

Hadron thermodynamics from imaginary chemical potentials

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The virial expansion

- ▶ Taylor expansion of the pressure in the fugacity $e^{\mu/T}$
- ▶ Starting from a grand canonical ensemble (for simplicity with one conserved charge N and chemical potential μ):

$$\begin{aligned} p &= \frac{T}{V} \log Z = \frac{T}{V} \log \text{Tr} \left(e^{-(H-\mu N)/T} \right) \\ &= \frac{T}{V} \log \sum_N e^{\mu N/T} \text{Tr}_N e^{-H/T} \\ &= \sum_N A_N e^{\mu N/T} \end{aligned}$$

- ▶ $A_N =$ virial coefficient

The virial expansion

- ▶ In many respects, very similar to the Taylor expansion in μ
 - ▶ Can be used to extrapolate to small finite μ
 - ▶ Sensitive to criticality
 - ▶ Sensitive to the degrees of freedom of the matter
- ▶ From my p.o.v., has two advantages:
 - ▶ Conceptual: contributions to a particular coefficient come from a Hilbert subspace with a given value of the conserved charges
 - ▶ Technical: can be naturally calculated with imaginary μ simulations
- ▶ And one disadvantage:
 - ▶ Harder to connect with heavy ion experiments

S-matrix formalism for the virial expansion

- ▶ The master formula (Dashen, Ma, Bernstein 1969):

$$A_{B,S,Q} - A_{B,S,Q}^{\text{free}} = \sum a_v$$
$$a_v = \left(\frac{T}{8\pi^3 i} \right) \int_{M_v}^{\infty} dE E^2 K_2(E/T) \text{Tr}_v \left(S^{-1} \overleftrightarrow{\frac{\partial}{\partial E}} S \right)_c$$

Here:

- ▶ v = channel, e.g. $B = 1, S = 0, Q = 0$ include:
 $\rho\pi^-, \rho\pi^-\pi^+\pi^-, \rho\pi^-\pi^+\pi^-\pi^+\pi^-, \dots,$
 $\rho\pi^-, \rho\pi^-\rho\bar{\rho}, \dots$
 $n\pi^0, \dots$
- ▶ M_v = sum of all masses in the given channel
- ▶ $K_2(E/T) \sim \sqrt{\frac{\pi T}{2E}} e^{-E/T}$ as $E \rightarrow \infty$ a “Boltzmann factor”

What about the cross-over?

- ▶ Virial expansion at high T
 - ▶ The virial expansion itself is not invalidated at high T , but the HRG does not give the correct coefficients
 - ▶ Since there is no true phase transition at $\mu = 0$, any physics should be describeable in either hadronic or partonic language. At least in principle.
 - ▶ The S-matrix formula should still be true, but not very useful.

“You may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you’ll be sorry.”

— Steven Weinberg, Third Law of Progress in Theoretical Physics

Virial coefficients from imaginary μ

- ▶ Real μ virial coefficients:

$$\rho(T, \mu_B) \ni p_1^B(T) \cosh(\mu_B/T)$$

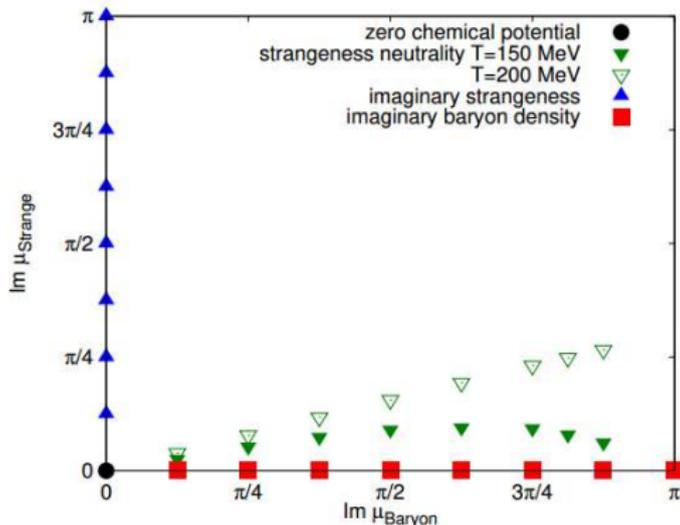
- ▶ Become imaginary μ Fourier coefficients:

$$\rho(T, i\mu_I) \ni p_1^B(T) \cos(\mu_I/T)$$

- ▶ At imaginary μ , there is no sign problem.
- ▶ A large body of literature uses analytical continuation from imaginary chemical potentials: de Forcrand, Philipsen 2002; D'Elia, Lombardo 2003; Bonati et al 2015; J. Günther's talk earlier today
- ▶ This is NOT what I do. I merely use the fact, that there is an exact relation between the Fourier coefficients and the density of states/S-matrix to answer yes-or-no type questions about models.

Lattice details

- ▶ Lattice action
 - ▶ 4 levels of stout smearing in the fermion action, with the smearing parameter $\rho = 0.125$
 - ▶ $N_t = 8, 10, 12, 16$
 - ▶ Aspect ratios $LT = 4$



The hadronic phase - strangeness sectors

- ▶ HRG suggests a particular truncation of the virial expansion:

$$\begin{aligned} P(\hat{\mu}_B, \hat{\mu}_S, \hat{\mu}_Q = 0) &= P_{00}^{BS} + P_{10}^{BS} \cosh(\hat{\mu}_B) + P_{01}^{BS} \cosh(\hat{\mu}_S) \\ &+ P_{11}^{BS} \cosh(\hat{\mu}_B - \hat{\mu}_S) + P_{12}^{BS} \cosh(\hat{\mu}_B - 2\hat{\mu}_S) \\ &+ P_{13}^{BS} \cosh(\hat{\mu}_B - 3\hat{\mu}_S) + \dots, \end{aligned}$$

with all positive coefficients $P_{ij}^{BS} > 0$.

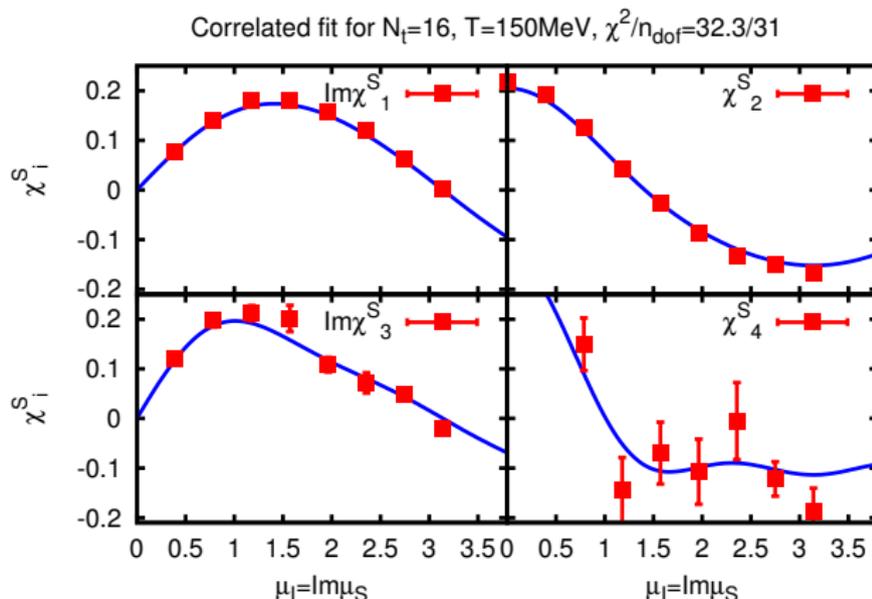
- ▶ From this:

$$\begin{aligned} P_{01}^{BS} &= \chi_2^S - \chi_{22}^{BS} \\ P_{13}^{BS} &= \frac{1}{8} \left(\chi_4^S - \chi_2^S + 3\chi_{13}^{BS} + 3\chi_{22}^{BS} \right) \end{aligned}$$

