Chiral Condensate and Dirac Spectrum at Nonzero θ Angle

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Acknowledgments

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Tilo Wettig (Regensburg)

Relevant Papers

- J.J.M. Verbaarschot and T. Wettig, The Spectrum of the Dirac Operator for QCD with on flavor at fixed θ angle, Phys. Rev. D90 (2014) 070 [arXiv:1410.0883[hep-lat]].

  P.H. Damgaard, Topology and the Dirac Operator Spectrum in Finite Volume Gauge Theory, Nucl. Phys. B556 327 (1999).

  H. Leutwyler and A. Smilga, Spectrum of Dirac Operator and Role of Winding Number in QCD, Phys. Rev. D 46 (1992) 5607.

  M. Creutz, One Flavor QCD, Ann. Phys. 322 (2007) 1518 [arXiv[hep-th/0609187]].

  V. Azcoiti, E. Follana, E. Royo-Amondarain, G. Di Carlo and A. Vaquero Avilés-Casco, θ-Dependence of the Massive Schwinger Model, PoS (LATTICE2016) (2016) 252.

- J.J.M. Verbaarschot and T. Wettig, The Chiral Condensate of One-Flavor QCD and the Dirac Spectrum at $\theta = 0$, PoS LATTICE2014 (2014) 072 [arXiv:1412.5483 [hep-lat]].

- M.K. Kieburg, J.J.M. Verbaarschot and T. Wettig, The Dirac Spectrum and the θ Dependence of the Chiral Condensate (2017), in preparation.

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II. QCD at nonzero θ Angle

One flavor QCD

QCD with two flavors

Chiral condensate

Silver Blaze phenomenon

One Flavor QCD Partition Function

$$Z(m,\theta) = \left\langle \sum_\nu e^{i\nu\theta} m^\nu \prod_{\lambda_k \neq 0} (i\lambda_k + m) \right\rangle .$$

The average is over gauge field configurations with topological charge ν , m is the quark mass and $i\lambda_k$ the eigenvalues of the anti-Hermitian Dirac operator.

- A negative quark mass can be interpreted as $\theta = \pi$ and visa versa.
- Partition function for fixed topological charge

$$Z_\nu(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-i\nu\theta} Z(m,\theta).$$

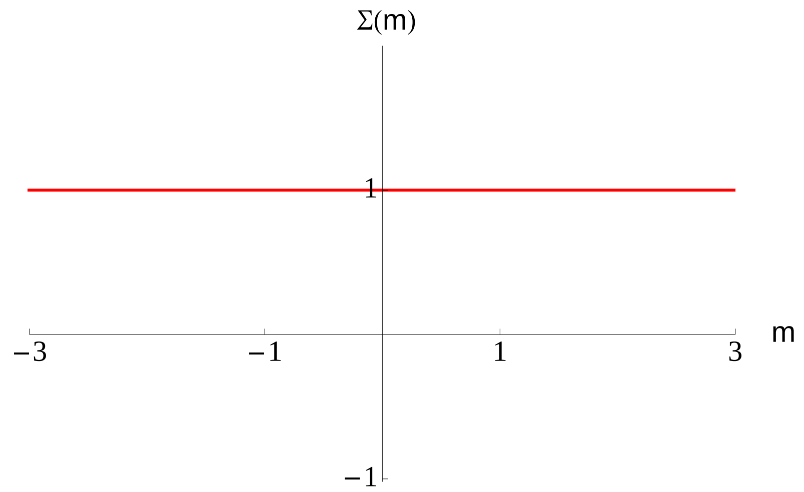
One Flavor QCD

- Chiral symmetry is broken by the anomaly.
- There is no spontaneous symmetry breaking and there are no Goldstone bosons.
- The mass dependence of the one flavor QCD partition function is given by

$$Z = e^{mV\Sigma\cos\theta + O(m^2V)}.$$

- Among others, the chiral condensate vanishes for $\theta = \pi/2$.

Chiral Condensate



Behavior of the chiral condensate for $N_f = 1$.

$$\Sigma(m) \quad = \quad -\langle \bar{q}q \rangle = \frac{1}{V}\frac{d}{dm}\log Z(m) = \left\langle \frac{1}{V}\sum_k \frac{m}{\lambda_k^2 + m^2} \right\rangle .$$

The motivation to study QCD at nonzero θ -angle came from the desire to understand this behavior of the chiral condensate in terms of the Dirac spectrum, also because of statements in the literature that seemed puzzling. Creutz-2005

Two-Flavor QCD

Partition function $Z(m_1, m_2, \theta)$

$$\left\langle \sum_\nu e^{i\nu\theta}(m_1 m_2)^\nu \prod_k (i\lambda_k + m_1)(i\lambda_k + m_2) \right\rangle ,$$

where ν is the topological charge and $i\lambda_k$ are the nonzero eigenvalues of the Dirac operator.

These results can be understood simply from the chiral Lagrangian. The mean field result for two-flavor partition function is given by

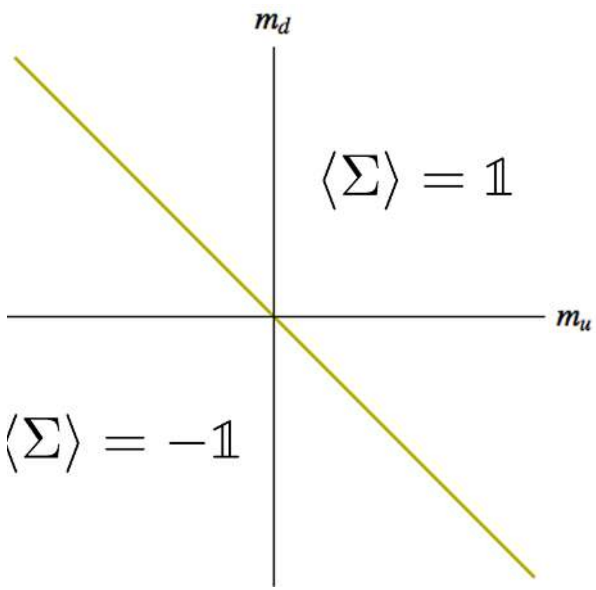
$$Z(m_1, m_2) = \int_{U \in \mathrm{SU}(2)} dU \exp[V\Sigma \mathrm{Tr}\, \mathrm{diag}(m_1, m_2)(U + U^{-1})].$$

