Numerical Study of the 2+1d Thirring Model with U(2N)-invariant fermions

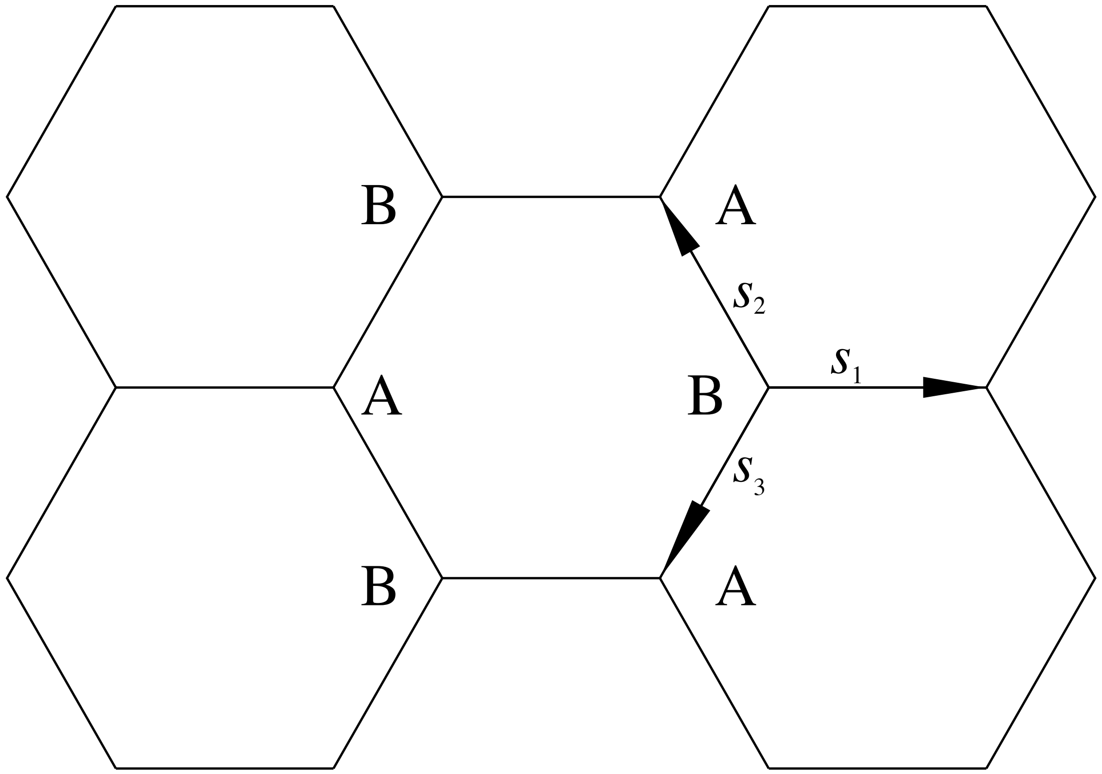
Simon Hands
Swansea University

The Leverhulme Trust

JHEP **1509** (2015) 047
PLB **754** (2016) 264
JHEP **1611** (2016) 015

Lattice 2017, Granada, 22nd June 2017

Relativistic Fermions in 2+1d



Several applications
in condensed matter physics

- Nodal fermions in *d*-wave superconductors
- Spin liquids in Heisenberg AFM
- surface states of topological insulators
- ….and graphene

Free reducible fermions in 3 spacetime dimensions

$$\mathcal{S} = \int d^3x \; \bar{\Psi}(\gamma_\mu \partial_\mu)\Psi \; + \; m\bar{\Psi}\Psi$$

$$\mu = 0, 1, 2$$
$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$
$$\mathrm{tr}(\gamma_\mu \gamma_\mu) = 4$$

For m=0 S is invariant under global U(2N) symmetry generated by

(i) $\Psi \mapsto e^{i\alpha}\Psi\,; \qquad \bar{\Psi} \mapsto \bar{\Psi}e^{-i\alpha},$ (ii) $\Psi \mapsto e^{i\alpha\gamma_5}\Psi\,; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{i\alpha\gamma_5}$

(iii) $\Psi \mapsto e^{\alpha\gamma_3\gamma_5}\Psi\,; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{-\alpha\gamma_3\gamma_5}$ (iv) $\Psi \mapsto e^{i\alpha\gamma_3}\Psi\,; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{i\alpha\gamma_3}$

For $m \neq 0$ γ_3 and γ_5 rotations no longer symmetries

⇒ U(2N) → U(N)⊗U(N)

Mass term $m\bar{\Psi}\Psi$ is hermitian & invariant under parity $x_\mu \mapsto -x_\mu$

Two physically equivalent antihermitian “twisted” mass terms:

$$im_3\bar{\Psi}\gamma_3\Psi; \quad im_5\bar{\Psi}\gamma_5\Psi$$

The “Haldane” mass $m_{35}\bar{\Psi}\gamma_3\gamma_5\Psi$ is not parity-invariant

The Thirring Model in 2+1d

four-fermi form $$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i + \frac{g^2}{2N_f}(\bar{\psi}_i\gamma_\mu\psi_i)^2$$

bosonised form $$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + i\frac{g}{\sqrt{N_f}}A_\mu\gamma_\mu + m)\psi_i + \frac{1}{2}A_\mu A_\mu$$

- Interacting QFT
- expansion in g^2 non-renormalisable
- Hidden Local Symmetry $\psi \mapsto e^{i\alpha}\psi; \quad A_\mu \mapsto A_\mu + \partial_\mu\alpha; \quad \varphi \mapsto \varphi + \alpha$

  if Stueckelberg scalar field ϕ introduced
- expansion in $1/N_f$ exactly renormalisable for $2<d<4$

  $\langle A_\mu A_\nu \rangle \propto \delta_{\mu\nu}/k^{d-2}$ in "Feynman gauge"
- dynamical chiral symmetry breaking for $g^2 > g_c^2$; $N_f < N_{fc}$?
- Quantum Critical Point at $g_c^2(N<N_{fc})$?

Determination of N_{fc} is a non-perturbative problem

eg. $N_{fc}=4.32$ strong coupling Schwinger-Dyson (ladder approximation)

Itoh, Kim, Sugiura & Yamawaki Prog. Theor. Phys. **93** (1995) 417

Numerical Lattice Approach

Del Debbio, SJH, Mehegan
NP**B502** (1997) 269; **B552** (1999) 339

Staggered fermions

$$S_{latt} = \frac{1}{2}\sum_{x\mu i} \bar{\chi}^i_x \eta_{\mu x}(1 + iA_{\mu x})\chi^i_{x+\hat{\mu}} - \bar{\chi}^i_x \eta_{\mu x}(1 - iA_{\mu x-\hat{\mu}})\chi^i_{x-\hat{\mu}}$$

$$+ \; m\sum_{xi} \bar{\chi}^i_x \chi^i_x + \frac{N}{4g^2}\sum_{x\mu} A^2_{\mu x}$$

auxiliary boson couples linearly

$A_{\mu x}$ auxiliary vector field defined on **link** between x and $x+\mu$

$$\eta_{\mu x} \equiv (-1)^{x_0+\cdots+x_{\mu-1}} \Rightarrow \prod_{\square} \eta\eta\eta\eta = -1$$