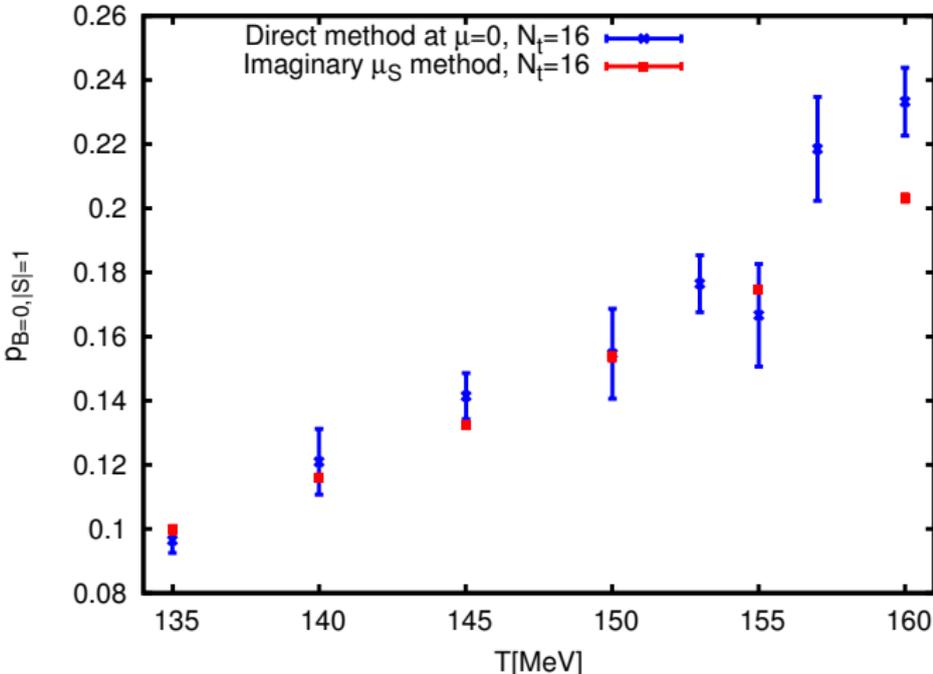
This is the method used e.g. Bazavov et al, 1404.6511. It has limited accuracy because of large cancellations in the linear combinations.

Strangeness sectors - fit procedure

- In our study we instead use the HRG ansatz for a correlated fit on lattice data for the fluctuations for imaginary μ_S



Comparison with the Taylor expansion method



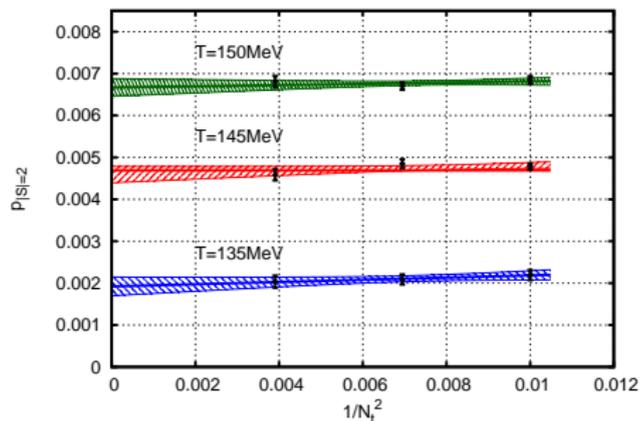
Statistics explains only a factor of 2 difference in the errors

Strangeness sectors - continuum limit

- ▶ Fits at fix N_t , followed by continuum extrapolation
- ▶ Systematic error from:
 - ▶ scale setting: w_0 and f_π
 - ▶ interpolation in temperature
 - ▶ tree level improvement or not
 - ▶ continuum limit ansatz: $a + b/N_t^2$ or $1/(a + b/N_t^2)$
- ▶ AIC weights and the histogram method

