How to Identify Zero Modes for Improved Staggered Fermions

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Zero Modes and Chirality

- Chirality operators for staggered fermions : $[\gamma_5 \otimes \mathbb{1}], \epsilon \equiv [\gamma_5 \otimes \xi_5]$

$$\langle f | (\mathcal{O}_1 \otimes \mathcal{O}_2) | f \rangle \equiv \int d^4x \; f^\dagger(x_A) \overline{(\mathcal{O}_1 \otimes \mathcal{O}_2)}_{AB} U(x_A, x_B) f(x_B) \quad (1)$$

with

$$\overline{(\mathcal{O}_1 \otimes \mathcal{O}_2)}_{AB} = \frac{1}{4} \mathrm{Tr}\,(\gamma_A^\dagger \mathcal{O}_1 \gamma_B \mathcal{O}_2^\dagger)\,, \quad (2)$$

$$U(x_A, x_B) = \mathbb{P}_{\mathrm{SU(3)}} \left[\sum_{p \in \mathcal{C}} V(x_A, x_{p_1}) V(x_{p_1}, x_{p_2}) V(x_{p_2}, x_{p_3}) V(x_{p_3}, x_B) \right] \quad (3)$$

- $x_A = 2x + A$, $x_B = 2x + B$
  where A, B are the hypercubic vectors with $A_\mu, B_\mu \in \{0, 1\}$
- $V(x, y)$: (HYP-smeared) gluon link from x to y
- $\mathbb{P}_{\mathrm{SU(3)}}$: SU(3) projection
- $\mathcal{C}$: complete set of the shortest paths from A to B

Eigenmodes for Staggered Fermions

- $D_s^\dagger = -D_s$: eigenvalues are purely imaginary or zero:

$$D_s | f_\lambda(x) \rangle = i\lambda | f_\lambda(x) \rangle \quad (4)$$

- $\varepsilon D_s = -D_s \varepsilon$ for $\varepsilon = [\gamma_5 \otimes \xi_5]$: nonzero eigenvalues exist as $\pm$ pairs.
- Taste symmetry brings four-fold degeneracy.
- We use Lanczos algorithm to calculate eigenvalues and eigenvectors for a Hermitian matrix $D_s^\dagger D_s$.

$$D_s^\dagger D_s | \tilde{f}_{\lambda^2}(x) \rangle = |\lambda|^2 | \tilde{f}_{\lambda^2}(x) \rangle \quad (5)$$

- $|f_{\pm\lambda}\rangle$ can be extracted from $|\tilde{f}_{\lambda^2}\rangle$:

$$|\tilde{f}_{\lambda^2}\rangle = c_+ |f_{+\lambda}\rangle + c_- |f_{-\lambda}\rangle \quad (6)$$