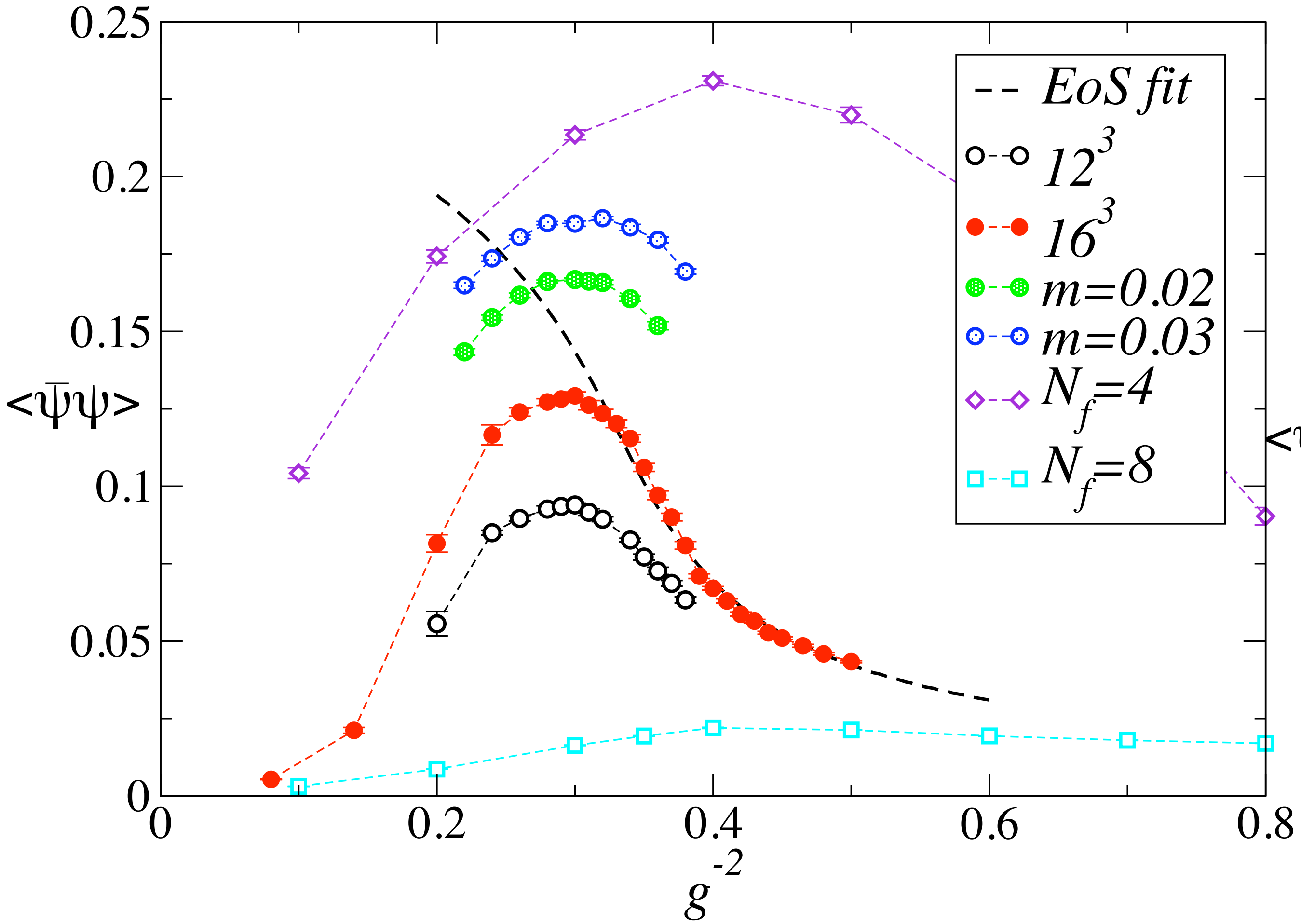
π-flux

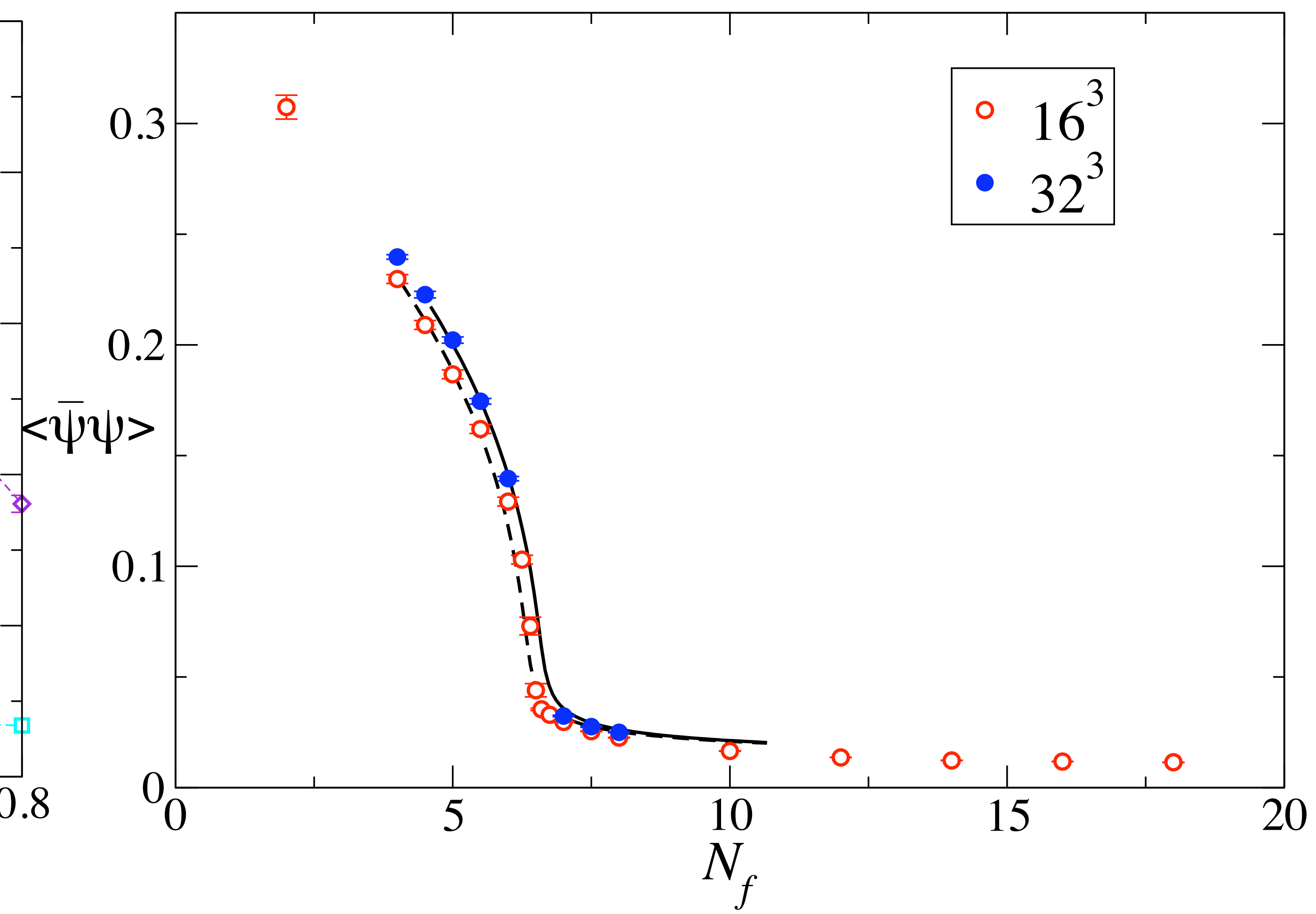
Chiral symmetry: U(N)⊗U(N) → U(N) (if $m, \Sigma \neq 0$)

In weak coupling continuum limit
U($2N_f$) symmetry is recovered, with $N_f = 2N$

Results in effective strong-coupling limit

Christofi, SJH, Strouthos, PR**D75** (2007) 101701





N_{fc}=6.6(1), $\delta(N_{fc})$=6.90(3)

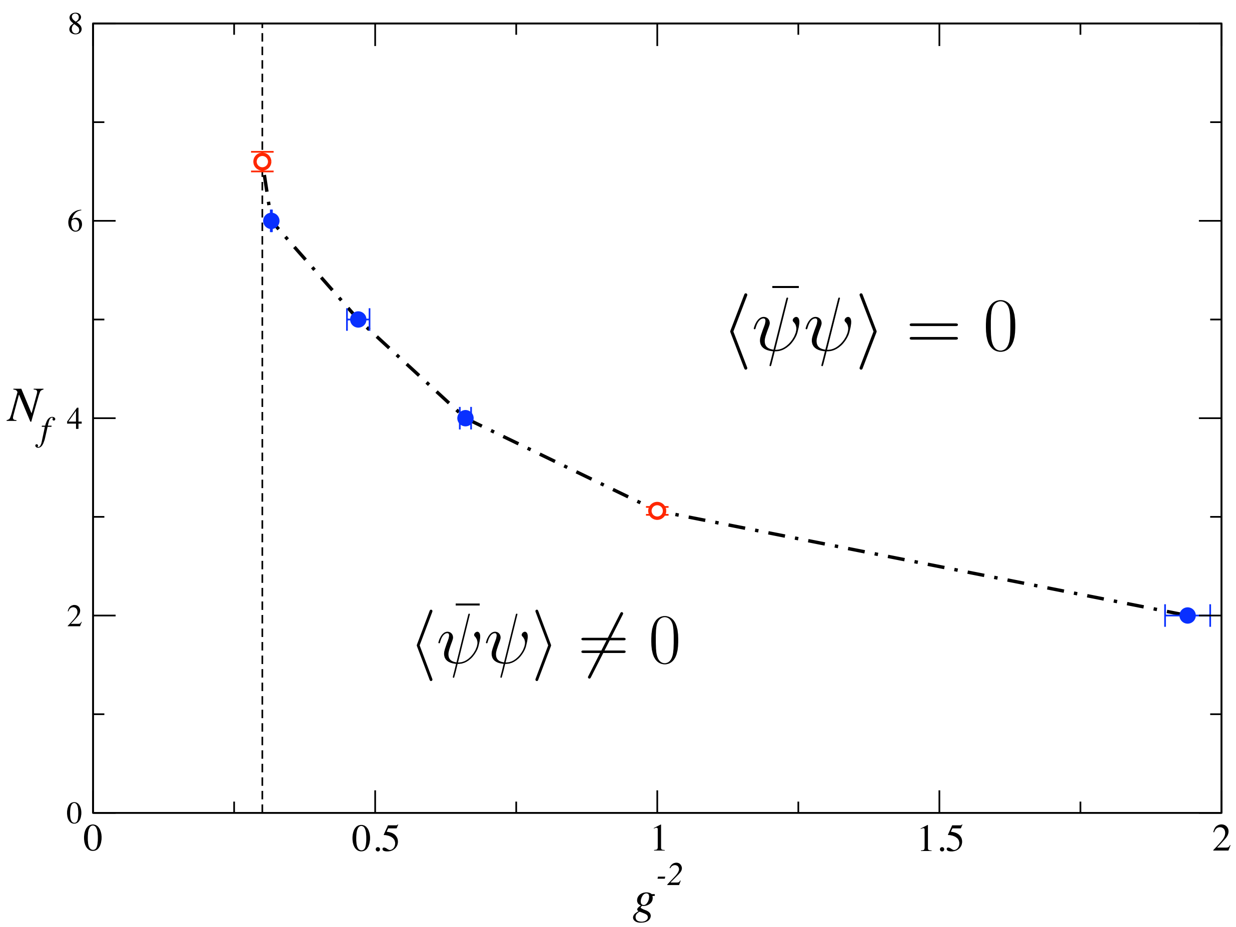
Chiral symmetry unbroken for all g^2 for $N_f > N_{fc}$

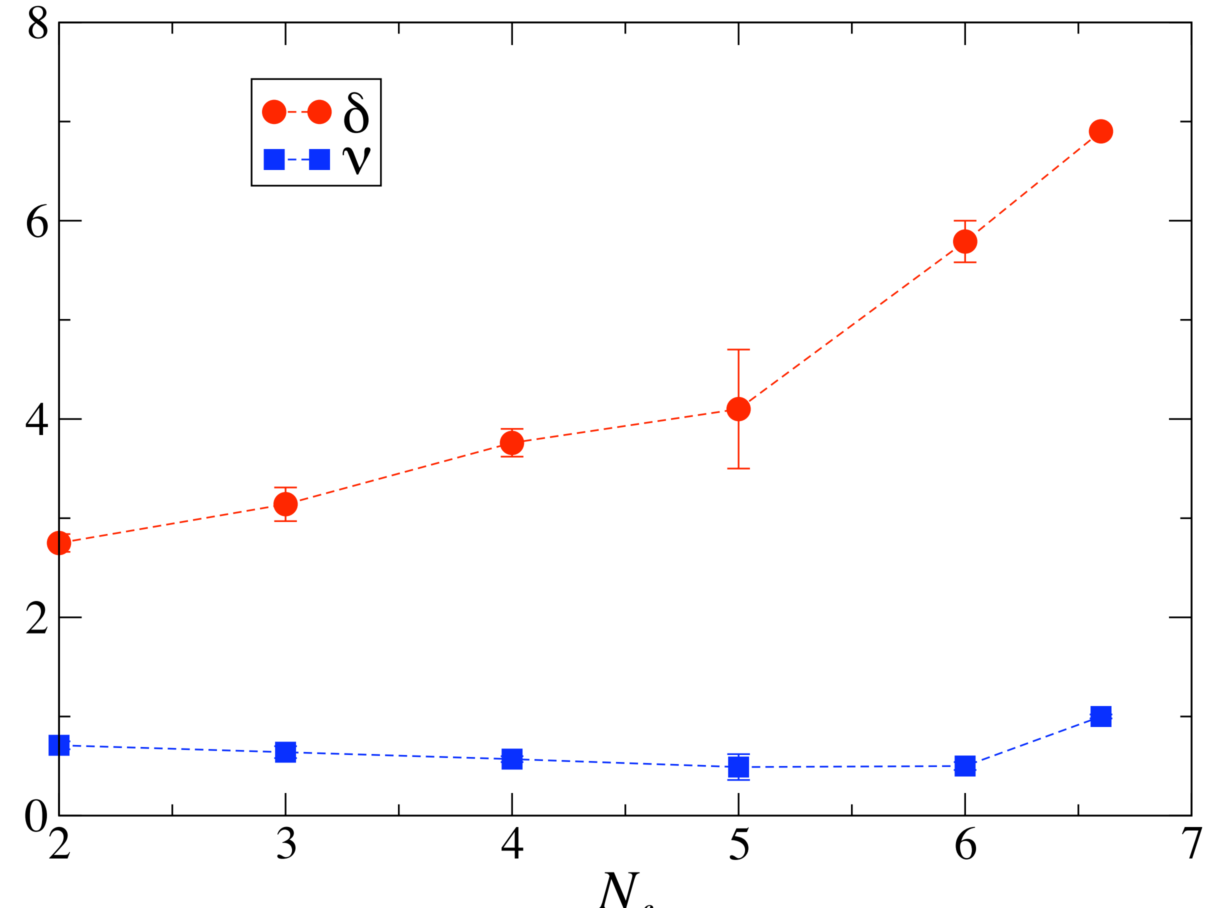
Cf. SDE: N_{fc}=4.32, $\delta(N_{fc})$=1