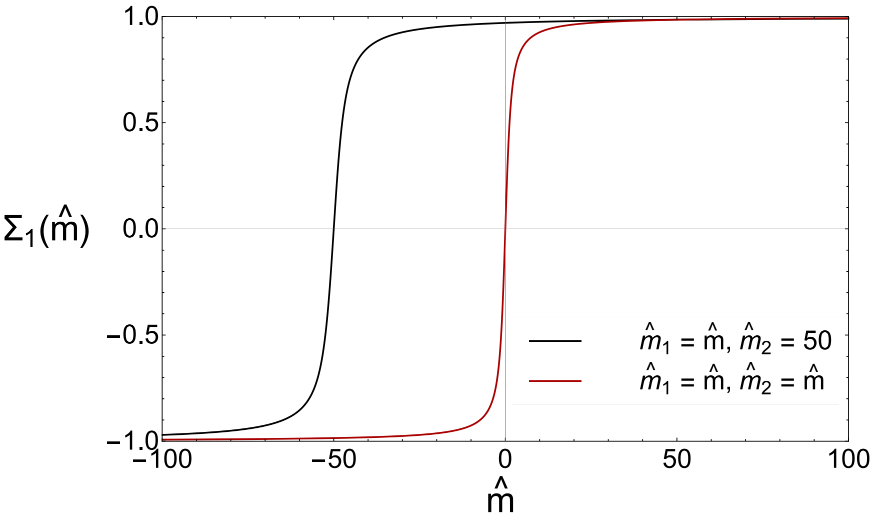
In the thermodynamical limit U aligns with the mass term resulting in

$$Z(m_1, m_2) = e^{mV\Sigma|m_1 + m_2|}$$

Leutwyler-Smilga-1992

Chiral Condensate of Two-Flavor QCD





Horkel-Sharpe-2015

Phase diagram (left) and mass dependence (right) of the chiral condensate for two flavors with $\hat{m} = mV\Sigma$.

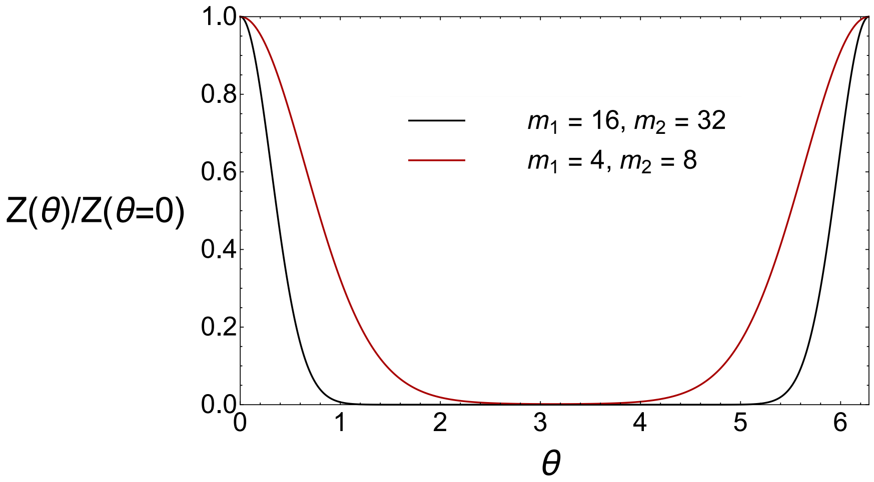
Chiral condensate

$$\Sigma(m) = -\langle \bar{q}q \rangle = \frac{1}{V}\frac{d}{dm}\log Z(m, m_2).$$

Sign Problem at Nonzero θ Angle

Measure for the severity of the sign problem

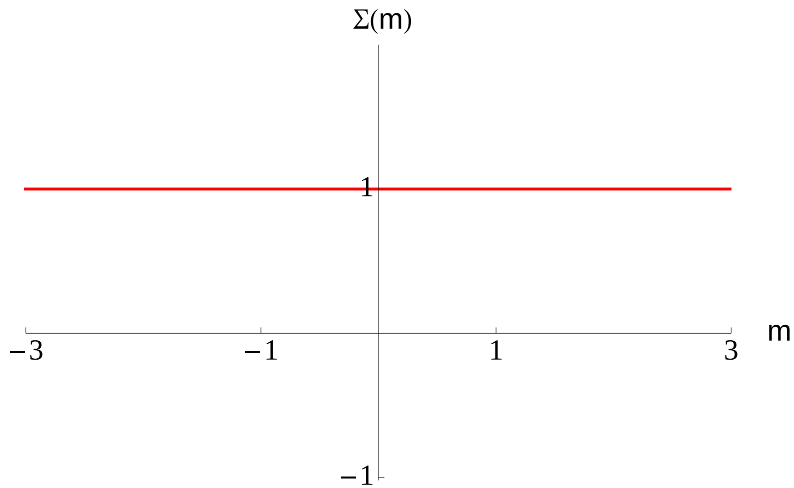
$$\frac{Z(m_1, m_2, \theta)}{Z(m_1, m_2, \theta = 0)}.$$