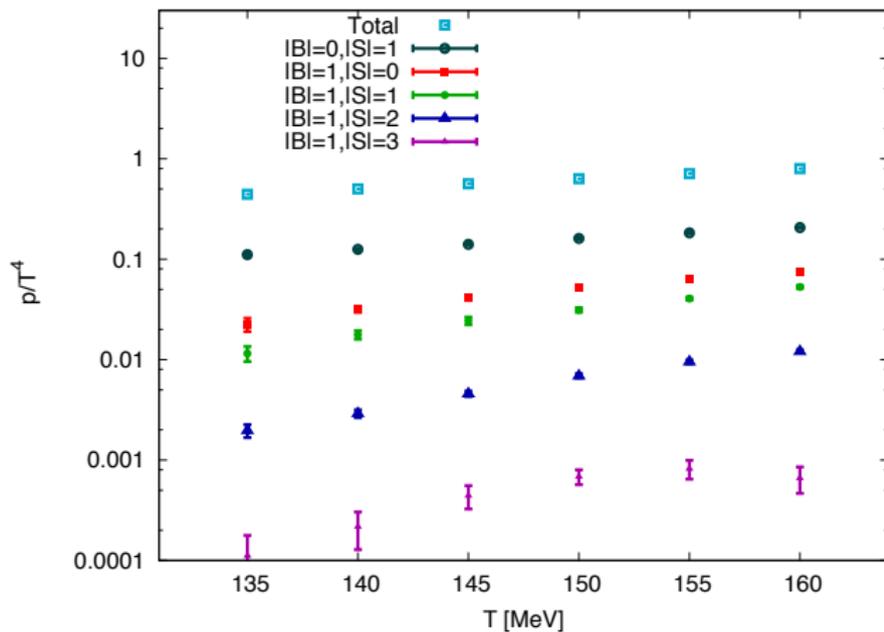
Strangeness sectors - continuum limit

- ▶ Continuum limit for $B > 0$

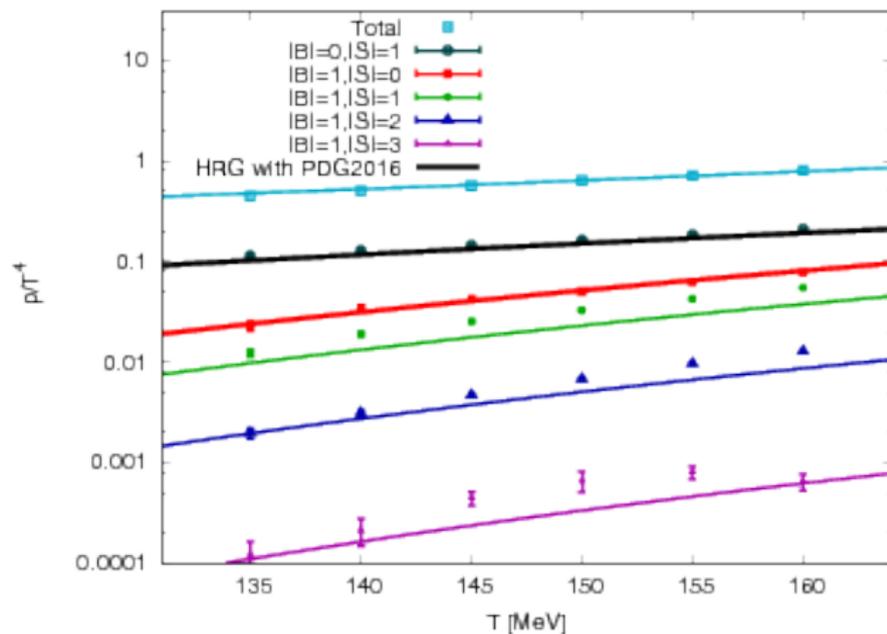


- ▶ For the strange mesons, only estimate from the $N_t = 12$ and 16

Strangeness sectors - results



Strangeness sectors - results



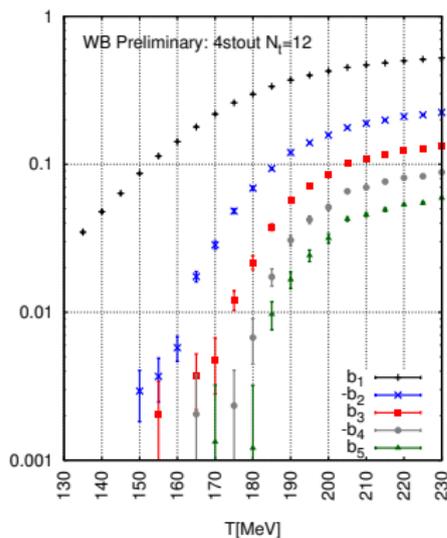
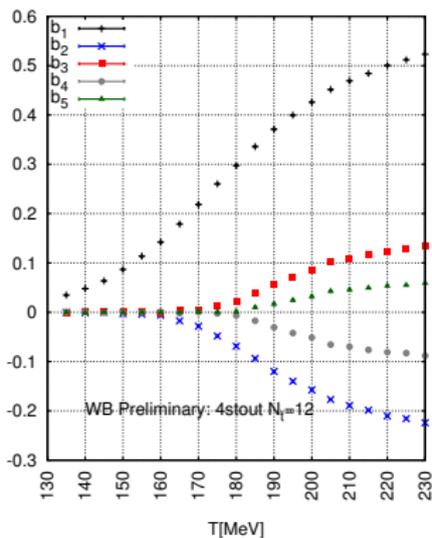
What about the crossover?

- ▶ The HRG ansatz will not hold above the crossover temperature
- ▶ In particular, new terms, like $B = 2$, will appear in the virial expansion
- ▶ Some terms will have a negative sign
- ▶ To study this issue we use imaginary baryon chemical potentials, and perform a discrete Fourier transform:

$$\chi_1^B(i \operatorname{Im} \mu_B) = i \sum k P_k^B \sin(k \operatorname{Im} \mu_B)$$

Higher Fourier components

$$\chi_1^B(i \text{Im} \mu_B) = i \sum k P_k^B \sin(k \text{Im} \mu_B) = i \sum b_k \sin(k \text{Im} \mu_B)$$