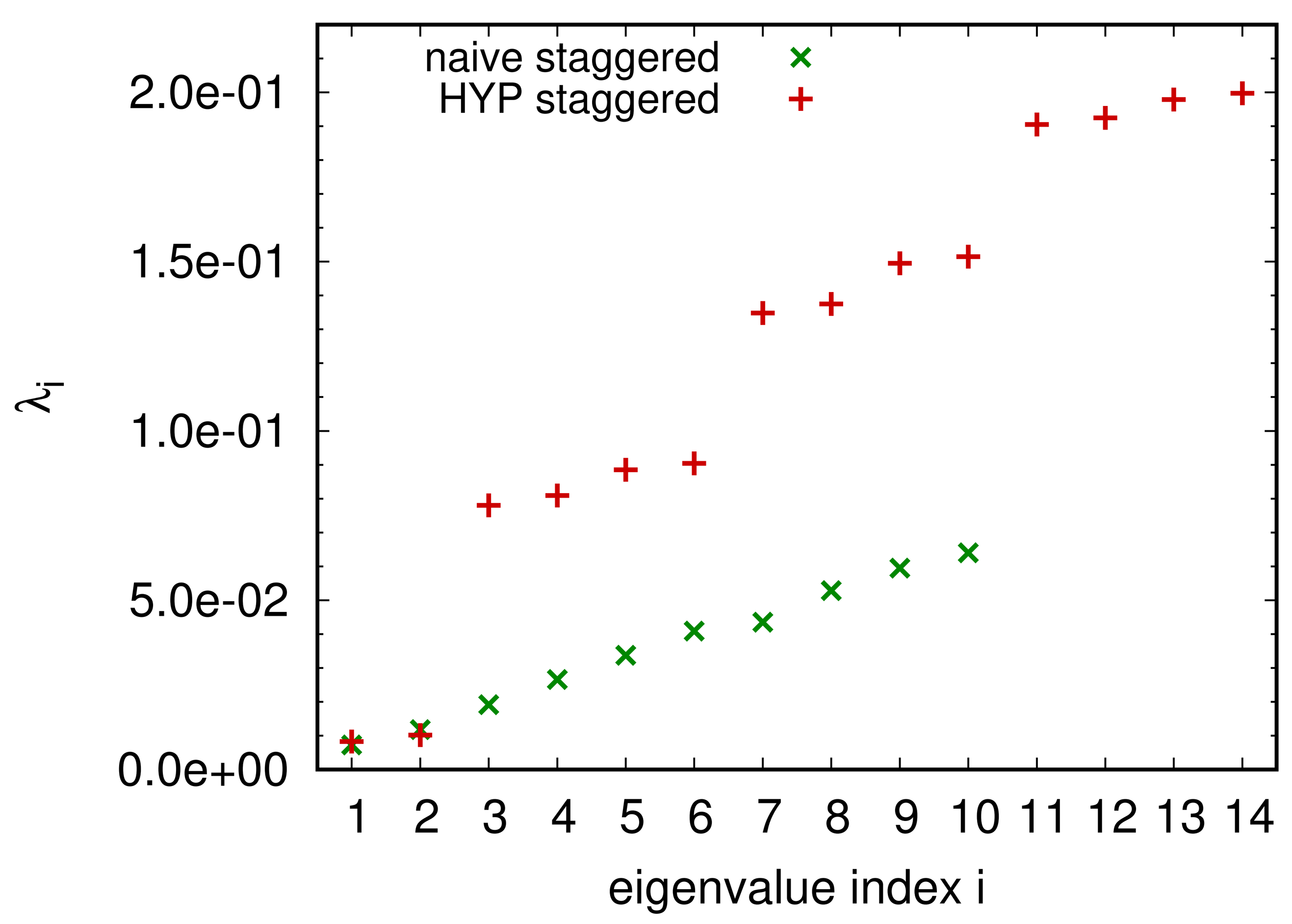
$$\Rightarrow \begin{cases} (D_s + i\lambda)\, |\tilde{f}_{\lambda^2}\rangle = 2\, c_+ |f_{+\lambda}\rangle \\ (D_s - i\lambda)\, |\tilde{f}_{\lambda^2}\rangle = 2\, c_- |f_{-\lambda}\rangle \end{cases} \quad (7)$$

$$\Rightarrow \; |f_{\pm\lambda}\rangle = \frac{1}{2\, c_\pm} (D_s \pm i\lambda) |\tilde{f}_{\lambda^2}\rangle \quad (8)$$

- We can obtain $|f_{\pm\lambda}\rangle$ by normalizing $(D_s \pm i\lambda)|\tilde{f}_{\lambda^2}\rangle$