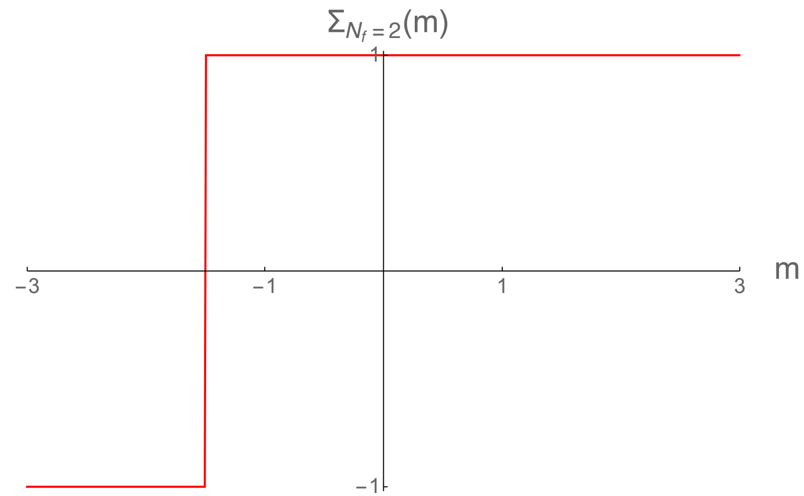


Azcoiti-Di Carlo-Galante-Laliena-2002

Severity of the sign problem for two flavors as a function of the θ -angle. Results are to leading order in chiral perturbation theory.

Chiral Condensate





Behavior of the chiral condensate for $N_f = 1$ (left) and $N_f = 2$ (right).

Silver Blaze Behavior © Cohen-2003

- Chiral Condensate of one and two flavor QCD remains the same when the quark mass crosses the line of eigenvalues.
- This seems to violate the Banks-Casher relation

$$\Sigma = \frac{1}{V} \left\langle \sum_k \frac{2m}{\lambda_k^2 + m^2} \right\rangle .$$

- Apparently, the sign gets compensated by the sign from the determinant which appear when $m < 0$ or $m_u m_d < 0$.
- What is the solution of this “Silver Blaze Phenomenon”?
- A similar problem arose in QCD at nonzero chemical potential where the chiral condensate remains constant while the Dirac spectrum is strongly altered by the chemical potential.

II. Dirac Spectrum and Chiral Condensate at $\theta \neq 0$

Dirac spectrum

Decomposition

Microscopic Domain

Dirac Spectrum

The spectral density of the Dirac operator is given by

$$\rho(x,m,\theta) = \frac{\sum_\nu P_\nu \rho_\nu(x,m)}{\sum_\nu e^{i\nu\theta} Z_\nu(m)}, \qquad P_\nu = \frac{e^{i\nu\theta} Z_\nu(m)}{\sum_\nu e^{i\nu\theta} Z_\nu(m)}.$$

The partition function at fixed topology is given by

$$Z_\nu(m) = m^\nu \langle \prod_k (\lambda_k^2 + m^2) \rangle_\nu .$$

For $\theta \neq 0$ the partition function is not positive definite and $\rho(x, m < 0, \theta)$ may become negative.

Contributions to the Chiral Condensate

The chiral condensate is given by

$$\Sigma(m) = \int_{-\infty}^{\infty} d\lambda \frac{m}{m^2 + \lambda^2} \rho(\lambda).$$

The spectral density can be decompose into a zero mode part and a nonzero mode part which is the sum of a “quenched” part and a “dynamical” part,

$$\begin{aligned} \rho(\lambda) &= \rho_{\rm ZM}(\lambda) + \rho_{\rm NZM}(\lambda). \\ &= \delta(\lambda) \frac{\sum_\nu |\nu| e^{i\nu\theta} Z_\nu}{\sum_\nu e^{i\nu\theta} Z_\nu} + \rho^{\rm q}_{\rm NZM}(\lambda) + \rho^{\rm d}_{\rm NZM}(\lambda), \end{aligned}$$

where $\rho^{\rm q}_{\rm NZM}(\lambda)$ is the quenched spectral density and $\rho^{\rm d}_{\rm NZM}(\lambda)$ is the correction due to the fermion determinant.

The Microscopic Domain of QCD

We will do our calculations in the microscopic domain of QCD, and the explicit results we quoted before were already in this domain.

In this domain, also know as the ϵ -domain, the quark mass and the Dirac eigenvalues scale in the thermodynamic limit as

$$m \sim \frac{1}{V}, \qquad \lambda \sim \frac{1}{V}.$$

Correction terms will enter when $m, \lambda \approx 1/\Lambda_{\rm QCD}\sqrt{V}$.

In this domain, the spectral density can be evaluated analytically.

Spectral Density at Fixed ν for $N_f = 1$

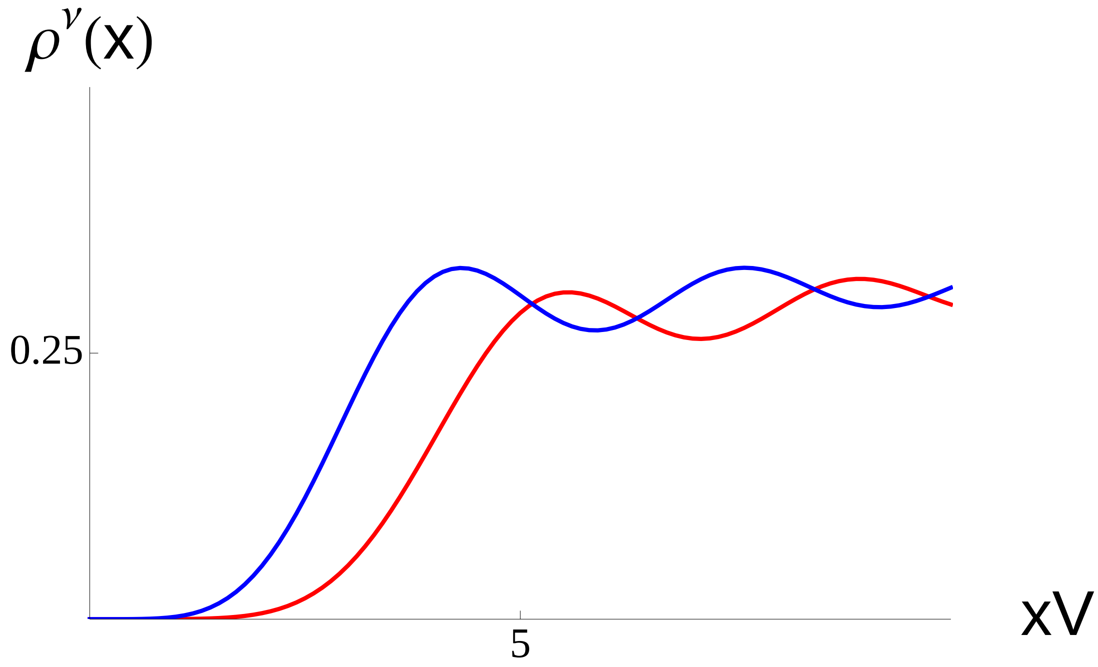
The one-flavor spectral density in the ϵ -domain is given by

