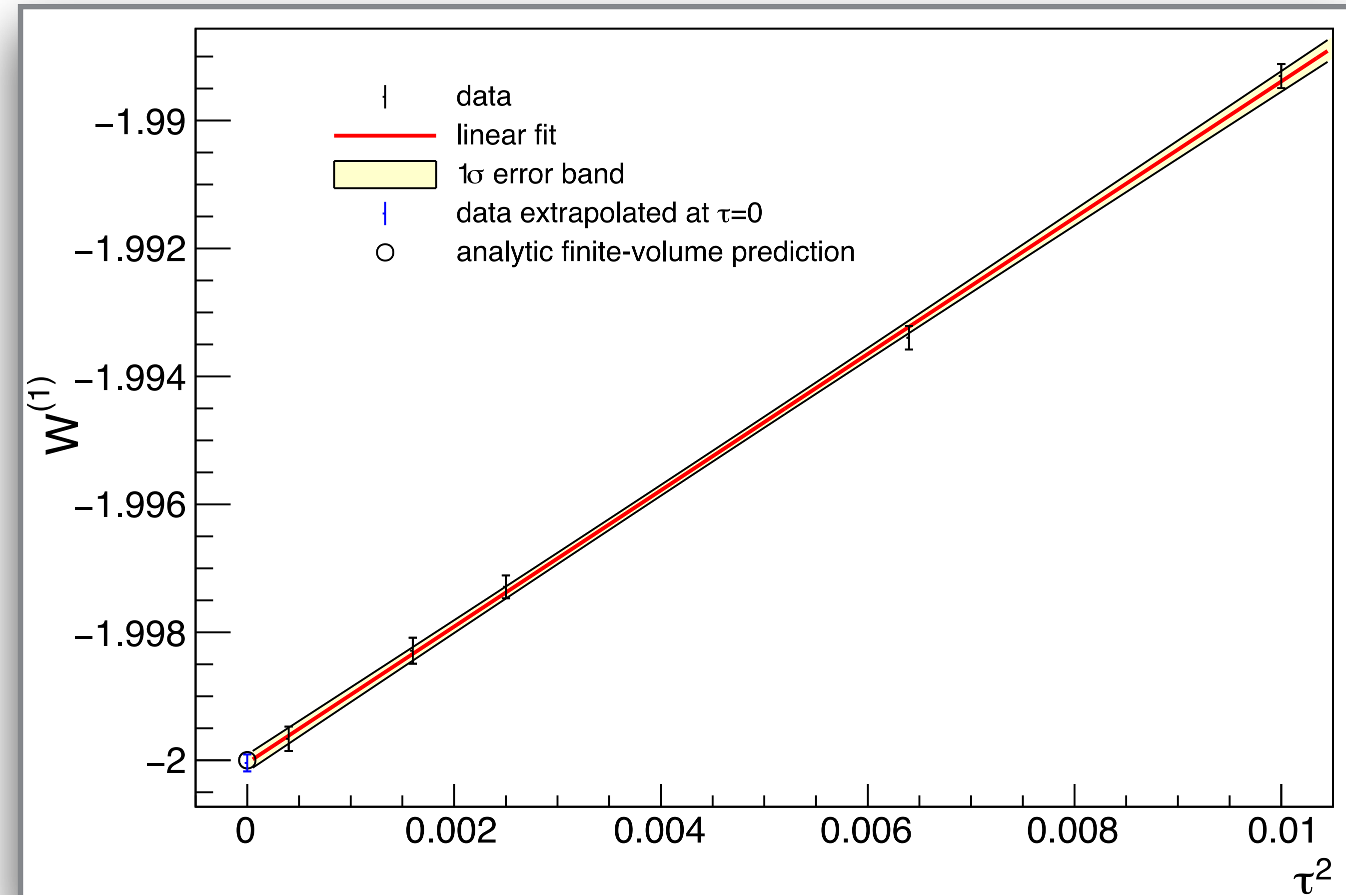
$$U' = e^{-\left(\tau\nabla S[U_n]+\sqrt{\tau}\,\eta\right)}\,U_n$$

$$F[U_n,\eta] = \frac{\tau}{2}\left(\nabla S[U_n] + \nabla S[U']\right) + \frac{C_A}{6}\,\tau^2\,\nabla S[U'] + \sqrt{\tau}\,\eta$$

→ results extrapolate quadratically in τ.

First example of second-order integrator for a complete dynamics with fermions.