The appearance of a negative sign $B = 2$ contribution clearly signals the breakdown of HRG.

Higher Fourier components - S matrix formalism?

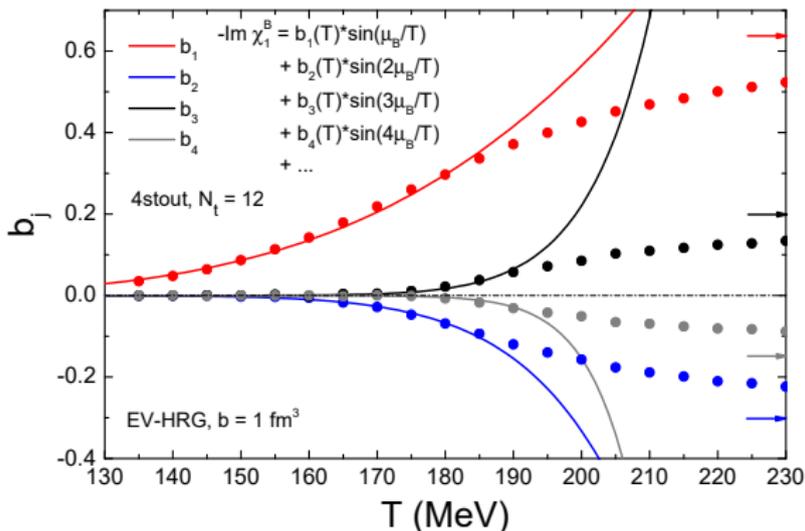
- ▶ In the S-matrix formalism, this second virial coefficient gets contributions from many channels: NN, NN $\pi\pi$, N $\pi\pi\pi\pi$, ... N Λ , N $\Lambda\pi\pi$, ... $\Lambda\Lambda$, ...
- ▶ Assuming two-body elastic scatterings dominate we get contributions of the type:

$$a_{NN} = \frac{T}{2\pi^3} \int dE E^2 K_2(E/T) \sum_{\ell, l} (2\ell + 1)(2l + 1) \frac{\partial \delta_{\ell}^l}{\partial E}$$

- ▶ A repulsive interaction leads to a decreasing phase shift \rightarrow a negative second virial coefficient/

Excluded volume HRG

- ▶ A simple way to account for repulsion between baryons is the excluded volume version of the hadron resonance gas.
- ▶ Interestingly this very simple approach can account for the always different sign structure of the higher virial coefficients