"conformal phase transition"

Staggered Thirring Summary

SJH, Lucini, PL**B461** (1999) 263 Christofi, SJH, Strouthos, PR**D75** (2007) 101701





- Chiral symmetry broken for small N_f, large g^2
- Each point (for N_f integer) defines a UV fixed point of RG
- Distinct critical exponents ⇔ distinct interacting QFT
- δ increases with N_f, $\delta(N_{fc})\approx 7$
- Non-covariant form used as EFT for graphene $\Longrightarrow N_{fc}\approx 5$

SJH, Strouthos, PR**B78** (2008)165423; Armour, SJH, Strouthos, PR**B81** (2010)125105

Fermion Bag Algorithm with minimal $N_f = 2$

Chandrasekharan & Li, PRL **108** (2012) 140404; PR**D88** (2013) 021701

Thirring Model: $\nu=0.85(1)$, $\eta=0.65(1)$, $\eta_\psi=0.37(1)$

U(1) GN Model: $\nu=0.849(8)$, $\eta=0.633(8)$, $\eta_\psi=0.373(3)$ ($N_f \to \infty$: $\nu=\eta=1$)

Interactions between staggered fields $\chi, \bar{\chi}$ spread over elementary cubes.
Only difference between Thirring & GN is body-diagonal term

Staggered fermions not reproducing expected distinction between models near strongly-coupled fixed point…

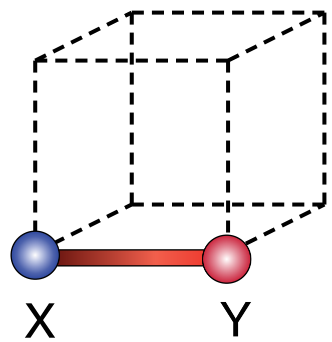
see also Schmidt, Wellegehausen & Wipf, PoS LATTICE2015 (2016) 050

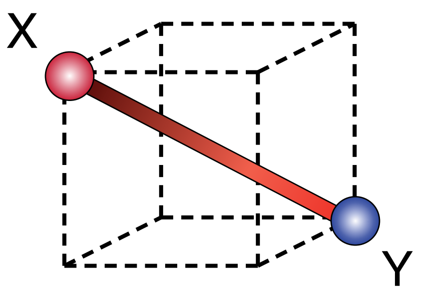
Fermion Bag Algorithm with minimal $N_f = 2$

Chandrasekharan & Li, PRL **108** (2012) 140404; PR**D88** (2013) 021701

Thirring Model: $\nu=0.85(1)$, $\eta=0.65(1)$, $\eta_\psi=0.37(1)$

U(1) GN Model: $\nu=0.849(8)$, $\eta=0.633(8)$, $\eta_\psi=0.373(3)$ ($N_f \to \infty$: $\nu=\eta=1$)





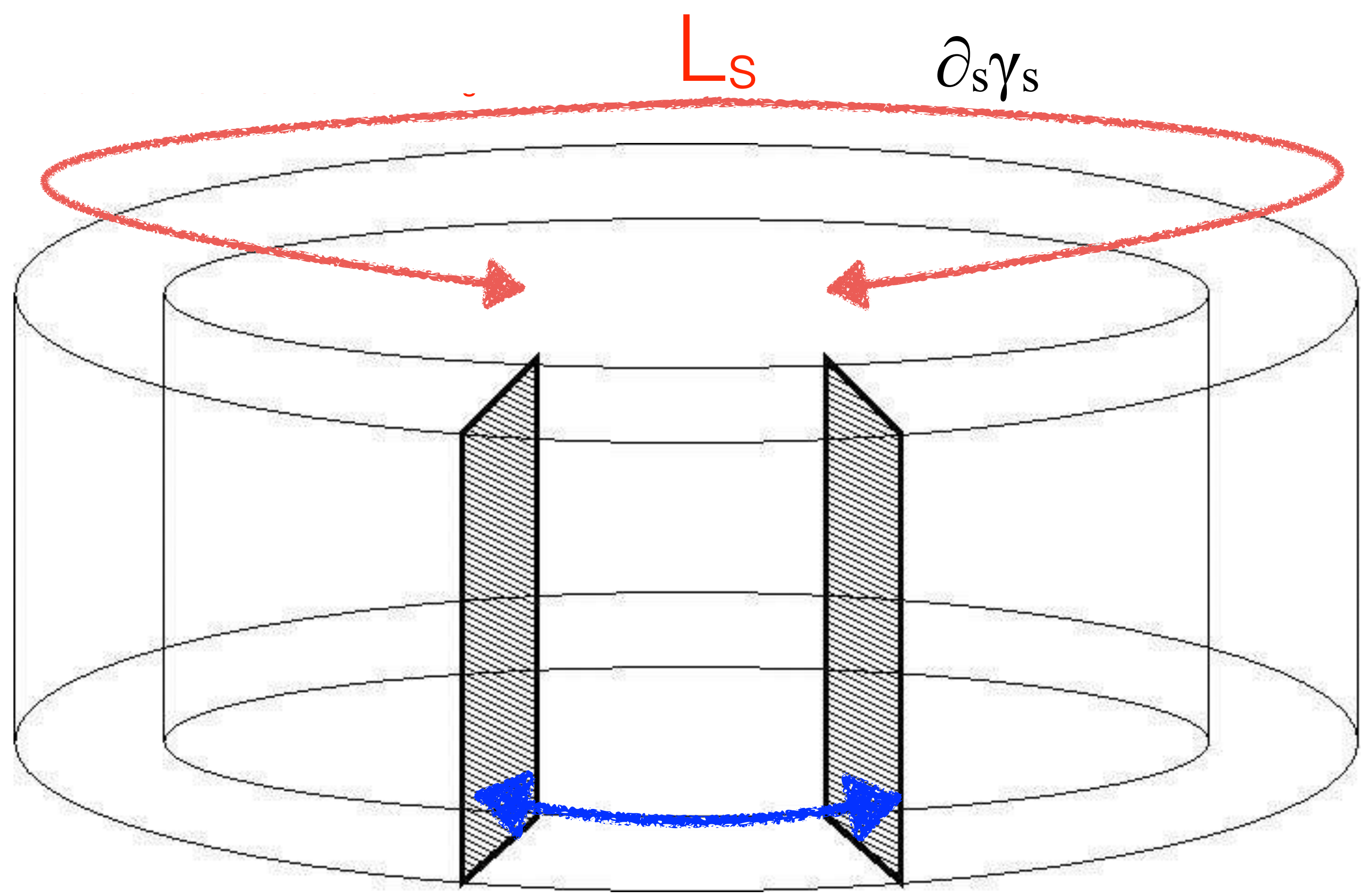
Interactions between staggered fields $\chi, \bar{\chi}$ spread over elementary cubes.
Only difference between Thirring & GN is body-diagonal term

Staggered fermions not reproducing expected distinction between models near strongly-coupled fixed point…

… so we need better lattice fermions

see also Schmidt, Wellegehausen & Wipf, PoS LATTICE2015 (2016) 050

Domain Wall Fermions



Fermions propagate freely along a fictitious third direction of extent L_s with open boundaries

coupling between the walls proportional to explicit massgap m

Basic idea as $L_s \rightarrow \infty$:

- zero-modes of D_{DWF} localised on walls are $\pm$ eigenmodes of γ_s
- Modes propagating in bulk can be decoupled (with cunning)

"Physical" fields in target space

$$\psi(x) = P_- \Psi(x,1) + P_+ \Psi(x, L_s);$$
$$\bar{\psi}(x) = \bar{\Psi}(x, L_s) P_- + \bar{\Psi}(x,1) P_+.$$

with $P_\pm = \frac{1}{2}(1 \pm \gamma_s)$

Are DWF in 2+1+1d U(2N) symmetric?

Issue: wall modes are eigenstates of γ_3 as $L_s \to \infty$,

but: U(2N) symmetry demands equivalence under rotations generated by both γ_3 and γ_5

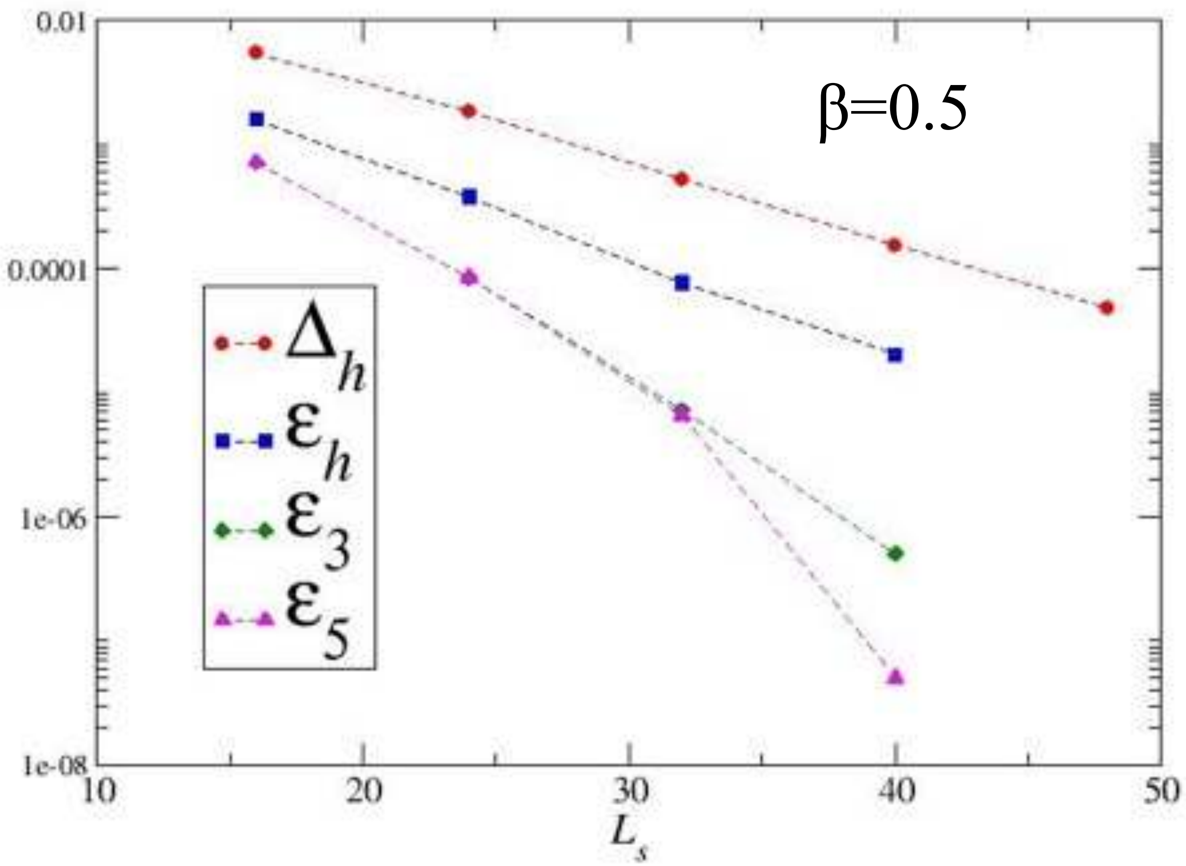
ie. U(2N) → U(N)⊗U(N) symmetry-breaking mass terms