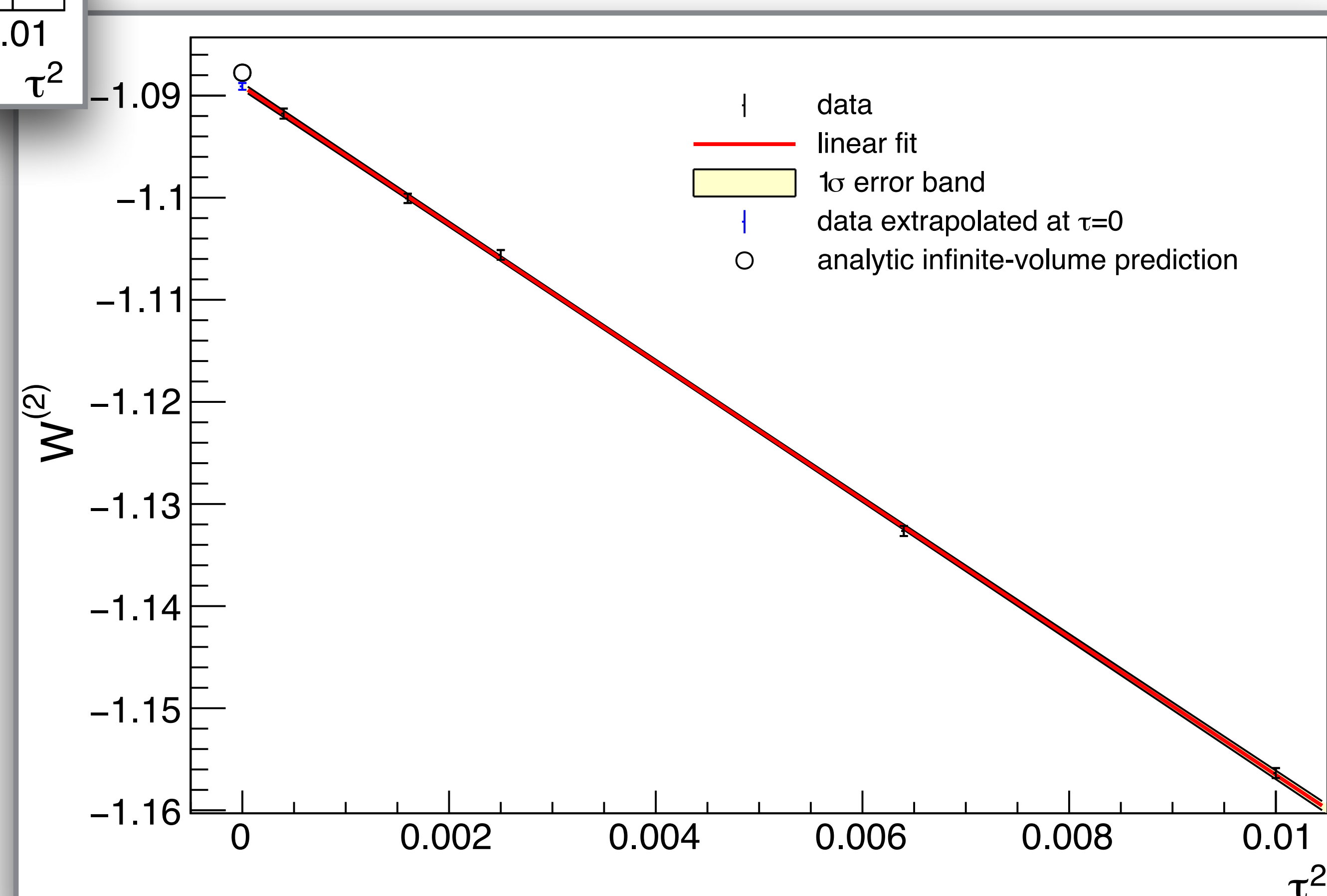
Plaquette with second order integrator



Twisted boundary conditions on a plane

$L^4=8^4$, $N_c=3$, $N_f=2$

Measured values scale like τ^2, as expected.



Quadratic regime seems to hold up to $\tau=0.1$

At β^{-2} only the infinite-volume plaquette is known.

Fermion bare mass is set to zero

Critical mass

Massless Wilson fermions do not have chiral symmetry, they require an additive mass renormalisation.

