Isospin Breaking Corrections to the HVP with Domain Wall Fermions

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Introduction

- hadronic vacuum polarisation (HVP) is leading hadronic contribution to $\mathbf{a}_\mu$
- lattice calculation aiming at $\mathbf{1\%}$ precision requires to include isospin breaking
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((\mathbf{m_d} - \mathbf{m_u})/\Lambda_{QCD}))$
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)

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  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)

- Euclidean path integral including QED

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}]\mathcal{D}[U]\mathcal{D}[A] \; O \; e^{-S_F[\Psi,\overline{\Psi},U,A]} \, e^{-S_G[U]} \, e^{-S_\gamma[A]}$$

- non-compact photon action

$$S_\gamma [A] = \frac{a^4}{4} \sum_x \sum_{\mu,\nu} \left(\partial_\mu A_\nu (x) - \partial_\nu A_\mu (x)\right)^2$$

- this study: two approaches
  - stochastic QED using **U(1)** gauge configurations
    [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
  - perturbative QED by expanding the path integral in α
    [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

stochastic method

- Feynman gauge

$$S_\gamma^{\mathrm{Feyn}}[A] = -\frac{a^4}{2}\sum_x\sum_\mu A_\mu(x)\partial^2 A_\mu(x) \qquad \text{with} \quad \partial^2 = \sum_\mu \partial_\mu^*\partial_\mu$$

- in momentum space

$$S_\gamma^{\mathrm{Feyn}}[A] = \frac{1}{2N}\sum_{k,\vec{k}\neq 0}\hat{k}^2\sum_\mu\left|\tilde{A}_\mu(k)\right|^2 \qquad \hat{k}_\mu = \frac{2}{a}\sin\left(\frac{ak_\mu}{2}\right)$$

- remove all spatial zero modes $\rightarrow$ $\mathrm{QED_L}$
  [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)]
- draw $\tilde{A}_\mu(k)$ from Gaussian distribution with variance $2N/\hat{k}^2$
- electro quenched approximation
- multiply $SU(3)$ gauge links with $U(1)$ photon fields

$$U_\mu(x) \rightarrow e^{ieA_\mu(x)}U_\mu(x)$$

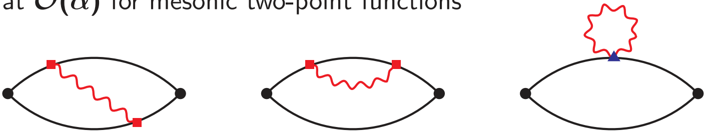
- remove $\mathcal{O}(e)$ noise by averaging over $+e$ and $-e$
- QED correction to all orders in α

perturbative method

- expand path integral in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

$$\langle O \rangle = \langle O \rangle_0 + \frac{1}{2} e^2 \frac{\partial^2}{\partial e^2} \langle O \rangle \Big|_{e=0} + \mathcal{O}(\alpha^2)$$

- at $\mathcal{O}(\alpha)$ for mesonic two-point functions



■ conserved vector current, ▲ tadpole operator

- photon propagator Feynman gauge

$$\Delta_{\mu\nu}(x-y) = \delta_{\mu\nu} \frac{1}{N} \sum_{k, \vec{k} \neq 0} \frac{e^{ik\cdot(x-y)}}{\hat{k}^2}$$

- calculate diagrams using sequential propagators

perturbative method

- method 1: $\left\langle \eta(\mathbf{u})\eta^\dagger(\mathbf{y})\right\rangle_\eta = \delta_{\mathbf{uy}}$

$$\mathbf{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \left\langle \sum_{\mathbf{u}} \mathbf{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{u})\eta(\mathbf{u})\eta^\dagger(\mathbf{y})\right\rangle_\eta = \left\langle \tilde{\mathbf{\Delta}}_{\mu\nu}(\mathbf{x})\eta^\dagger(\mathbf{y})\right\rangle_\eta$$

$\rightarrow$ sequential sources for every combination of $\{\mu,\nu\}$, e.g.