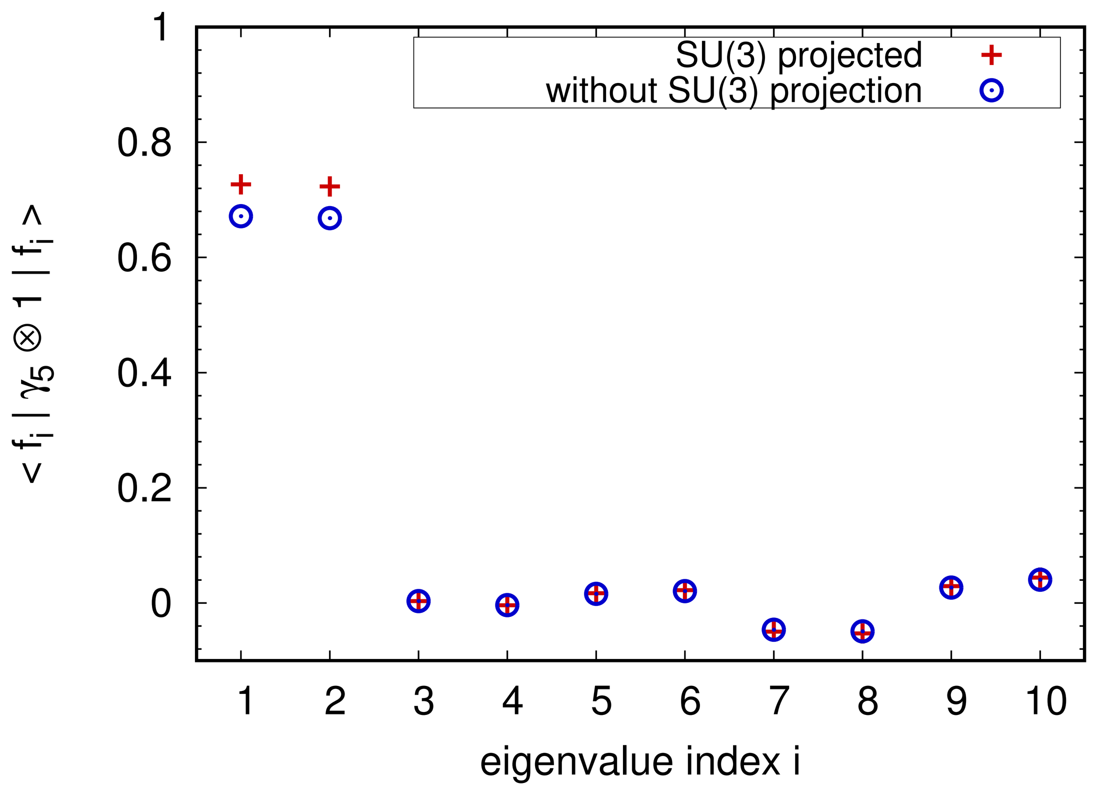
Eigenvalues for HYP staggered fermions

- 12^4 quenched lattice at $a \cong 0.125 fm$

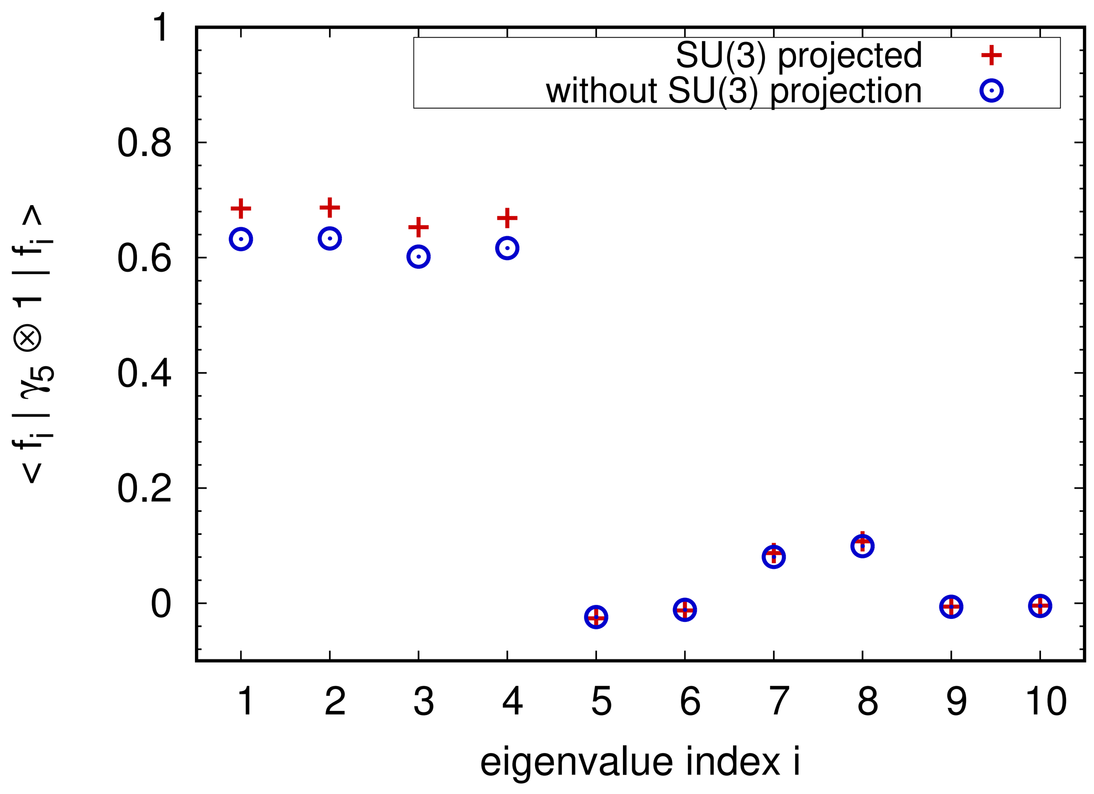


Chirality for HYP staggered fermions

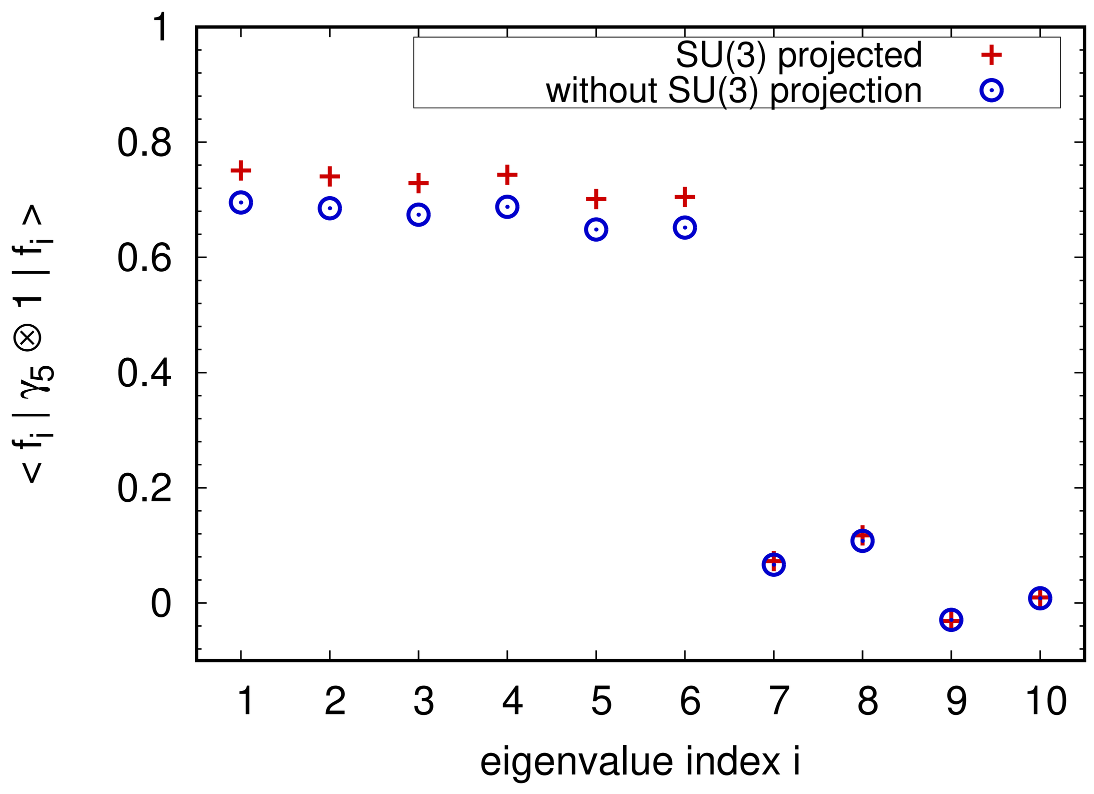
- 12^4 quenched lattice at $a \cong 0.125 fm$
- (topological charge) : $Q_t = -1$



- $Q_t = -2$



- $Q_t = -3$



Spectral Flow

- Consider a *Hermitian* operator H_s:

$$H_s \equiv -iD_s + \mu[\gamma_5 \otimes \mathbb{1}] \quad (9)$$

- Eigenvalues of H_s are real: $H_s f_s(x, \mu) = \lambda_s(\mu) f_s(x, \mu)$.

$$\lambda_s(\mu) = \langle f_s(\mu) | H_s | f_s(\mu) \rangle \quad (10)$$

- $\lambda_s(\mu = 0) = \lambda$ and $f_s(x, \mu = 0) = f_\lambda(x)$ for λ and f_λ in Eq. (4).
- Taking derivative with respect to μ,

$$\lambda_s'(\mu) = \langle f_s(\mu) | [\gamma_5 \otimes \mathbb{1}] | f_s(\mu) \rangle \quad (11)$$

- We define the chirality of the zero modes by

$$\mathrm{sign}\left(\lim_{\lambda \to 0} \left[\lim_{\mu \to 0} \lambda_s'(\mu) \right] \right) = \pm 1\,, \quad (12)$$

- Chirality can be determined by **spectral flow**. (doing this.)