Since the inverse of the Wilson propagator is

$$aS(ap)^{-1} = i\sum_\mu \gamma_\mu \sin^2(ap_\mu) + 2\sum_\mu \sin^2\left(\frac{ap_\mu}{2}\right) + am + a\Sigma(ap, am, \beta^{-1})$$

in an on-shell scheme we define the **critical mass** m_c as the value of the bare mass such that the renormalised mass of the fermion is zero,

$$m_c = -\Sigma(ap = 0, am_c, \beta^{-1})$$

Expanding both sides in the coupling, we are defining the mass counterterms order by order in perturbation theory:

$$m_c = \sum_k m_c^{(k)} \beta^{-k}$$

They can be computed from the fermion self-energy at zero momentum.

Determination of the critical mass

Strategy:

- collect gauge configurations from NSPT at different Langevin time steps τ
- fix the Landau gauge
- measure the propagator and perform Monte Carlo average

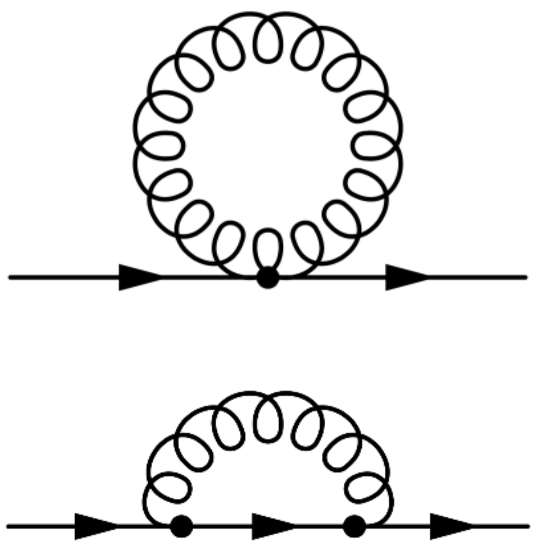
$$S(p) = \sum_p M[U]^{-1}_{pq} \delta_{pq}$$

- extrapolate S(p) to zero time step
- invert the propagator
- project onto the identity in Dirac space: $\Gamma(ap) = \frac{1}{4} \mathrm{Re}\,\mathrm{Tr}\, aS(ap)^{-1}$
- extract m_c from $\Gamma(ap)$ at zero momentum

Everything is done automatically order by order in perturbation theory!

Need to determine $m_c^{(k)}$ first before $m_c^{(k+1)}$.

Zero momentum extrapolation



We computed Γ(ap) and the critical mass at one-loop in twisted lattice perturbation theory.

A good (gauge invariant) definition at finite volume requires to set the external momentum to zero (which is equivalent to send a→0).

Because of fermion antiperiodic boundary conditions, the lowest momentum we are able to measure is π/L.