$\rightarrow$ **17** inversions in Feynman gauge

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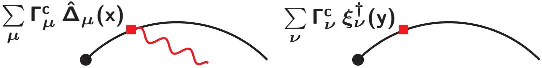
- method 2: $\left\langle \xi_\sigma(\mathbf{u})\xi^\dagger_\nu(\mathbf{y}) \right\rangle_\xi = \delta_{\mathbf{uy}}\delta_{\sigma\nu}$ [RM123 Collaboration]

$$\Delta_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \left\langle \sum_{\mathbf{u}} \sum_{\sigma} \Delta_{\mu\sigma}(\mathbf{x}-\mathbf{u})\xi_\sigma(\mathbf{u})\xi^\dagger_\nu(\mathbf{y}) \right\rangle_\xi = \left\langle \hat{\Delta}_{\mu}(\mathbf{x})\xi^\dagger_\nu(\mathbf{y}) \right\rangle_\xi$$

$\rightarrow$ sequential sources summed over μ or ν, e.g.



$\rightarrow$ **5** inversions

setup of the run

- $N_f = 2+1$ Domain Wall Fermions
- 64×24^3 lattice with $a^{-1} = 1.78$ GeV
- $L_s = 16$, $M_5 = 1.8$
- **87** gauge configurations
- **16** source positions
- isospin symmetric pion mass $m_\pi = 340$ MeV
- physical valence strange quark mass [T. Blum *et al*, Phys. Rev. D93, 074505 (2016)] (not re-tuned in presence of QED)

- stochastic method:
  - one $U(1)$ photon configuration per gauge configuration
- perturbative method:
  - one $\mathbb{Z}_2$ noise for the stochastic insertion of the photon propagator per gauge configuration and source position

The hadronic vacuum polarisation

- construction of the HVP from (see e.g. [RBC/UKQCD, JHEP 1604 (2016) 063] for details)

$$\mathbf{C}_{\mu\nu}(\mathbf{x}) = \mathbf{Z}_v\, \mathbf{q}_f^2 \left\langle \mathbf{V}^c_\mu(\mathbf{x}) \mathbf{V}^\ell_\nu(0) \right\rangle$$

- QED correction to vector two-point function

$$\delta \mathbf{C}_{\mu\nu}(\mathbf{x}) = \delta \mathbf{Z}_v\ \mathbf{q}_f^2 \left\langle \mathbf{V}^c_\mu(\mathbf{x}) \mathbf{V}^\ell_\nu(0) \right\rangle + \mathbf{Z}_v\, \mathbf{q}_f^2\ \delta \left\langle \mathbf{V}^c_\mu(\mathbf{x}) \mathbf{V}^\ell_\nu(0) \right\rangle$$

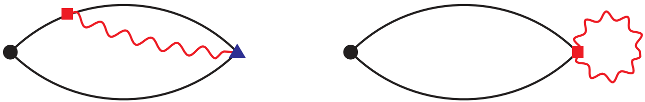
- conserved vector current depends on link variables

$$\mathbf{U}_\mu(\mathbf{x}) \to \mathbf{e}^{\mathbf{ieA}_\mu(\mathbf{x})} \mathbf{U}_\mu(\mathbf{x}) \qquad \text{and thus} \quad \mathbf{V}^c_\mu(\mathbf{x}) \to \mathbf{V}^{c,e}_\mu(\mathbf{x})$$