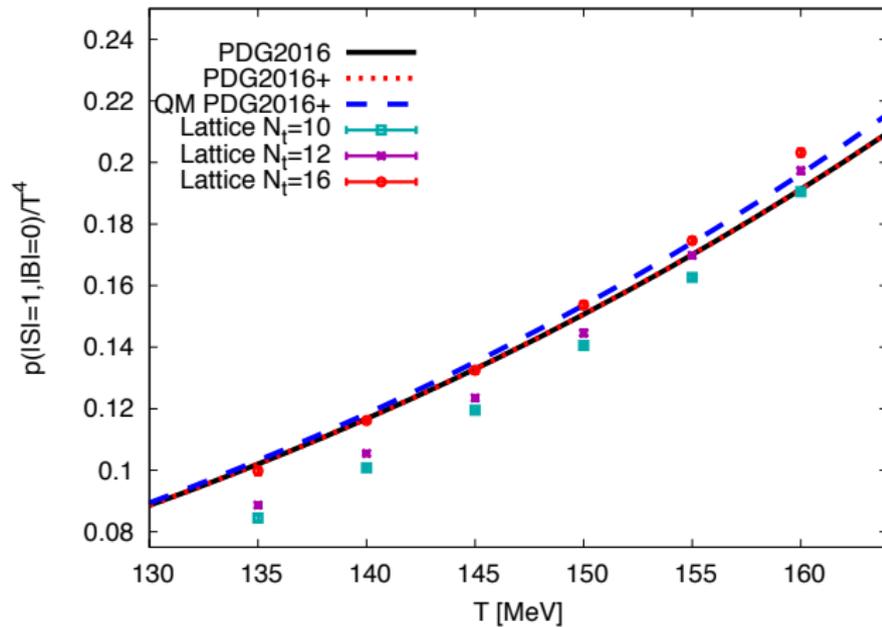


Model: Vovchenko, Gorenstein, Stoecker

Summary

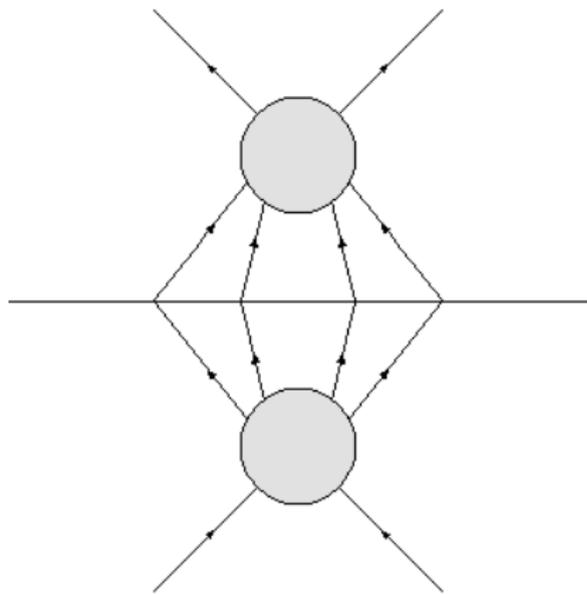
- ▶ The imaginary μ Fourier coefficients are nice quantities to know if you are interested in hadron chemistry
- ▶ $\text{Im} \mu_S$ allow for a clear separation of the strangeness sectors $S = 1, 2, 3$
- ▶ $\text{Im} \mu_S$ simulations in the confined phase suggest missing resonances in all strange sectors, even the mesons, but not in the non-strange sectors
- ▶ $\text{Im} \mu_B$ simulations can separate the different baryon number sectors
- ▶ Sign structure of the baryon sectors might be explained by repulsive interactions
- ▶ No analytical continuation is involved in any of my conclusions.