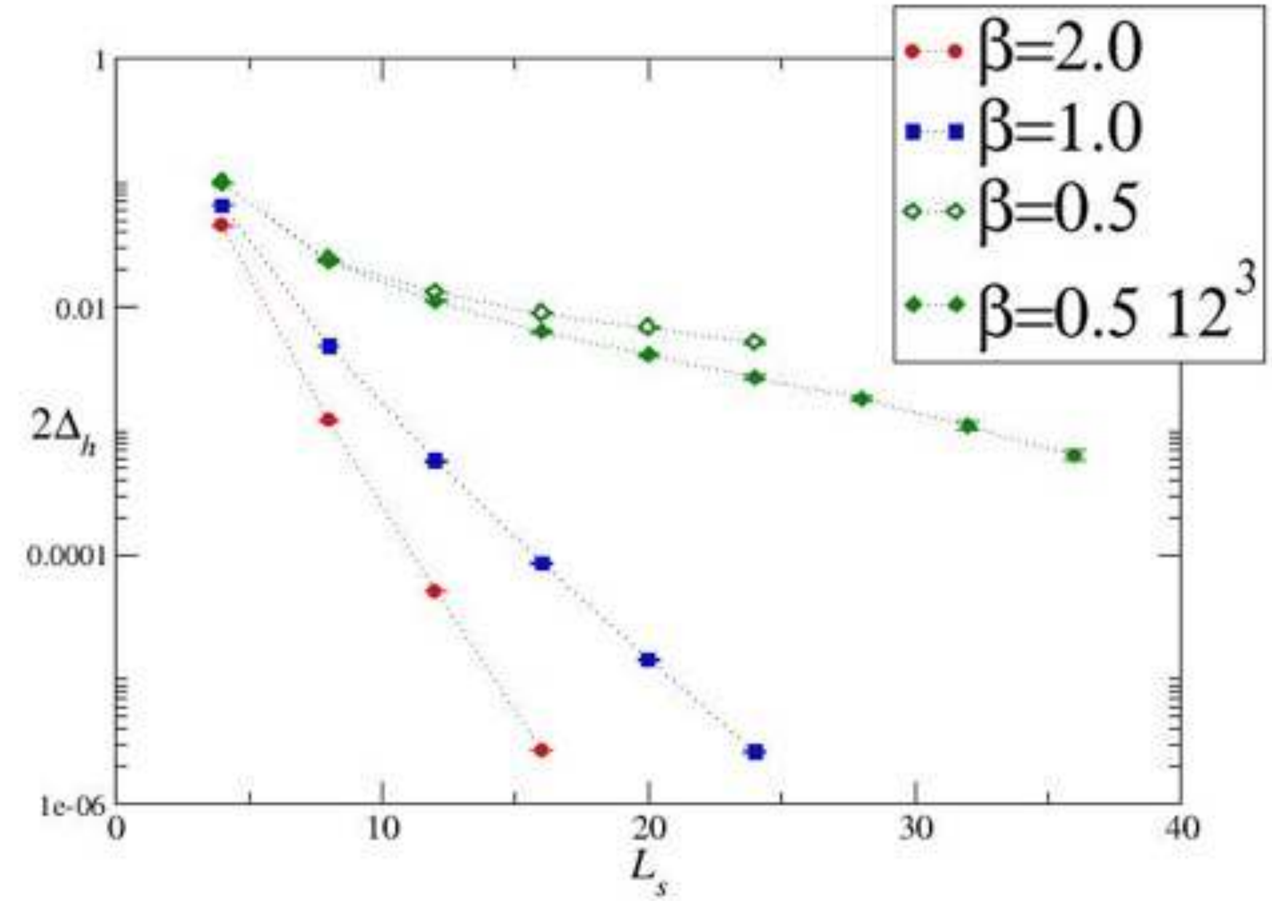
$$m_h \bar{\psi}\psi \qquad im_3 \bar{\psi}\gamma_3\psi \qquad im_5 \bar{\psi}\gamma_5\psi$$

should yield identical physics as $L_s \to \infty$

Non-trivial requirement
since m_h, m_3 couple Ψ, $\bar{\Psi}$ on *opposite* walls
while m_5 couples modes on *same* wall

Bilinear Condensates in Quenched QED_3 on $24^3 \times L_s$…





Define main *residual*: $i\langle\bar{\Psi}(1)\gamma_3\Psi(L_s)\rangle = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s} + i\Delta_h(L_s)$

real imaginary

and then…

$$\frac{1}{2}\langle\bar{\psi}\psi\rangle_{L_s} = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_S\to\infty} + \Delta_h(L_s) + \epsilon_h(L_s);$$

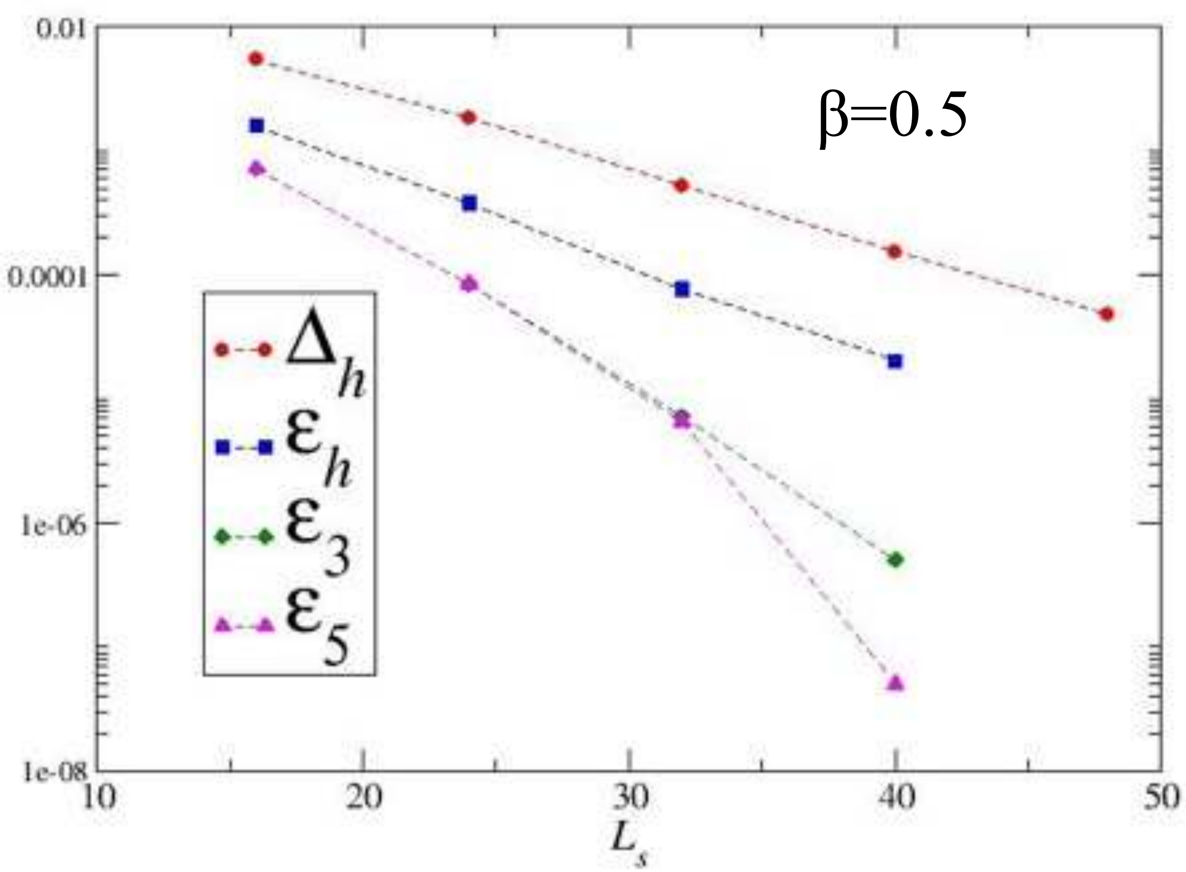
$$\frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s} = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_3(L_s);$$

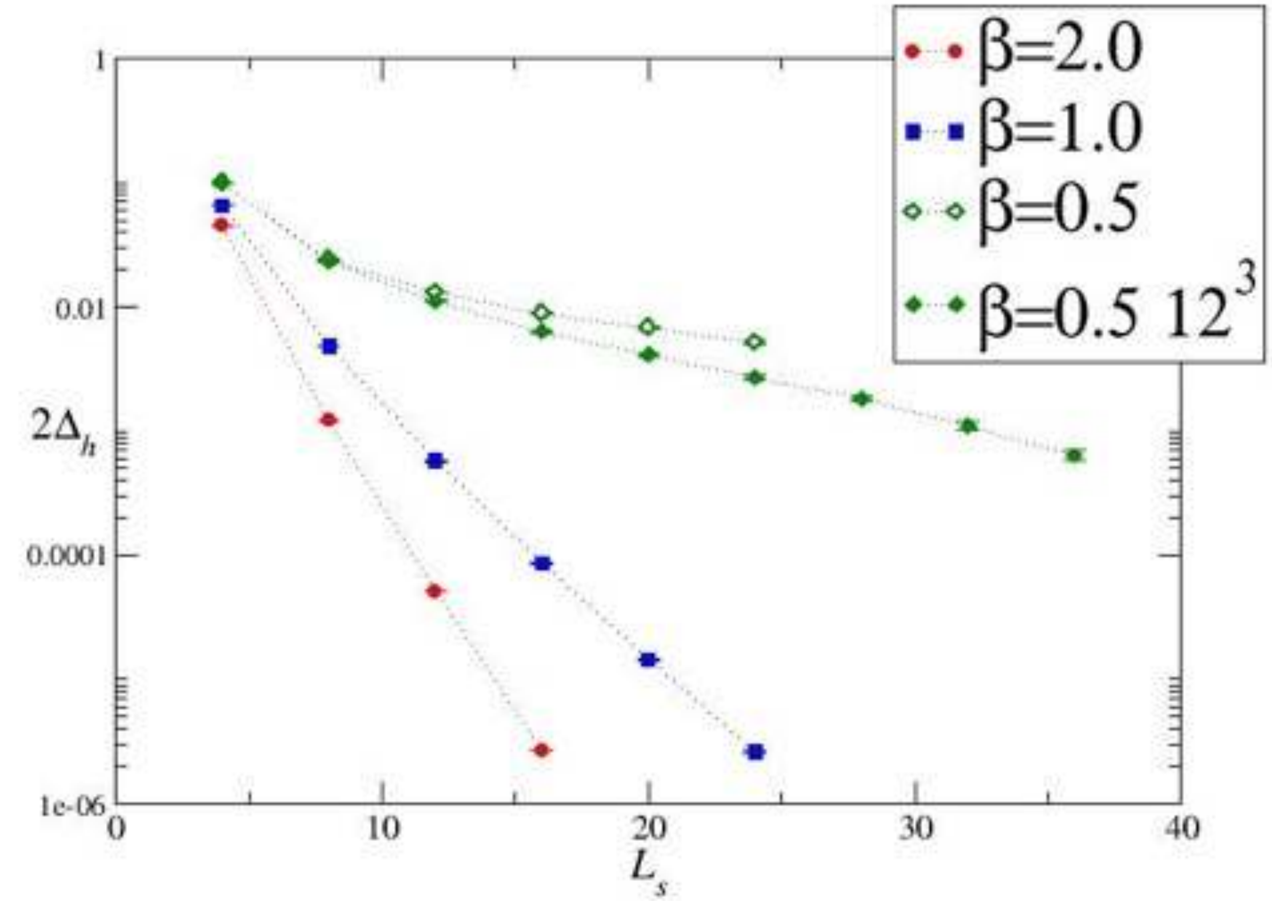
$$\frac{i}{2}\langle\bar{\psi}\gamma_5\psi\rangle_{L_s} = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_5(L_s).$$