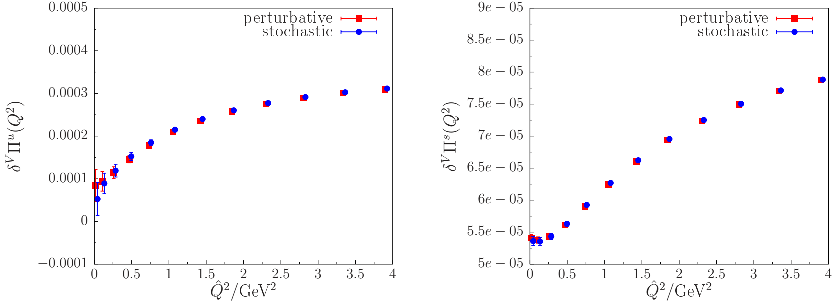
- perturbative expansion

$$\left\langle \mathbf{V}^{c,e}_\mu(\mathbf{x}) \mathbf{V}^\ell_\nu(0) \right\rangle = \left\langle \mathbf{V}^c_\mu(\mathbf{x}) \mathbf{V}^\ell_\nu(0) \right\rangle_0 + \frac{1}{2} \mathbf{e}^2 \frac{\partial^2}{\partial \mathbf{e}^2} \left\langle \mathbf{V}^{c,e}_\mu(\mathbf{x}) \mathbf{V}^\ell_\nu(0) \right\rangle \Bigg|_{\mathbf{e}=0}$$

- two additional types of diagrams



QED correction to the vector correlation function



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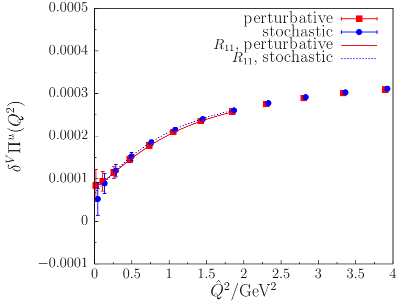

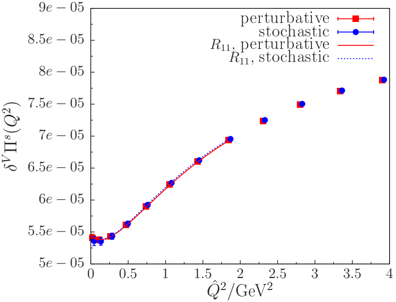

- Padé ansatz

$$\mathbf{R_{11}(\hat{Q}^2) = R_{11}^0 + \delta R_{11} = \left(\Pi_0^0 + \delta\Pi_0\right) + \hat{Q}^2 \left(\frac{(a^0 + \delta a)}{(b^0 + \delta b) + \hat{Q}^2} + \left(c^0 + \delta c\right)\right)}$$

	$a_\mu^0 \times 10^{10}$	$\delta a_\mu^{\mathrm{stoch}} \times 10^{10}$	$\delta a_\mu^{\mathrm{pert}} \times 10^{10}$
u	318 ± 11	0.65 ± 0.31	0.37 ± 0.33
s	47.98 ± 0.25	-0.0030 ± 0.0011	-0.0049 ± 0.0010

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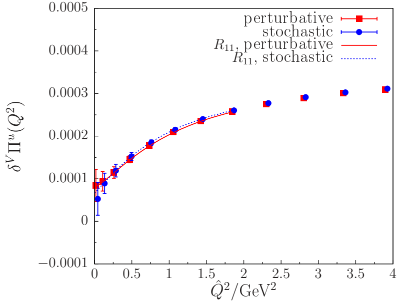

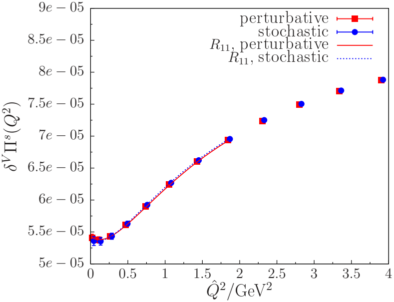

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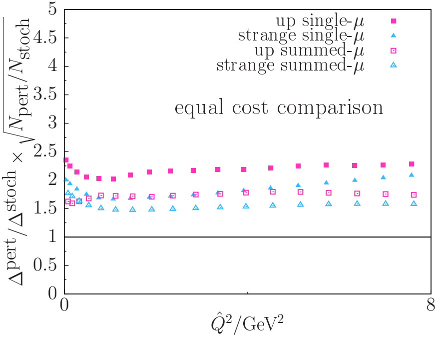
QED correction smaller than **1%**.