$$\begin{aligned}\rho_\nu(\lambda, m) \quad &= \quad \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x})J_{\nu-1}(\hat{x})) + |\nu|\delta(\hat{x}) \\ &\quad - \frac{\hat{x}}{\hat{m}^2 + \hat{x}^2}\left[\hat{x}J_\nu(\hat{x})J_{\nu+1}(\hat{x}) - \hat{m}\frac{I_{\nu+1}(\hat{m})}{I_\nu(\hat{m})}J_\nu^2(\hat{x})\right].\end{aligned}$$

Damgaard-Osborn-Toublan-JV-1999

$$\hat{x} \equiv \lambda \Sigma V, \qquad \hat{m} \equiv m \Sigma V$$

Microscopic Spectral Density at fixed ν



The one microscopic spectral density for $\nu = 2$ and $mV = 1$ (red) compared to the quenched result for $\nu = 2$ (blue).

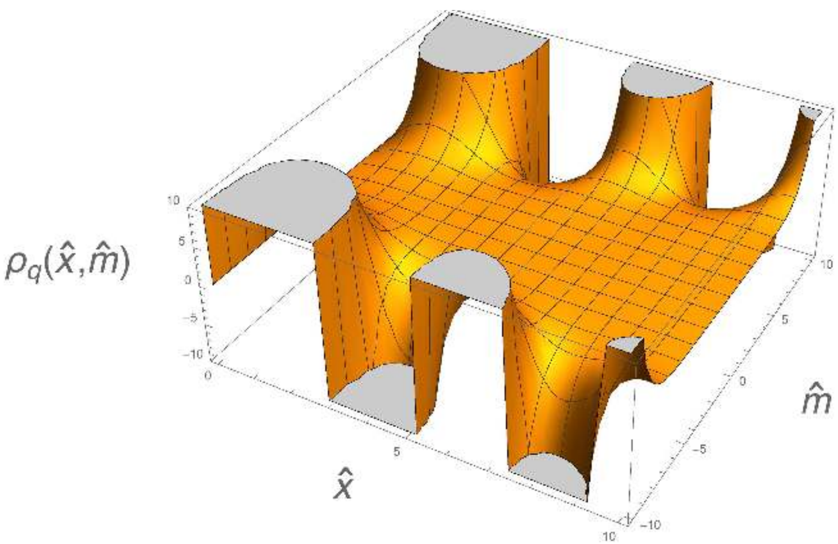
III. Dirac Spectrum and Chiral Condensate for $N_f = 1$

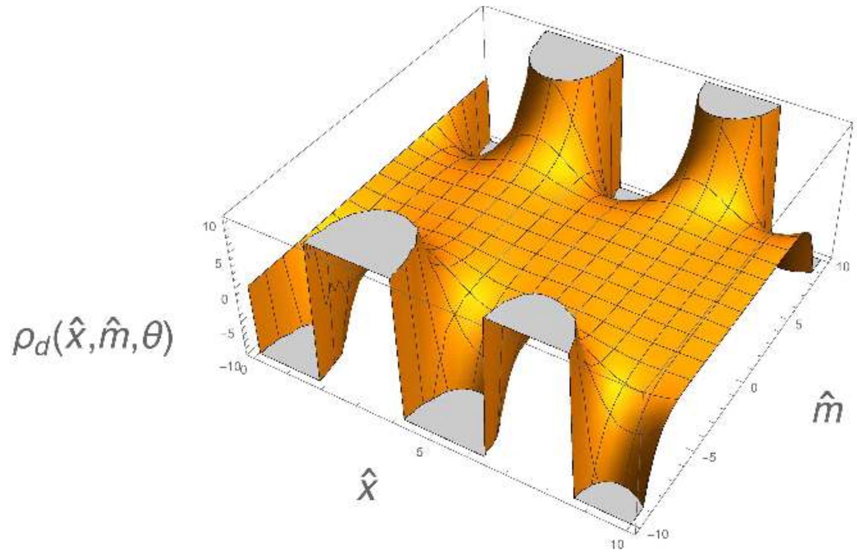
Dirac spectrum

Contribution of zero modes and nonzero modes

Chiral condensate

Spectral Density for $\theta = \pi/2$





The spectral density of the eigenvalues of the Dirac operator for one-flavor QCD. The quenched part is shown in the left figure and the dynamical part in the right figure. Kieburg-JV-Wettig-2017

The chiral condensate corresponding to this spectral density should vanish!