SJH JHEP **09**(2015)047,
PLB **754** (2016) 264

- exponentially suppressed as $L_s \to \infty$
- hierarchy: $\Delta_h > \varepsilon_h > \varepsilon_3 \equiv \varepsilon_5$

Bilinear Condensates in Quenched QED_3 on $24^3 \times L_s$…





Define main *residual*: $i\langle\bar\Psi(1)\gamma_3\Psi(L_s)\rangle = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_s} + i\Delta_h(L_s)$

real imaginary

and then…

$$\frac{1}{2}\langle\bar\psi\psi\rangle_{L_s} = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_S\to\infty} + \Delta_h(L_s) + \epsilon_h(L_s);$$

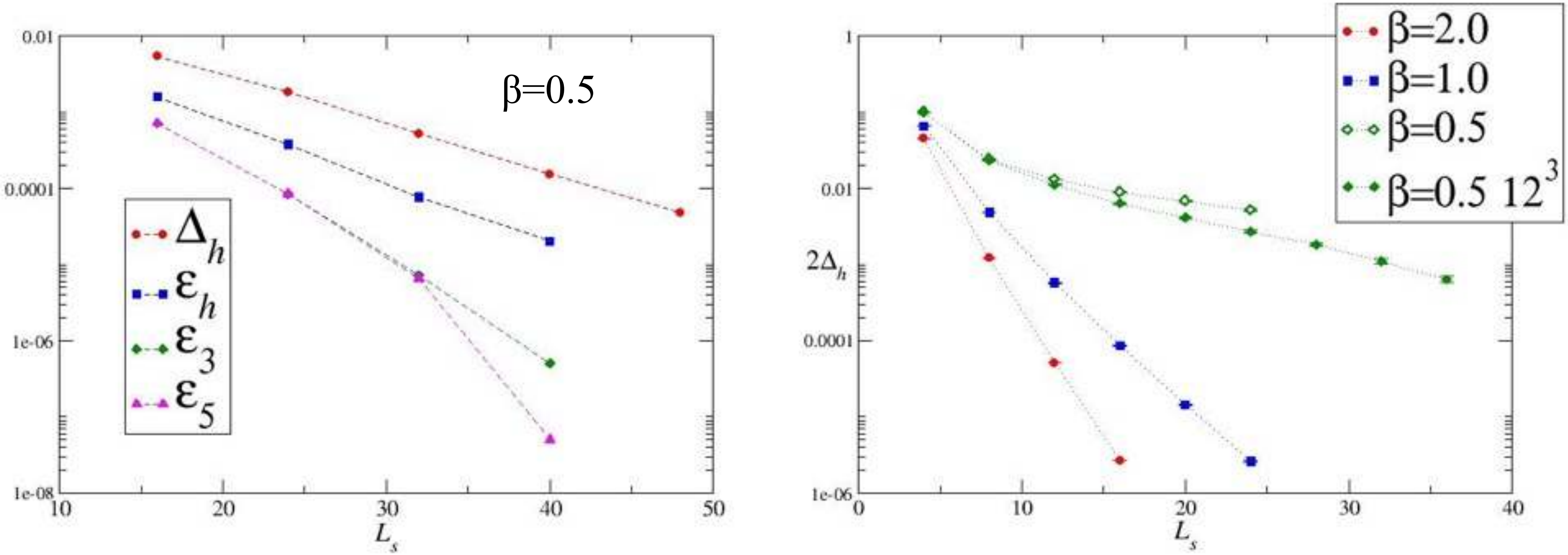
$$\frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_s} = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_3(L_s);$$

$$\frac{i}{2}\langle\bar\psi\gamma_5\psi\rangle_{L_s} = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_5(L_s).$$

SJH JHEP **09**(2015)047, PLB **754** (2016) 264

- exponentially suppressed as $L_s \to \infty$ ✓
- hierarchy: $\Delta_h > \varepsilon_h > \varepsilon_3 \equiv \varepsilon_5$

Bilinear Condensates in Quenched QED_3 on $24^3 \times L_s$…



Define main *residual*: $i\langle\bar{\Psi}(1)\gamma_3\Psi(L_s)\rangle = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s} + i\Delta_h(L_s)$

real imaginary

and then…

$$\frac{1}{2}\langle\bar{\psi}\psi\rangle_{L_s} = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_S\to\infty} + \Delta_h(L_s) + \epsilon_h(L_s);$$

$$\frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s} = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_3(L_s);$$

$$\frac{i}{2}\langle\bar{\psi}\gamma_5\psi\rangle_{L_s} = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_5(L_s).$$

SJH JHEP **09**(2015)047,
PLB **754** (2016) 264

- exponentially suppressed as $L_s \to \infty$ ✓
- hierarchy: $\Delta_h > \varepsilon_h > \varepsilon_3 \equiv \varepsilon_5$ ✓✓

Formulational issues

(a) Formulate interaction terms in terms of vector auxiliary $A_\mu(x)$ defined just on walls at $x_3 = 1, L_s$: "Surface"

Technical advantage: no need to introduce Pauli-Villars determinant to cancel bulk modes

. P. Vranas, I. Tziligakis and J.B. Kogut, Phys. Rev. D 62 (2000) 054507

(b) By analogy with QCD, formulate with $A_\mu(x)$ throughout bulk which are "static" ie. $\partial_3 A_\mu = 0$: "Bulk"

$$\mathcal{S} = \bar{\Psi}\mathcal{D}\Psi = \bar{\Psi}D_W\Psi + \bar{\Psi}D_3\Psi + m_i S_i$$ with $$\begin{aligned} D_W &= \gamma_\mu D_\mu - (\hat{D}^2 + M); \\ D_3 &= \gamma_3\partial_3 - \hat{\partial}_3^2, \end{aligned}$$

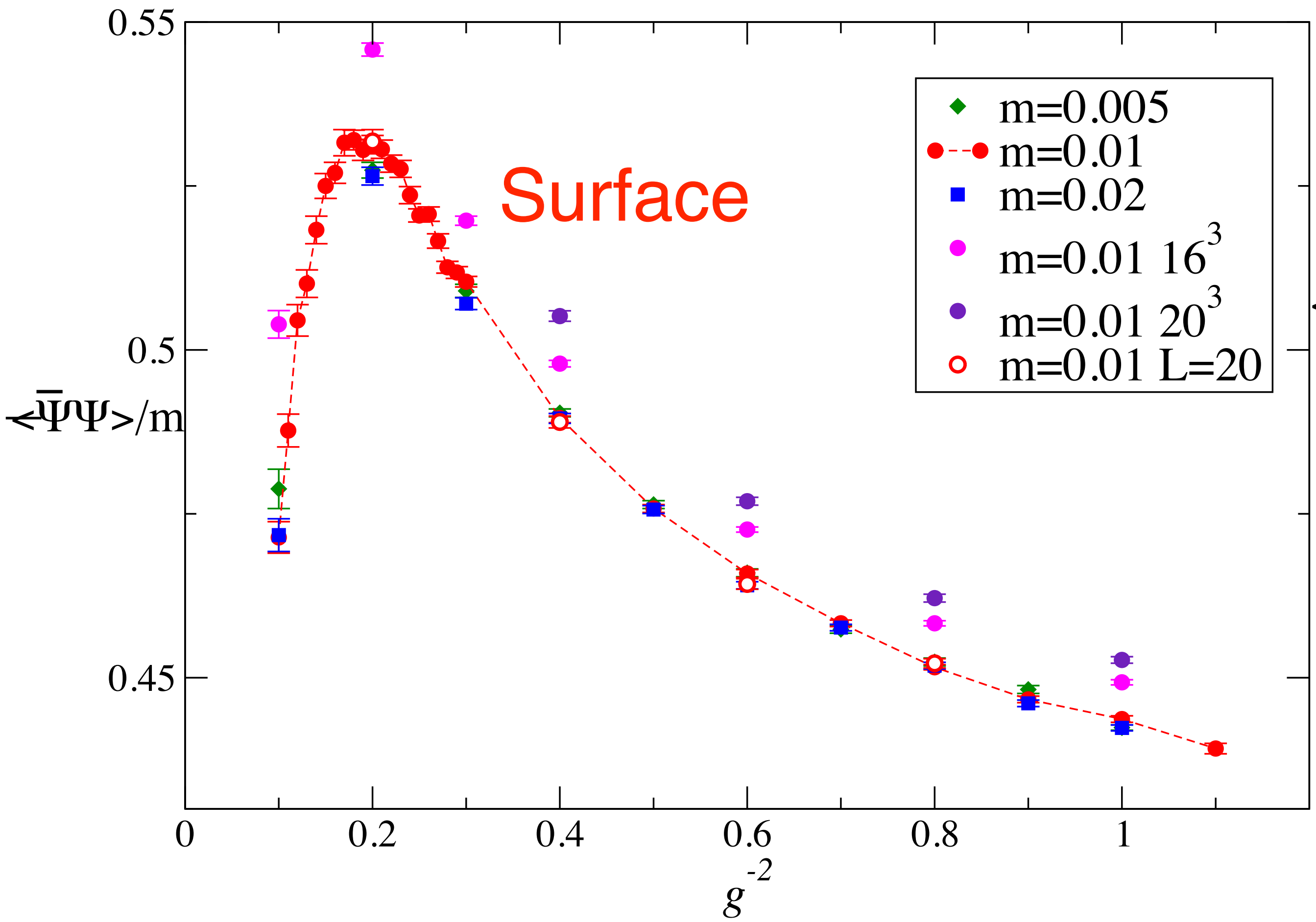
Bulk formulation $[\partial_3, D_\mu] = [\partial_3, \hat{D}^2] = 0$ but $[\partial_3, \hat{\partial}_3^2] \neq 0$ on walls