Comparison of statistical error

- numerical cost (i.e. number of inversions)
- stochastic method: $\mathbf{N_{stoch} = 3} \quad (\mathbf{e = 0, +e, -e})$
- perturbative method, single-μ insertion: $\mathbf{N_{pert} = 17}$
- perturbative method, summed-μ insertion: $\mathbf{N_{pert} = 5}$



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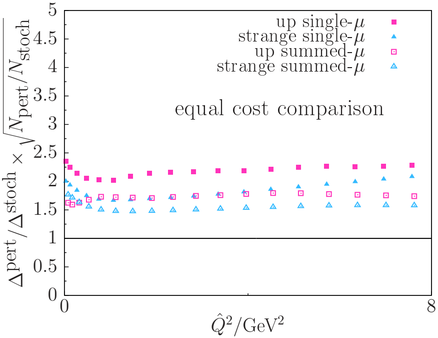
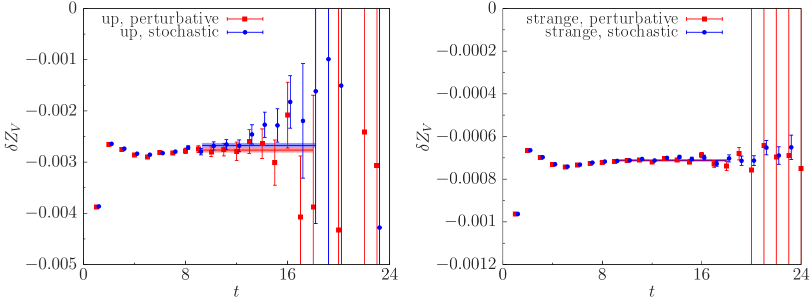


stochastic method gives **1.5 − 2** times smaller statistical errors for same cost

QED correction to $\mathbf{Z_V}$

- ratio of local-conserved $\mathbf{C^{lc}(t)}$ and local-local $\mathbf{C^{ll}(t)}$ vector two-point function
- QED correction to $\mathbf{Z_V}$ from

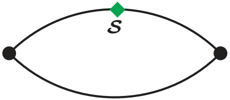
$$Z_V = Z_V^0 + \delta Z_V = \frac{C_0^{lc}(t) + \delta C^{lc}(t)}{C_0^{ll}(t) + \delta C^{ll}(t)} \quad \Rightarrow \quad \delta Z_V = \frac{\delta C^{lc}(t)}{C_0^{ll}(t)} - \frac{C_0^{lc}(t)}{C_0^{ll}(t)} \frac{\delta C^{ll}(t)}{C_0^{ll}(t)}$$

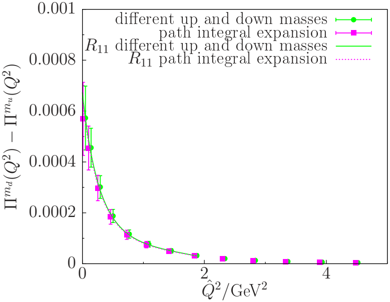


	$\mathbf{Z_V^0}$	$\mathbf{\delta Z_V^{stoch}}$	$\mathbf{\delta Z_V^{pert}}$
up	**0.70209 ± 0.00083**	**−0.002674 ± 0.000043**	**−0.002756 ± 0.000044**
strange	**0.69737 ± 0.00017**	**−0.0007102 ± 0.0000016**	**−0.0007139 ± 0.0000016**

strong isospin breaking correction to HVP

- use different quark masses for up and down quark to reproduce physical mass splitting [Z. Fodor *et al*, Phys. Rev. Lett. 117 (2016) 082001]
- expansion in $\mathbf{\Delta m = (m_u - m_d)}$ [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]