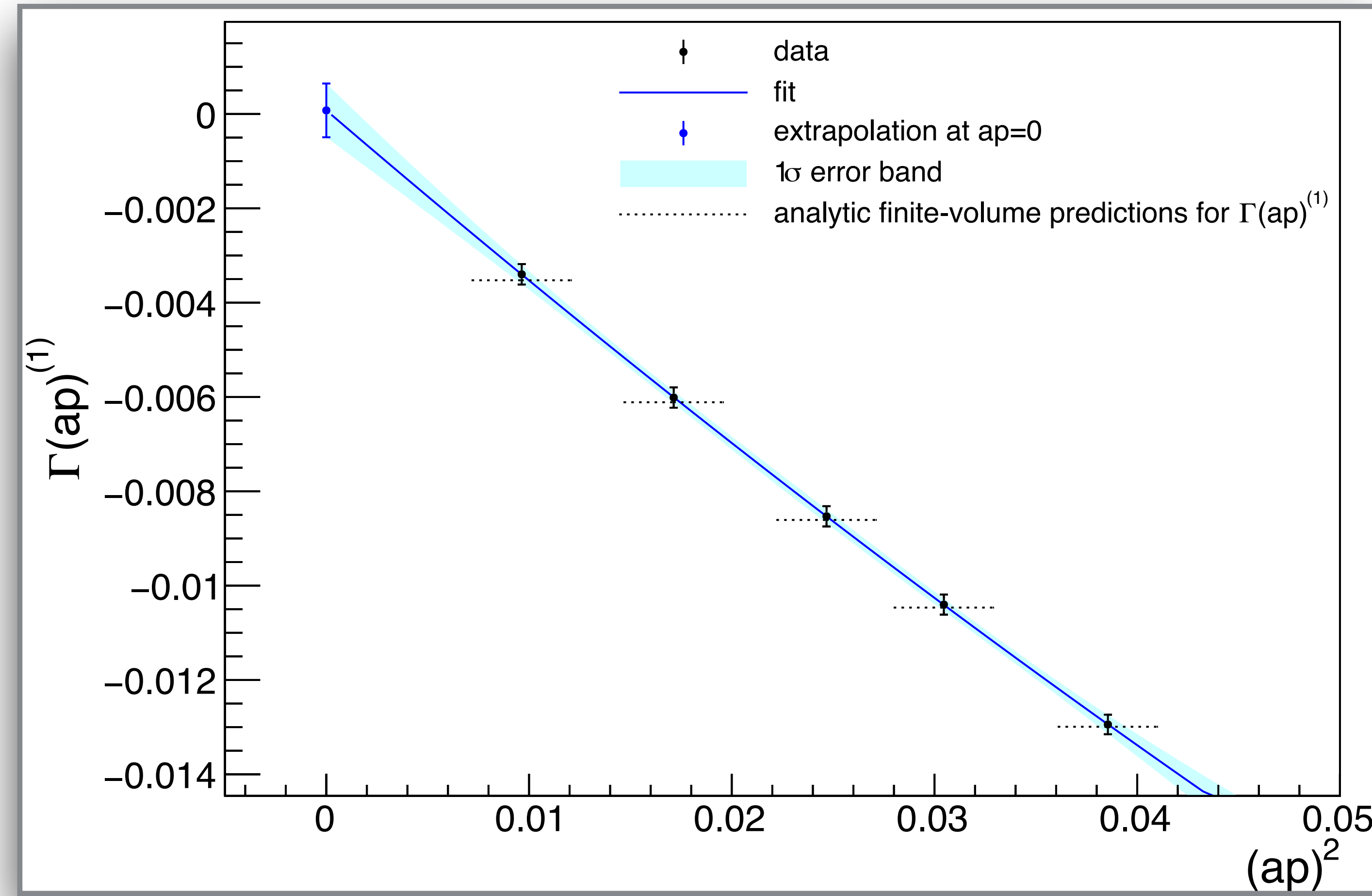
<u>"Traditional" approach</u>: hypercubic symmetric Taylor expansion around a=0.
In fact it does not converge enough to give a reliable fit, even for big volumes.

<u>Solution</u>: θ-boundary conditions for the "valence fermions" (i.e. for the inverse Dirac operator used for measuring the propagator)

$$\psi(x + L\hat{0}) = e^{i\theta}\psi(x)$$

Now the lowest momentum is θ/L.
Changing θ, we can span very low momenta in the 0 direction and then extrapolate with a polynomial fit.