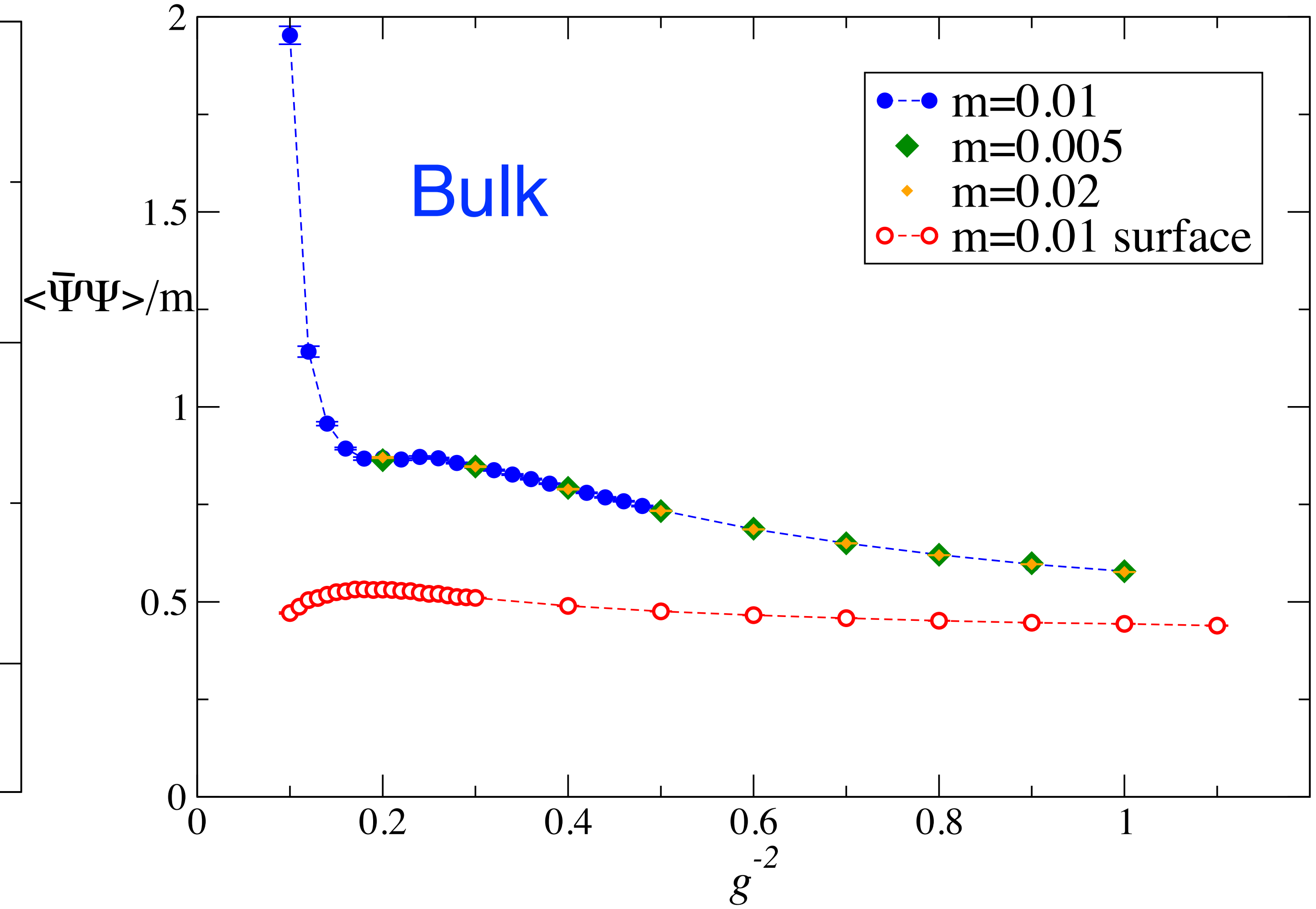
obstruction to proving $\det\mathcal{D} > 0$ for N=1

$\Rightarrow$ **need RHMC algorithm for N=1**

HMC Results with N_f=2 on 12^3×16

SJH JHEP **11**(2016)015

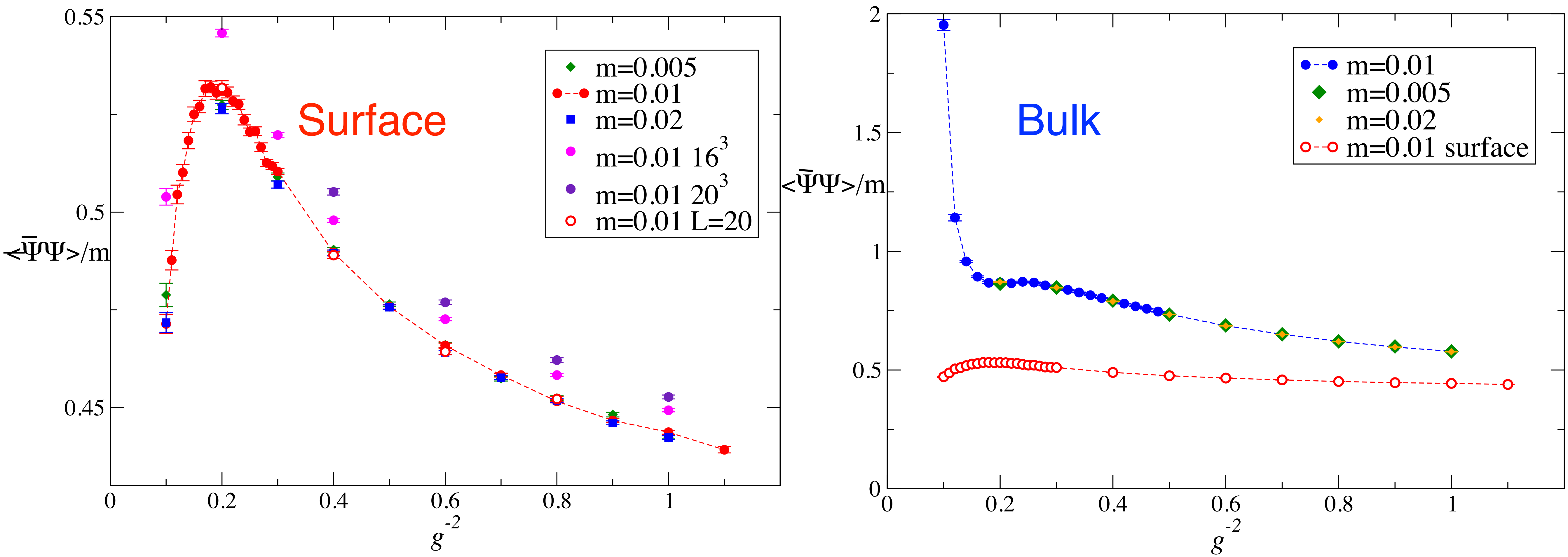




- Breakdown of reflection positivity for strong coupling $ag^{-2}\approx 0.2$?
- Strong volume dependence for surface model
- No evidence of spontaneous symmetry breaking anywhere along g^{-2} axis

HMC Results with N_f=2 on $12^3\times16$

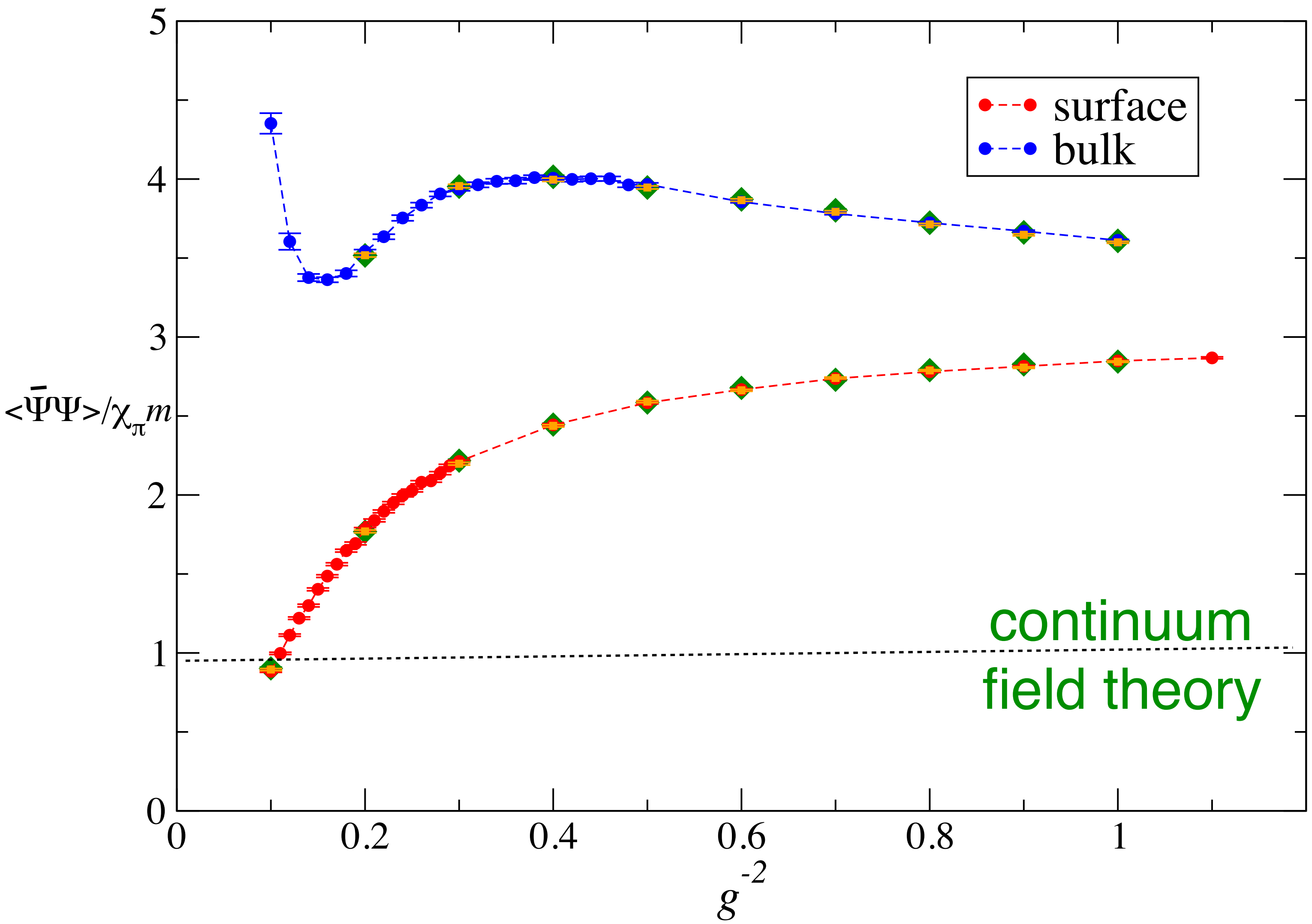
SJH JHEP **11**(2016)015



- Breakdown of reflection positivity for strong coupling $ag^{-2}\approx0.2$?
- Strong volume dependence for surface model
- No evidence of spontaneous symmetry breaking anywhere along g^{-2} axis

big disparity with previous staggered results

Axial Ward Identity



Ratio of order parameter to susceptibility is predicted constant by WI

$$\frac{\langle\bar{\psi}\psi\rangle}{m} = \sum_x \langle\bar{\psi}\gamma_3\psi(0)\bar{\psi}\gamma_3\psi(x)\rangle$$