Explicit Analytical Expressions

$$\rho^{\rm ZM}(x) = -\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{e^{\hat{m}\cos(\phi+\theta+\pi)} - e^{\hat{m}\cos\theta}}{2\cos^2\frac{\phi}{2} e^{\hat{m}\cos\theta}} \delta(\hat{x}),$$

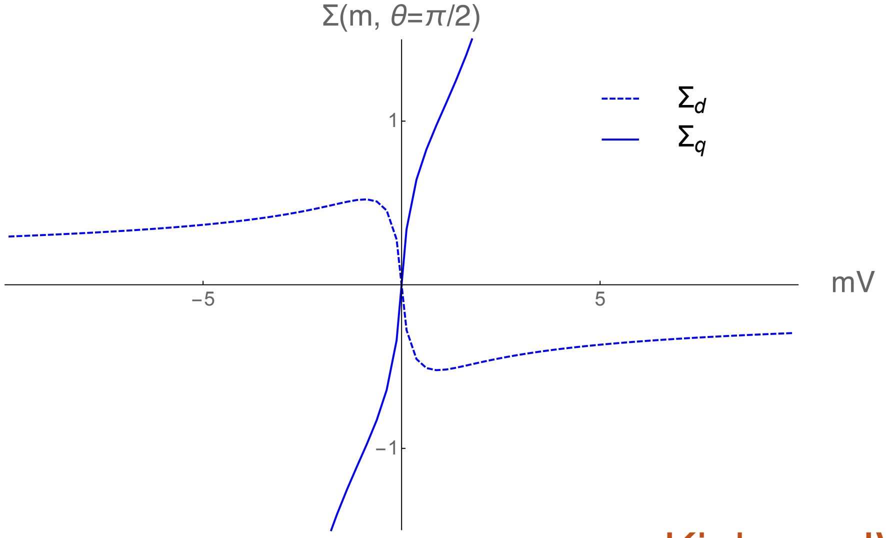
$$\rho^{\rm q}(x) = \int_{-\pi}^{\pi} \frac{d\phi}{4\pi\cos\frac{\phi}{2}} \frac{e^{\hat{m}\cos(\theta+\phi+\pi)}}{e^{\hat{m}\cos\theta}} J_1(2|\hat{x}|\cos\frac{\phi}{2}),$$

$$\rho^{\rm d}(x) = \frac{-|\hat{x}|}{\hat{x}^2+\hat{m}^2} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{e^{\hat{m}\cos(\theta+\phi+\pi)}}{e^{\hat{m}\cos\theta}} \left(\hat{x} e^{i\phi/2} J_1(2\hat{x}\cos\frac{\phi}{2}) - \hat{m} e^{i(\pi-\theta-\phi)} J_0(2|\hat{x}|\cos\frac{\phi}{2}) \right)$$

$$\hat{x} = xV\Sigma, \qquad \hat{m} = mV\Sigma.$$

Kieburg-JV-Wettig-2017

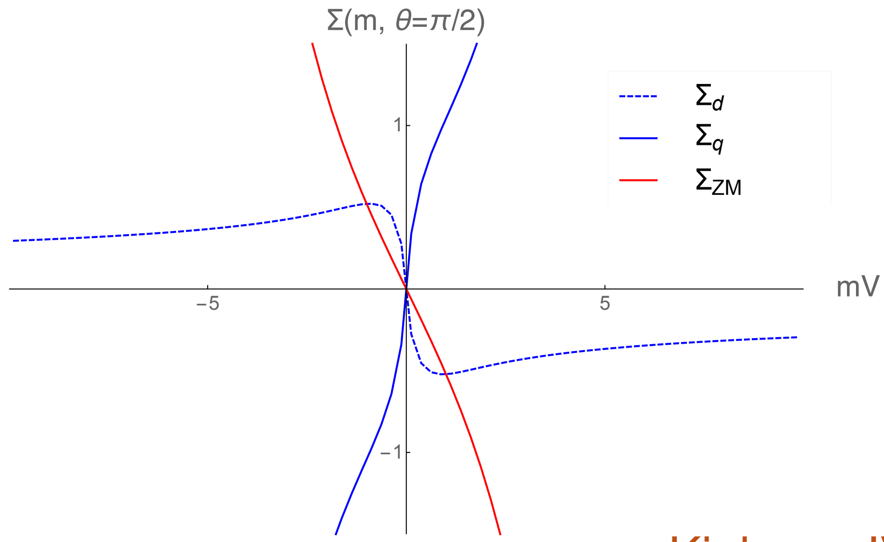
Chiral Condensate for $\theta = \pi/2$



Kieburg-JV-Wettig-2017

$$\Sigma_\mathrm{q} \sim \frac{1}{\sqrt{2\pi}|m|^{3/2}} e^{|m|V\Sigma}[1 + O(1/mV\Sigma)]$$

Chiral Condensate for $\theta = \pi/2$



Kieburg-JV-Wettig-2017

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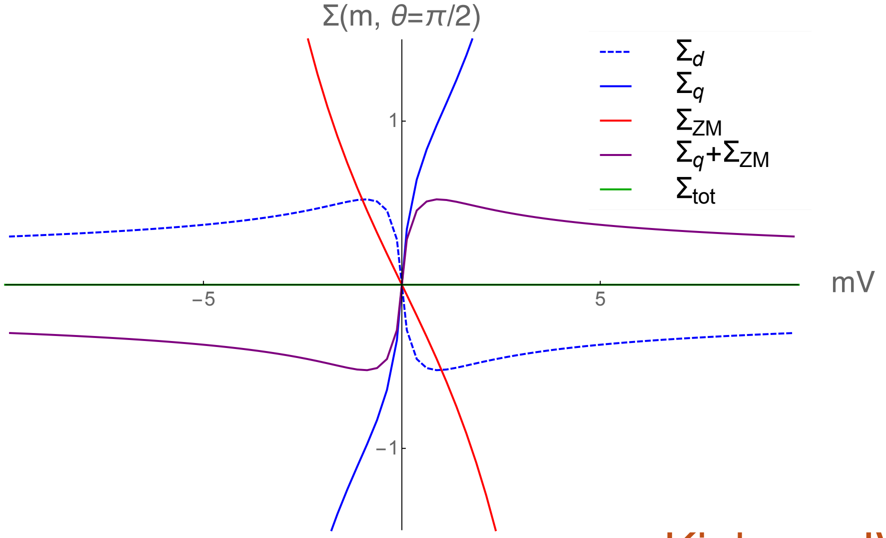