Critical mass at finite volume from NSPT



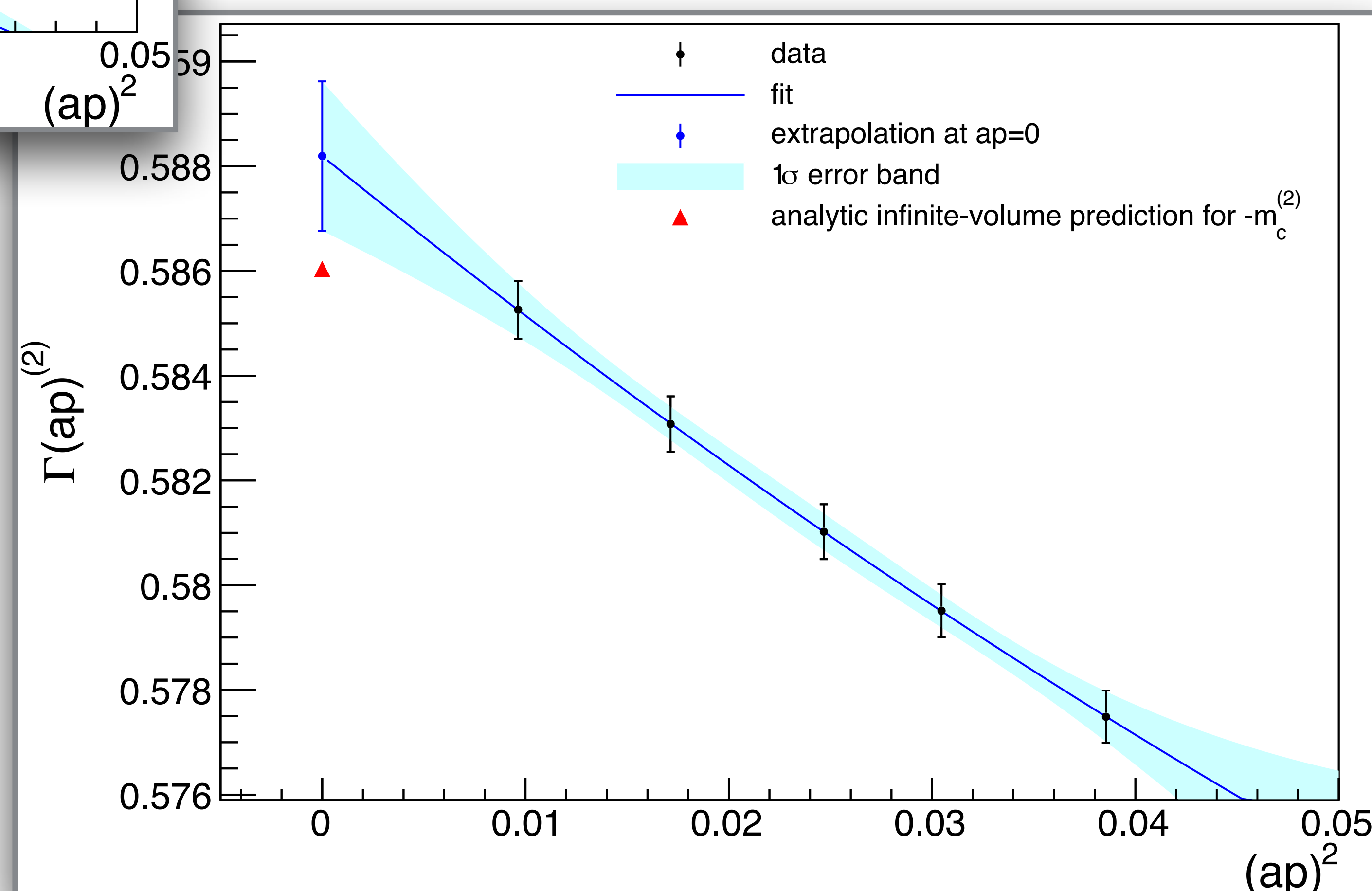
Twisted boundary conditions on a plane

$L^4=12^4$, $N_c=2$, $N_f=2$

Configurations are generated taking into account the finite-volume analytical value of $m_c^{(1)}$.