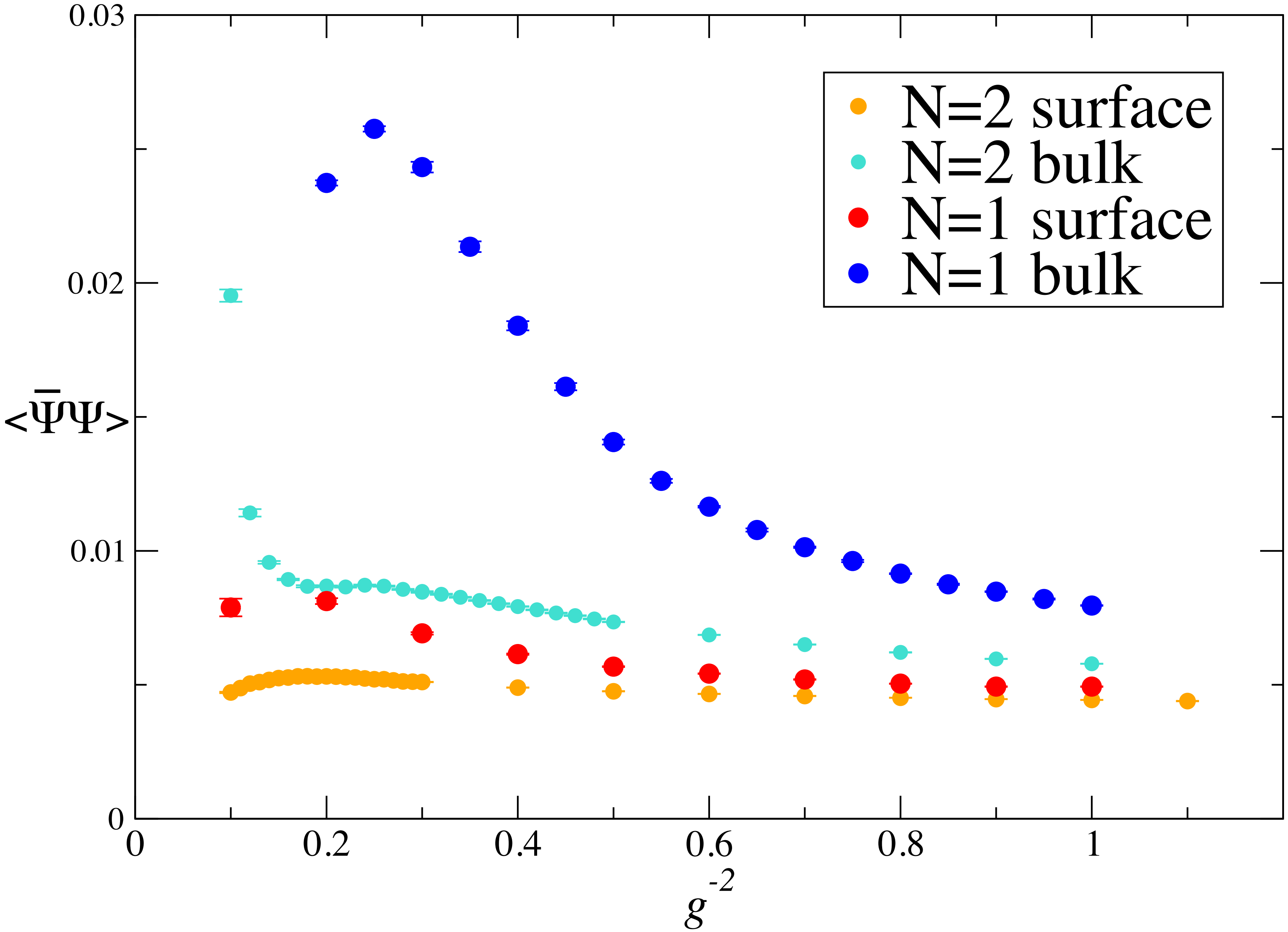
Strong-coupling behaviour suggests neither Surface nor Bulk model optimal

work still needed to specify 2+1 states ψ with control over normalisation

Cf. 2+1d Gross-Neveu model, where Ward Identity is respected, spectroscopy under control…

SJH JHEP **11**(2016)015

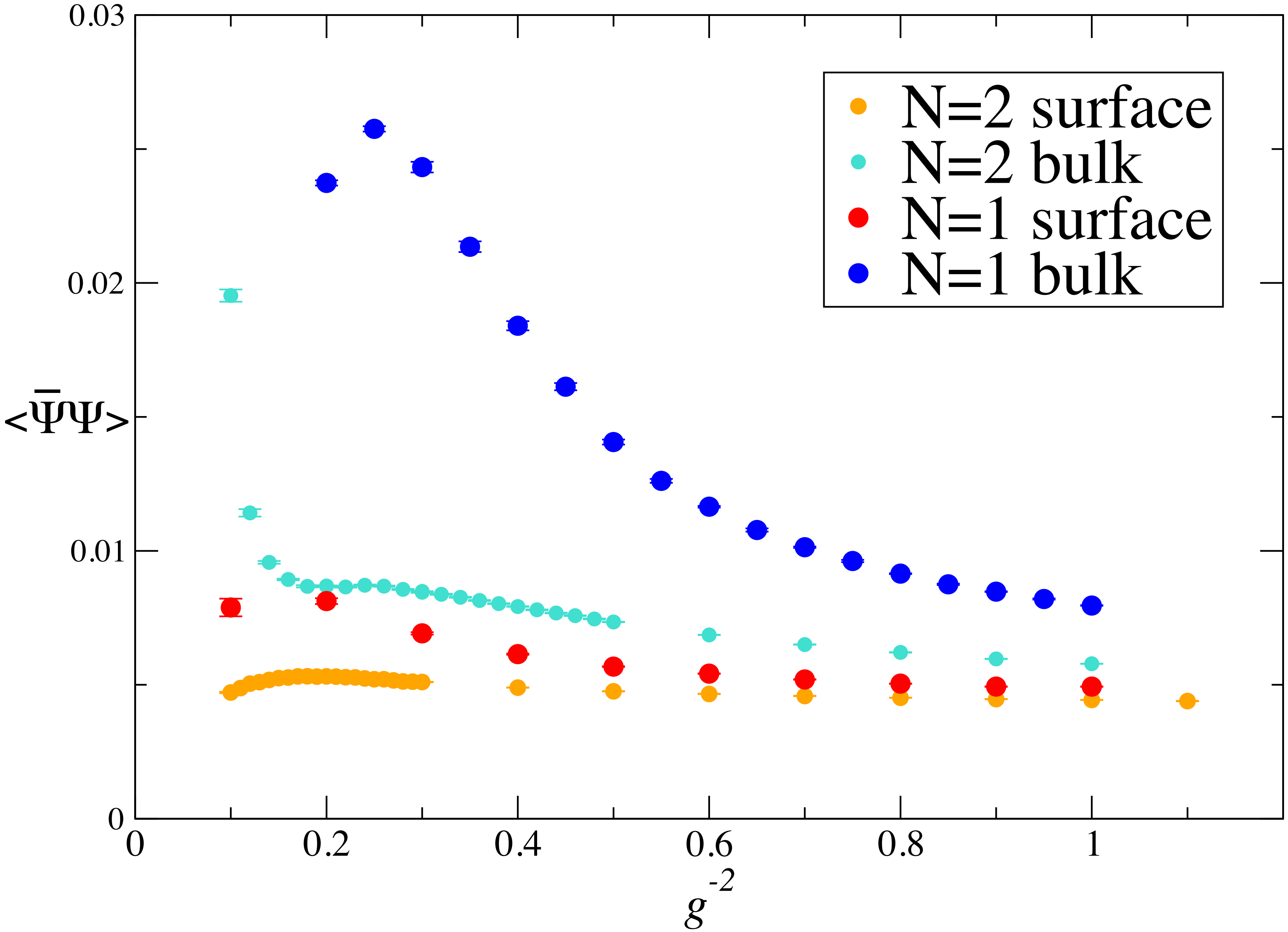
RHMC Results for N=1 (Preliminary, 12^3x8)



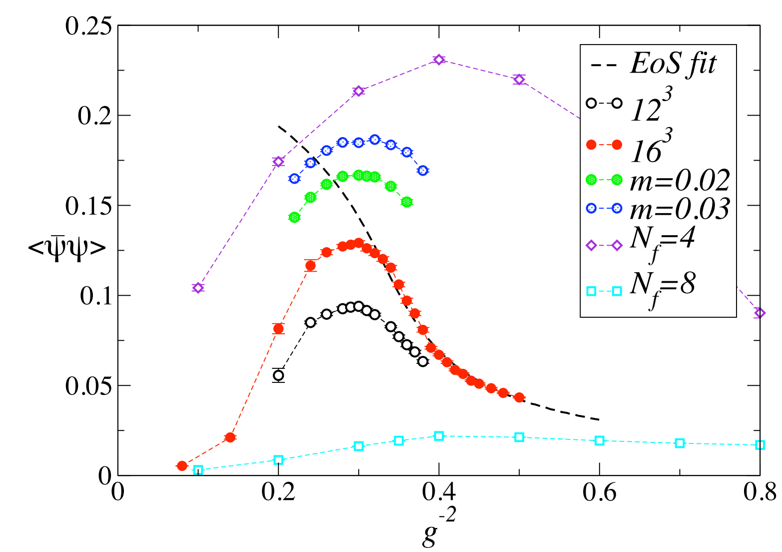
N=1 simulations performed with weight $\det(M^\dagger M)^{1/2}$ using **RHMC** algorithm with 25 partial fractions

Evidence for enhanced pairing for $ag^{-2}<0.5$?

RHMC Results for N=1 (Preliminary, 12^3x8)