Kieburg-JV-Wettig-2017

$$\Sigma_{\rm q} \sim \frac{1}{\sqrt{2\pi}|m|^{3/2}} e^{|m|V\Sigma}[1 + O(1/mV\Sigma)]$$

$$\Sigma_{\rm ZM} \sim -\frac{1}{\sqrt{2\pi}|m|^{3/2}} e^{|m|V\Sigma}[1 + O(1/mV\Sigma)]$$

This cancellation holds to all orders in $1/mV\Sigma$

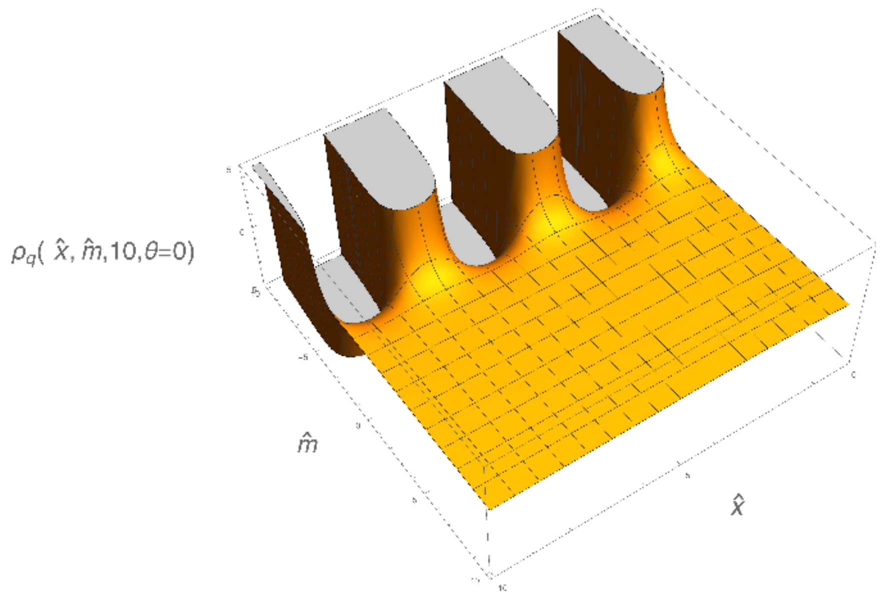
IV. Dirac Spectrum and Chiral Condensate for $N_f = 2$

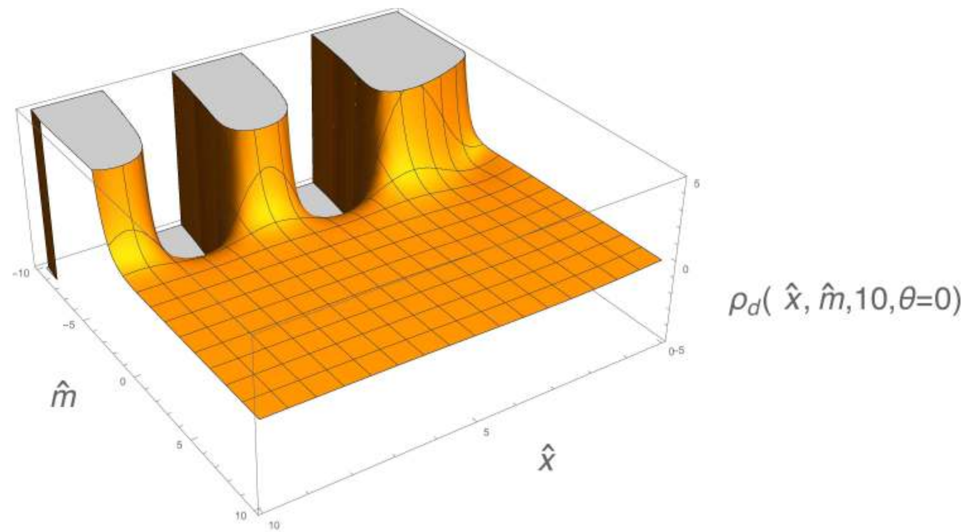
Dirac spectrum

Contribution of zero modes and nonzero modes

Chiral condensate

Spectral Density for Two Flavors





Spectral density of the QCD Dirac operator for $\theta = 0$ and $\hat{m}_2 = 10$. Left we show the quenched contribution and right the correction induced by the fermion determinant.

Analytical Expression

$$\rho(x, m_1, m_2, \theta) = \frac{|\hat{x}|}{2} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \Bigg(\Bigg(\frac{-2J_1(2\hat{x}\cos\phi/2)}{2\hat{x}\cos\phi/2}$$

$$- \frac{2\hat{x}e^{-i(\phi/2+\theta)}(2\hat{m}_1\hat{m}_2 - (\hat{m}_2^2 + \hat{m}_1^2)e^{i(\theta+\phi)})}{(\hat{x}^2 + \hat{m}_1^2)(\hat{x}^2 + \hat{m}_2^2)} J_1(2\hat{x}\cos\phi/2) \Bigg) \frac{Z(\hat{m}_1, \hat{m}_2, \theta + \phi + \pi)}{Z(\hat{m}_1, \hat{m}_2, \theta)}$$

$$- \frac{2J_0((\hat{m}_1\hat{m}_2 e^{-i(\theta+\phi)} - \hat{x}^2 e^{i\phi})2\hat{x}\cos\phi/2)}{(\hat{x}^2 + \hat{m}_1^2)(\hat{x}^2 + \hat{m}_2^2)} I_0(\sqrt{\hat{m}_1^2 + \hat{m}_2^2 - 2\hat{m}_1\hat{m}_2\cos(\theta + \phi)}) \Bigg)$$

Kieburg-JV-Wettig-2017

$$\hat{x} = xV\Sigma, \qquad \hat{m} = mV\Sigma.$$

Contribution of Zero Modes for Two Flavors

$$\rho(\lambda) = \frac{-\delta(\lambda)}{Z_2(m_1, m_2, \theta)} \int_{-\pi}^{\pi} d\phi \frac{Z_2(m_1, m_2, \theta + \phi + \pi) - Z_2(m_1, m_2, \theta)}{4\pi \cos^2(\phi/2)}$$

For large volume the integrand is dominated by $\alpha = \theta - \pi$ resulting in the asymptotic contribution to the chiral condensate

$$\Sigma_{\rm ZM}(m_1, m_2, \theta) \sim -\frac{1}{4\pi m_1 \cos^2 \frac{\theta+\pi}{2}} \frac{Z_2(m_1, m_2, \theta = 0)}{Z_2(m_1, m_2, \theta)},$$

which grows exponentially with the volume. Since the quenched part of the chiral condensate is finite for $V \to \infty$ it also should grow exponentially but with the opposite sign.