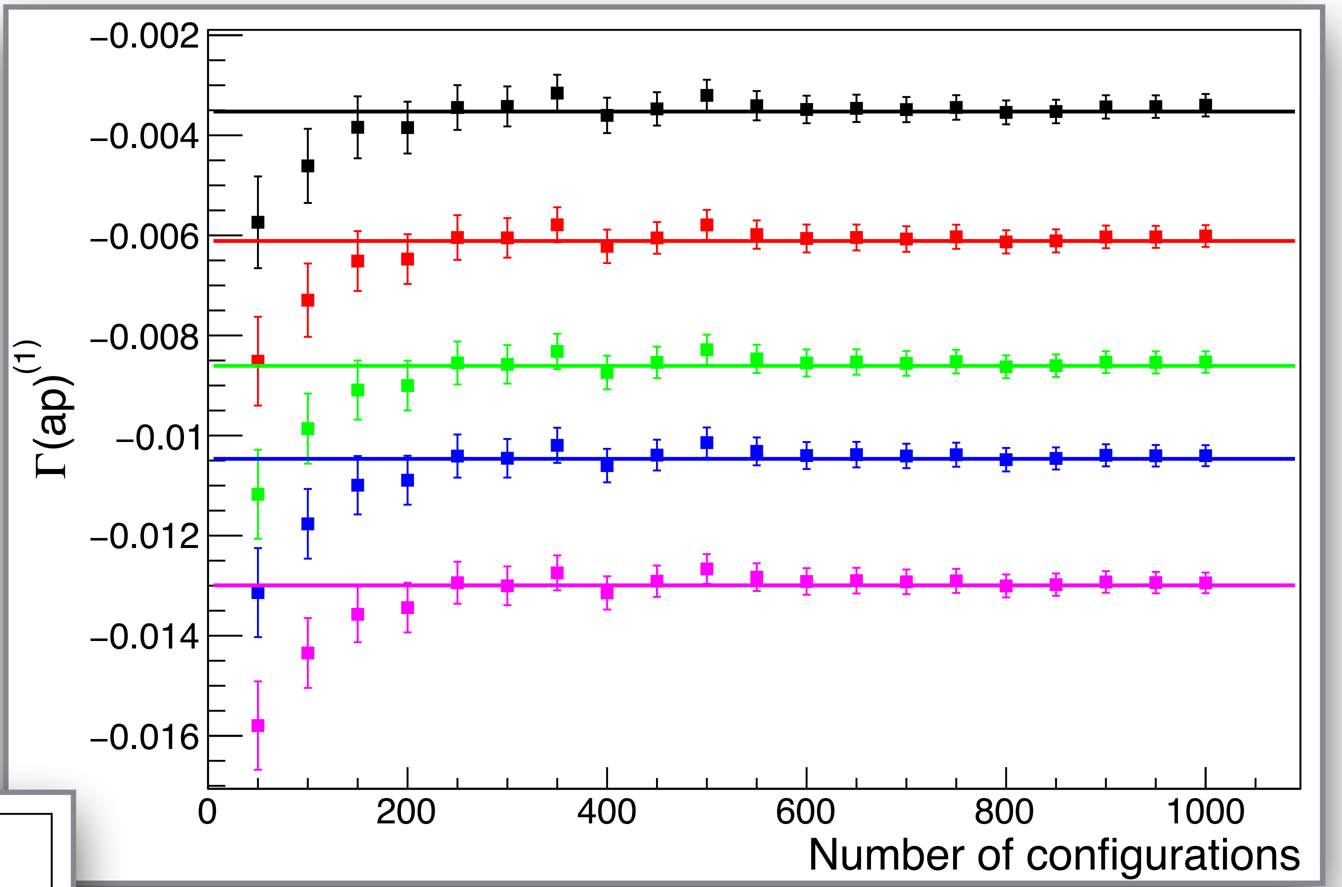
$\Gamma(ap)^{(1)}$ extrapolates to zero

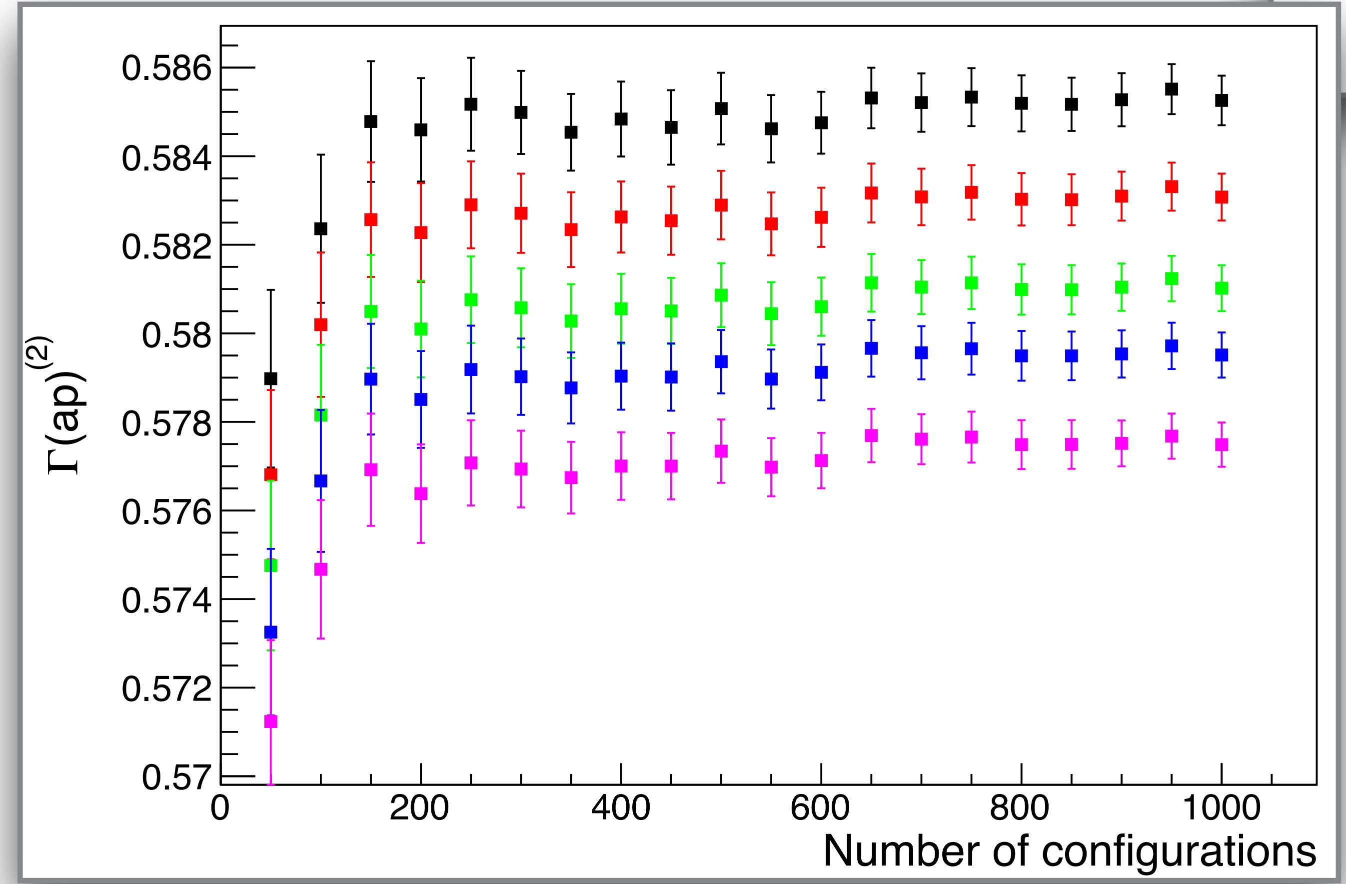
We determine $m_c^{(2)}$ from $\Gamma(ap)^{(2)}$ *(here quadratic fit)*