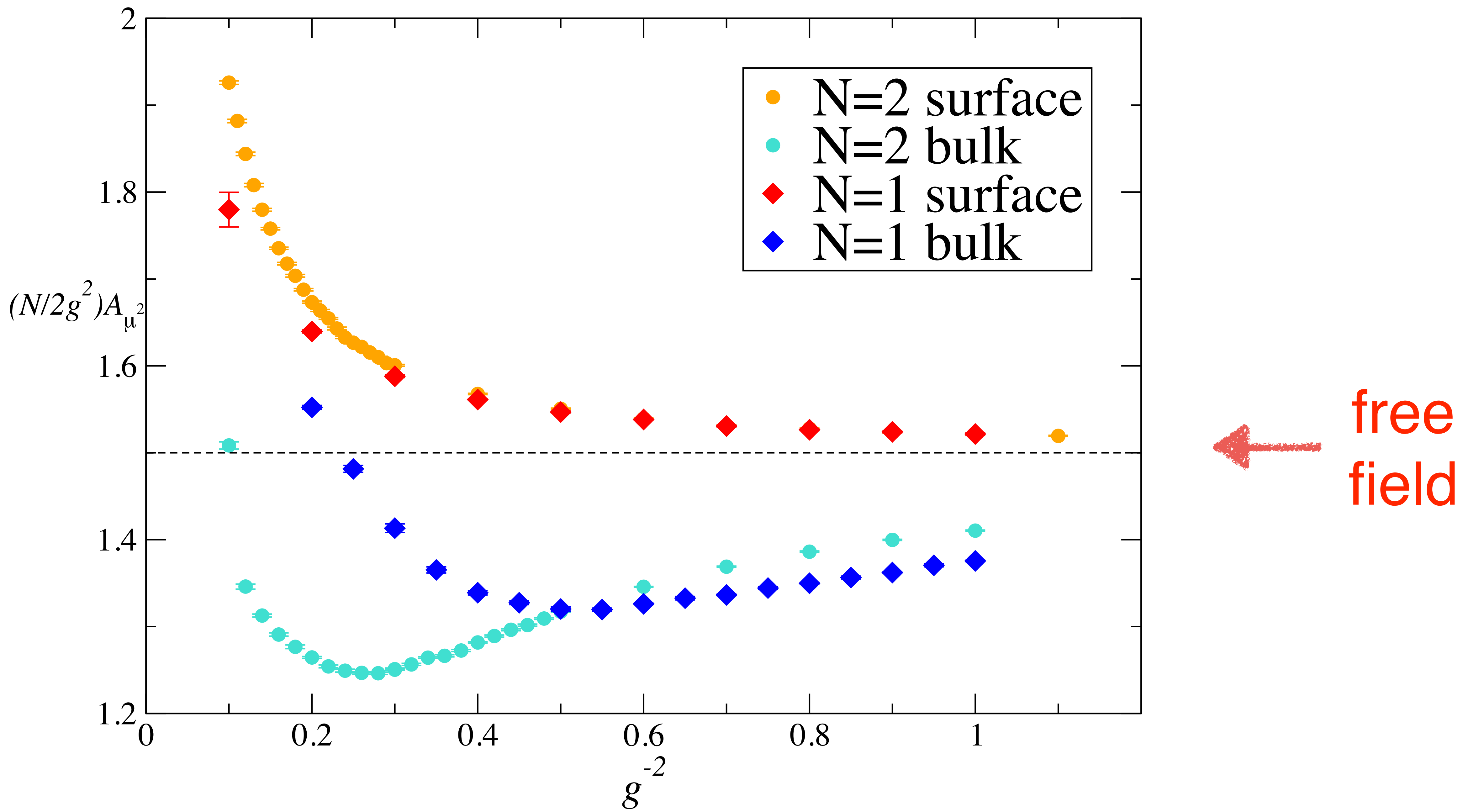


N=1 simulations performed with weight $\det(M^\dagger M)^{1/2}$ using **RHMC** algorithm with 25 partial fractions



Evidence for enhanced pairing for $ag^{-2}<0.5$?

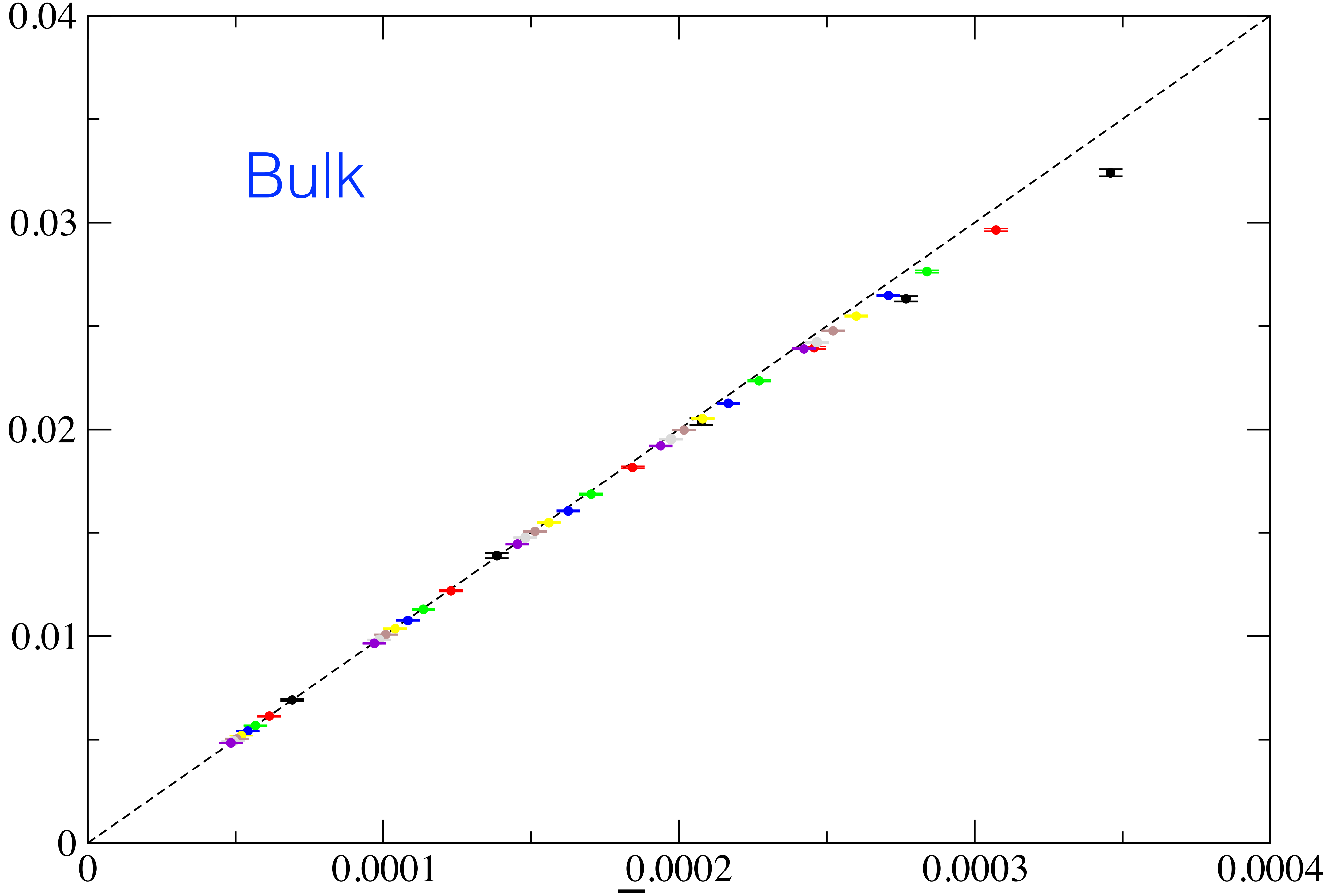
Boson Action



Surface and Bulk models show different behaviour

N=1: change of behaviour for $ag^{-2}<0.5$?

N=1 with ma=0.01,…,0.05

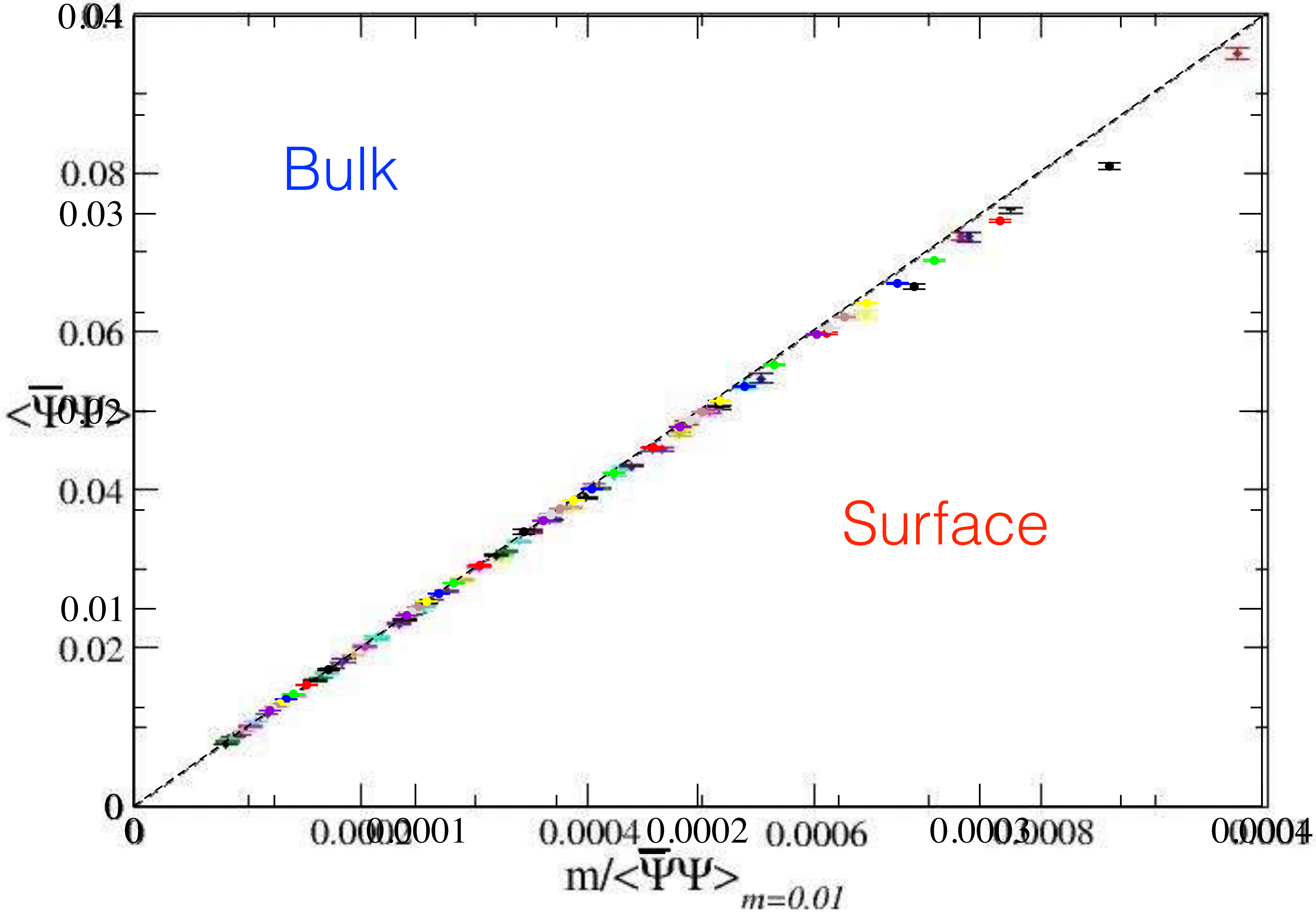


a trivial rescaling results in data collapse, suggesting

$$\lim_{m\to 0}\langle\overline{\Psi}\Psi\rangle=0$$

No evidence for singular behaviour

N=1 with ma=0.01,…,0.05



a trivial rescaling results in data collapse, suggesting

$$\lim_{m\to 0}\langle\overline{\Psi}\Psi\rangle = 0$$

No evidence for singular behaviour

Summary & Outlook

- No obstruction found to simulating U(2N) fermions
- "twisted mass" $im_3\bar{\psi}\gamma_3\psi$ optimises $L_s\to\infty$
- DWF capture very different physics to staggered fermions
- Bulk and surface models agree: $N_{fc}<1$ for pure Thirring ?

Cf. QED$_3$ $N_{fc}<1$ Karthik & Narayanan PR**D93** 045020, **D94** 065026 (2016)

- Need to check $L_s\to\infty$, $V\to\infty$, and effect of varying M_{wall}
- What does χSB look like for DWF in 2+1+1?
  Quenched study in progress
- Will need to examine locality of corresponding D_{ov}
- Investigate U(2N)-invariant "Haldane" term $-g_2(\bar{\psi}\gamma_3\gamma_5\psi)^2$ as indicated by FRG studies of QCP

Gehring, Gies & Janssen PR**D92** (2015) 085046