Note that for $\theta \neq 0$ the free energy of the denominator is less than the free energy of the numerator.

Kieburg-JV-Wettig-2017

Quenched Contribution

$$\Sigma(m_1, m_2, \theta) = \int_{-\pi}^{\pi} d\phi \frac{1 - 2|\hat{m}_1| \cos\frac{\phi}{2} K_1(2|\hat{m}_1| \cos\frac{\phi}{2})}{4\pi\hat{m}_1 \cos^2\frac{\phi}{2}} \frac{Z_2(\hat{m}_1, \hat{m}_2, \theta + \phi + \pi)}{Z_2(\hat{m}_1, \hat{m}_2, \theta = 0)}$$

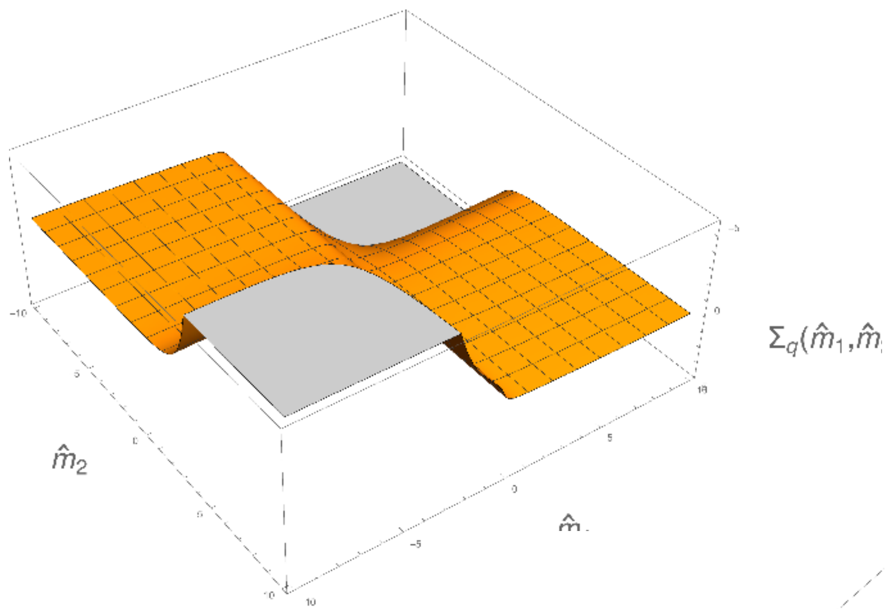
For large V this simplifies to

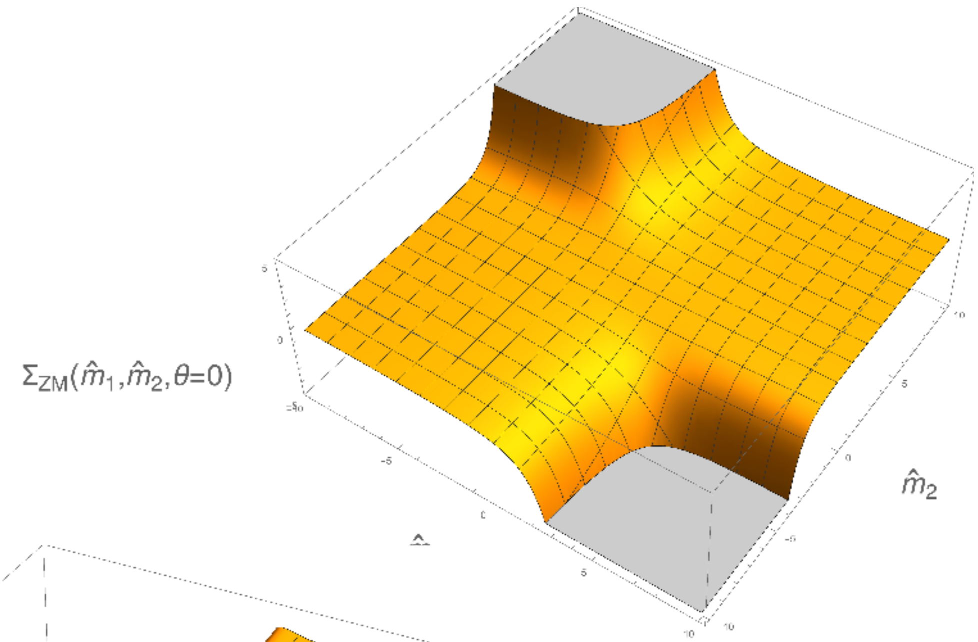
$$\Sigma[\hat{m}_1, \hat{m}_2, \theta] \sim \frac{1}{4\pi\hat{m}_1 \cos^2\frac{\theta+\pi}{2}} \frac{Z_2(\hat{m}_1, \hat{m}_2, \theta = 0)}{Z_2(\hat{m}_1, \hat{m}_2, \theta)}.$$