	$\delta_s a_\mu / q_f^2$ using Padé $\mathbf{R_{11}}$
diff u/d masses	$(-6.7 \pm 1.6) \times 10^{-10}$
path integral exp	$(-6.4 \pm 1.7) \times 10^{-10}$

strong Isospin Breaking correction $\delta_s a_\mu / a_\mu^{u,0} \approx \mathbf{0.9\%}$

Summary

- QED corrections to HVP at unphysical quark masses

		a_μ^0	$\delta^{VV} a_\mu^0$	$\delta^{Z_V} a_\mu^0$
stochastic	u	318(11)	0.65(31)	−1.181(75)
	s	47.98(25)	−0.0030(12)	−0.04938(27)
perturbative	u	318(11)	0.37(33)	−1.217(71)
	s	47.98(25)	−0.0049(11)	−0.04964(25)

$\rightarrow$ smaller than **1%** of isospin symmetric result

- Comparison of stochastic and perturbative method to include QED
  $\rightarrow$ stochastic method gives smaller statistical errors for the same cost
- strong isospin breaking correction $\approx$ **0.9%**

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- Comparison of stochastic and perturbative method to include QED
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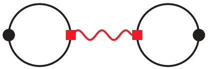
Outlook

- physical quark-masses [C. Lehner, Sat 11:30]
- quark-disconnected diagrams



- finite volume corrections [J. Harrison, Thu 17:50]

Backup

expansion of the Wilson-Dirac operator in e

- Including QED link variables the action is

$$S_W^e = \sum_x \Big[\overline{\Psi}(x)\,(M+4)\,\Psi(x) - \frac{1}{2}\,\overline{\Psi}(x)(1-\gamma_\mu)E_\mu(x)U_\mu(x)\Psi(x+\mu) - \frac{1}{2}\,\overline{\Psi}(x+\mu)(1-\gamma_\mu)U_\mu^\dagger(x)E_\mu^\dagger(x)\Psi(x) \Big]$$

with QED link variables

$$E_\mu(x) = e^{-iee_f A_\mu(x)} = 1 - iee_f A_\mu(x) + \frac{1}{2}(ee_f)^2 A_\mu(x)A_\mu(x) + \ldots$$

- Expanding the action in **e** one finds

$$S_W^e - S_W^0 = \sum_{x,\mu} \left\{ -iee_f\, A_\mu(x)\, V_\mu^c(x) + \frac{(ee_f)^2}{2}\, A_\mu(x)A_\mu(x)\;\; T_\mu(x) \right\}$$

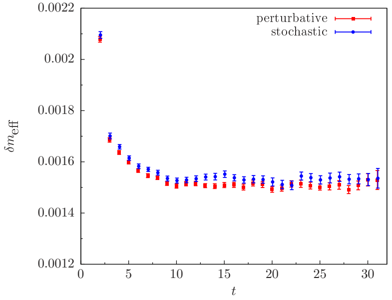
with the conserved vector current $V_\mu^c(x)$ and the tadpole operator $T_\mu(x)$

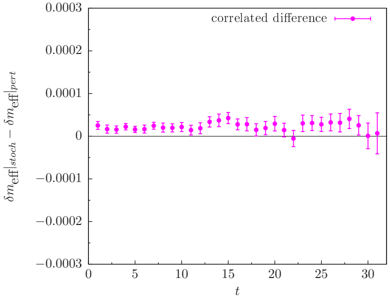
$$V_\mu^c(x) = \frac{1}{2}\Big[\overline{\Psi}(x+\mu)(1+\gamma_\mu)U_\mu^\dagger(x)\Psi(x) - \overline{\Psi}(x)(1-\gamma_\mu)U_\mu(x)\Psi(x+\mu)\Big]$$

$$T_\mu(x) = \frac{1}{2}\Big[\overline{\Psi}(x)(1-\gamma_\mu)U_\mu(x)\Psi(x+\mu) + \overline{\Psi}(x+\mu)(1+\gamma_\mu)U_\mu^\dagger(x)\Psi(x)\Big]$$