S=1 B=0



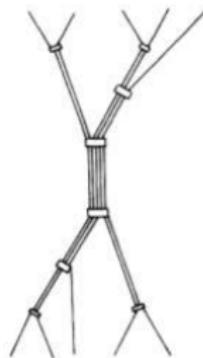
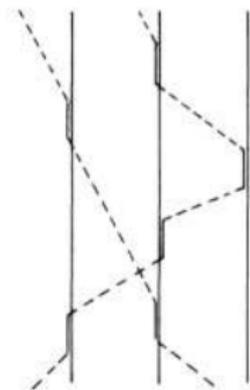
S-matrix formalism for the virial expansion

- ▶ The density of states in a given channel, expressed in term of the S-matrix $\text{Tr}_v \left(S^{-1} \frac{\overleftrightarrow{\partial}}{\partial E} S \right)$



The hadron resonance gas: formal approach

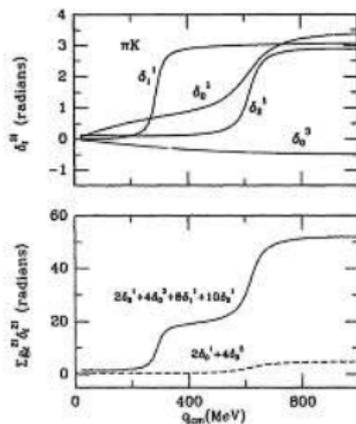
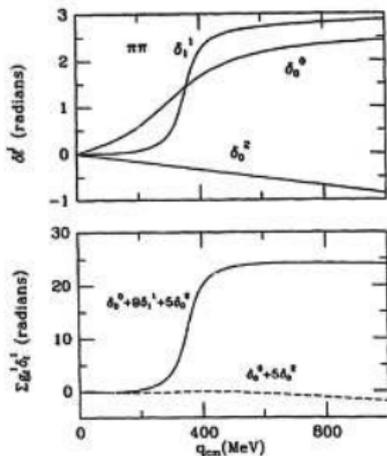
Dashen, Rajaraman 1974: scattering narrow ($\Gamma \rightarrow 0$) resonances + Unitarity \rightarrow all of the many particle scattering contributions in the previous formulas can be resummed, and lead to the HRG



The hadron resonance gas: phenomenological approach

Venugopalan, Prakash 1992: Consider only 2 particle elastic particle scattering, and use experimental phase shifts

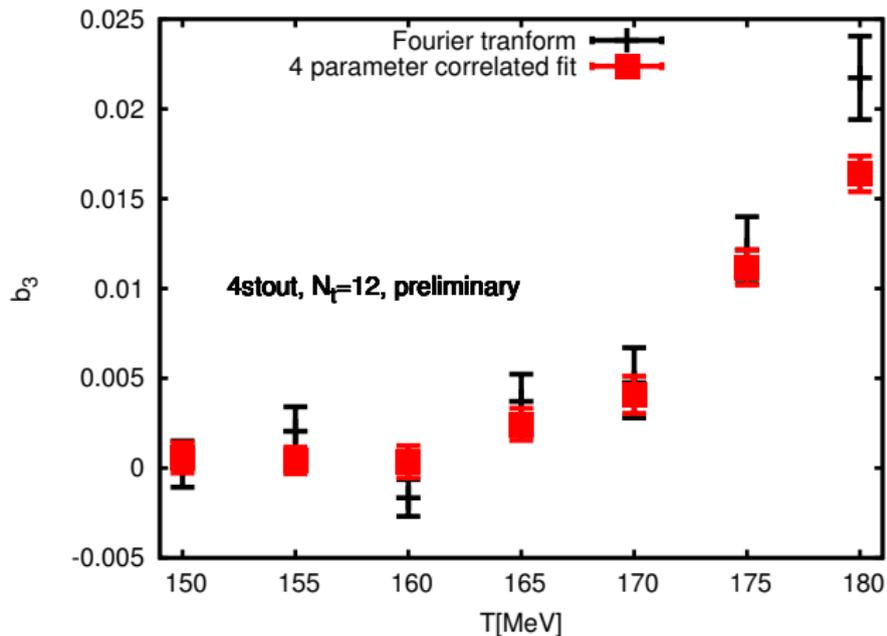
$$a_2 = \frac{T}{2\pi^3} \int dE E^2 K_2(E/T) \sum_{\ell, l} (2\ell + 1)(2l + 1) \frac{\partial \delta_\ell^l}{\partial E}$$



Only resonance contributions survive

Correlated fit for the baryon sectors

$$N_t = 12$$



Continuum for the baryon sectors?