Thermalisation

We are studying momenta lower than the "naturally allowed" ones: need to generate a good number of configurations before having agreement with theoretical predictions…





… but we can check when our measures are stable also at higher orders, without knowing Γ(ap) in advance.

Conclusions

- ✓ We have implemented NSPT for SU(2) and SU(3) gauge theories with an arbitrary number of fermions (with smell) in the fundamental representation and with twisted boundary conditions.
- ✓ A second-order Runge-Kutta integrator for this framework has been tested.
- ✓ We developed a way to extract reliably the mass counterterms at finite volume and therefore keep the Wilson fermion mass under control.
- ✓ All the measures have been found in very good agreement with one-loop twisted lattice perturbation theory predictions.

Next steps:

- implement Fourier accelerated gauge fixing
- measure critical masses at several perturbative orders / volumes
- add fermions in the adjoint representation
- investigate high-order behaviour of observables, e.g. the plaquette

BACKUP

Zero modes

There are three sources that could spoil the convergence of the stochastic process:

- **fermion zero mode**

  the massless Wilson propagator has a pole at zero momentum

  cured by *antiperiodic boundary conditions*

- **longitudinal gauge modes**

  the longitudinal component of the gauge field is not affected by the drift and would tend to diverge like a random walk

  cured by *stochastic gauge fixing*

- **gauge zero mode**

  same as the longitudinal modes
  (a similar problem arises in finite-volume perturbation theory)