QED correction to meson masses

- QED correction to effective mass of a Kaon





- difference between stochastic and perturbative is of $\mathcal{O}(\alpha^2)$

	stochastic		perturbative	
	δm /MeV	$\delta m^{\text{inf V}}$ /MeV	δm /MeV	$\delta m^{\text{inf V}}$ /MeV
$\delta m^{\gamma}_{\pi^+}$	3.504 ± 0.025	4.597 ± 0.025	3.459 ± 0.016	4.552 ± 0.016
$\delta m^{\gamma}_{\pi^0}$	1.555 ± 0.015	1.555 ± 0.015	1.538 ± 0.016	1.538 ± 0.016
$\delta m^{\gamma}_{K^+}$	2.722 ± 0.022	3.699 ± 0.022	2.677 ± 0.013	3.653 ± 0.013
$\delta m^{\gamma}_{K^0}$	0.547 ± 0.005	0.547 ± 0.005	0.548 ± 0.005	0.548 ± 0.005

Coulomb gauge

- projector for photon fields [Borsanyi et al., Science **347** (2015) 1452-1455]

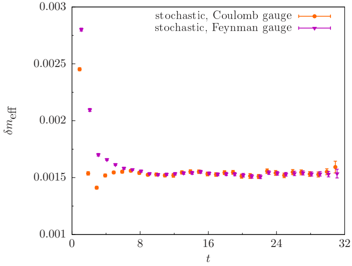
$$(\mathbf{P_C})_{\mu\nu} = \delta_{\mu\nu} - \left|\vec{\hat{\mathbf{k}}}\right|^{-2} \hat{\mathbf{k}}_\mu \left(0, \vec{\hat{\mathbf{k}}}\right)_\nu \qquad \text{with} \quad \tilde{\mathbf{A}}^{\text{Coul}}_\mu(\mathbf{k}) = (\mathbf{P_C})_{\mu\nu}\, \tilde{\mathbf{A}}^{\text{Feyn}}_\nu(\mathbf{k})$$

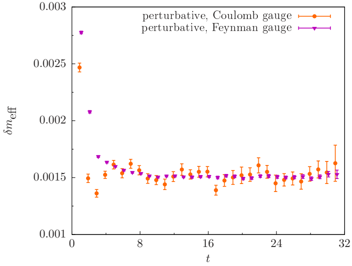
- Coulomb gauge photon propagator

$$\boldsymbol{\Delta}^{\text{Coul}}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \frac{1}{\mathbf{N}} \sum_{\mathbf{k}, \vec{\mathbf{k}} \neq 0} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} e^{i\mathbf{k}(\hat{\mu}-\hat{\nu})/2} \frac{1}{\hat{\mathbf{k}}^2} \left(\delta_{\mu\nu} - \frac{1}{\vec{\hat{\mathbf{k}}}^{\,2}} \left(\hat{\mathbf{k}}_\mu \tilde{\hat{\mathbf{k}}}_\nu + \tilde{\hat{\mathbf{k}}}_\mu \hat{\mathbf{k}}_\nu - \hat{\mathbf{k}}_\mu \hat{\mathbf{k}}_\nu\right)\right)$$

with $\tilde{\mathbf{k}}_\mu \equiv (0, \vec{\mathbf{k}})_\mu$

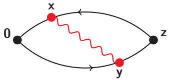
- QED correction to effective mass of charged Kaon



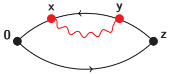


QED corrections to the HVP - perturbative data

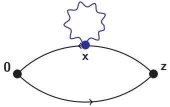
photon exchange



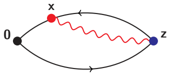
self energy



tadpole



contact 1



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