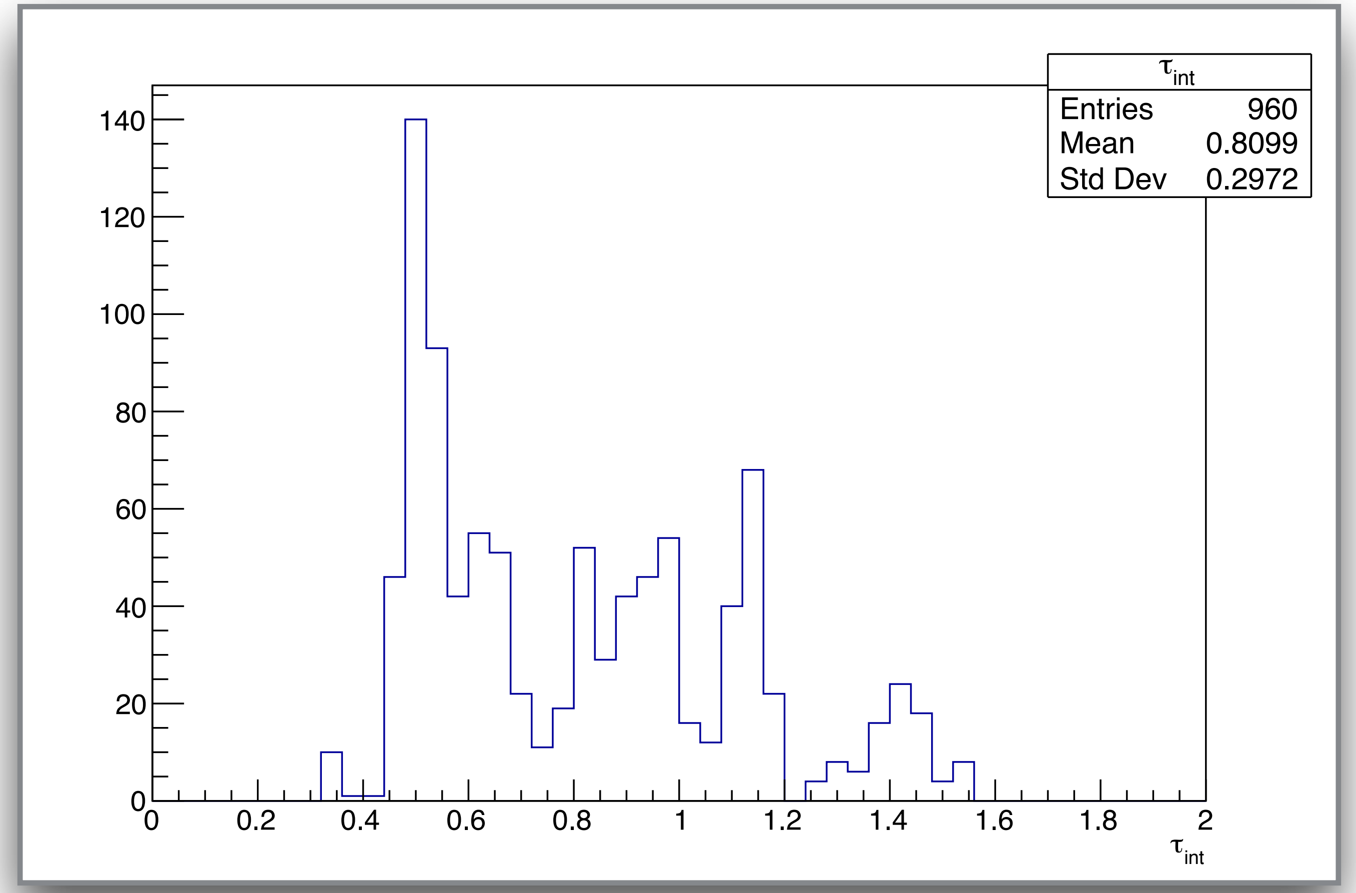
  cured by *twisted boundary conditions*

Autocorrelations

Autocorrelation are kept under control for each:

- perturbative order
- Dirac matrix element
- momentum
- Langevin time step

$L^4 = 12^4$, $N_c = 2$, $N_f = 2$



Twisted lattice perturbation theory

$$\Gamma(p)^{(1)} = \frac{1}{L^4}\sum_k \frac{1}{\hat{k}^2}\Bigg\{\frac{1}{2}\sum_\mu \cos p_\mu\left(1-(1-\alpha)\frac{\hat{k}_\mu^2}{\hat{k}^2}\right)+$$
$$+\frac{1}{\overline{(p-k)}^2+\frac{1}{4}\left(\widehat{(p-k)}^2\right)^2}\Bigg[\frac{1}{2}\widehat{(p-k)}^2\sum_\mu \cos(2p_\mu - k_\mu) - \overline{(p-k)}\cdot\overline{(2p-k)}+$$
$$-\frac{1-\alpha}{\hat{k}^2}\left(\frac{1}{2}\widehat{(p-k)}^2\sum_\mu \cos^2\left(p_\mu-\frac{k_\mu}{2}\right)\hat{k}_\mu^2 - \frac{1}{8}\widehat{(p-k)}^2\left(\widehat{(2p-k)}\cdot\hat{k}\right)^2+\right.$$
$$\left.-\widehat{(2p-k)}\cdot\hat{k}\sum_\mu \cos\left(p_\mu-\frac{k_\mu}{2}\right)\hat{k}_\mu\,\overline{(p_\mu-k_\mu)}\right)\Bigg]\Bigg\}$$

$$W_{\mu\nu}^{(1)} = -\frac{1}{2L^4}\sum_k \frac{\hat{k}_\mu+\hat{k}_\nu}{\hat{k}^2}$$