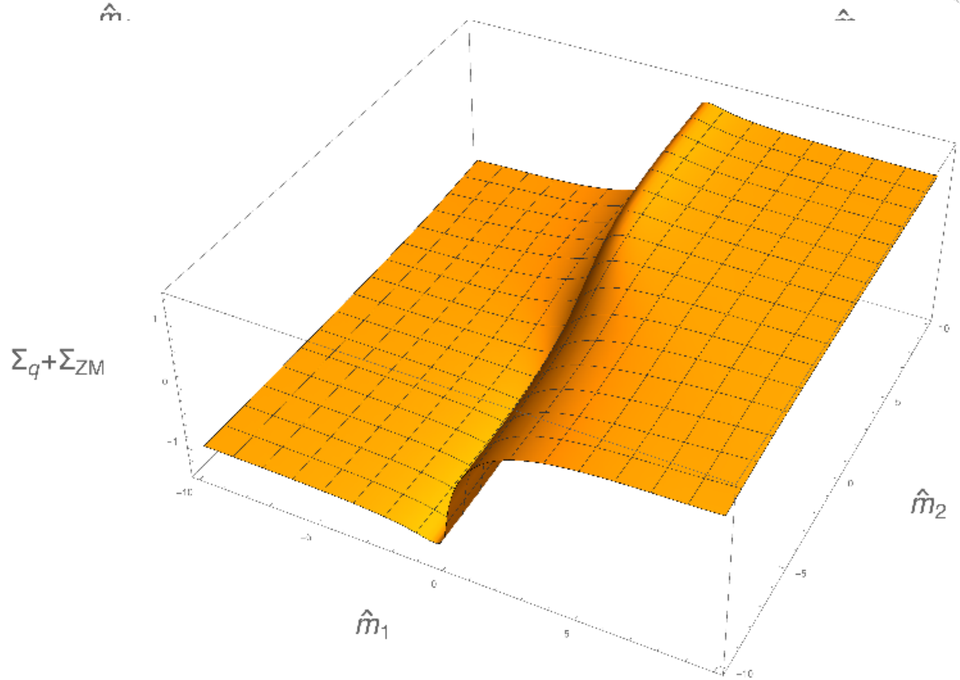
This exactly cancels the exponential increasing contribution from the zero modes.

One can show that this cancellation works to all orders.
Kieburg-JV-Wettig-2017

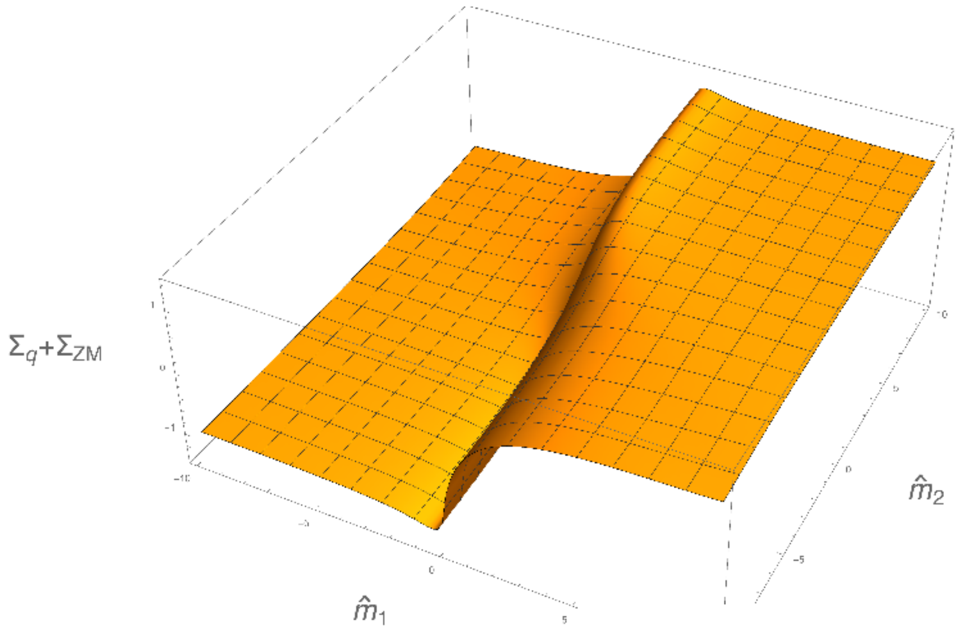
Contributions to the Chiral Condensate

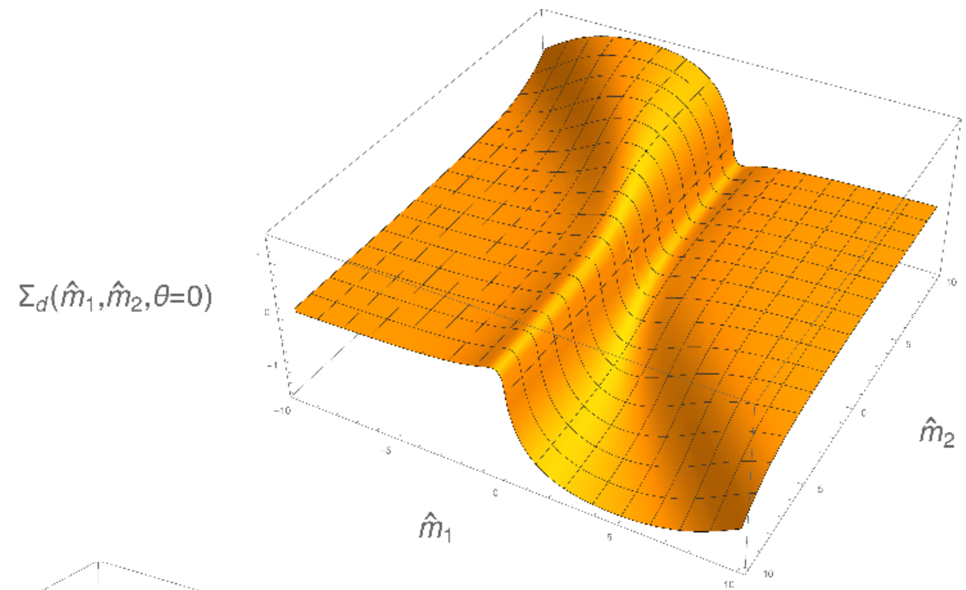


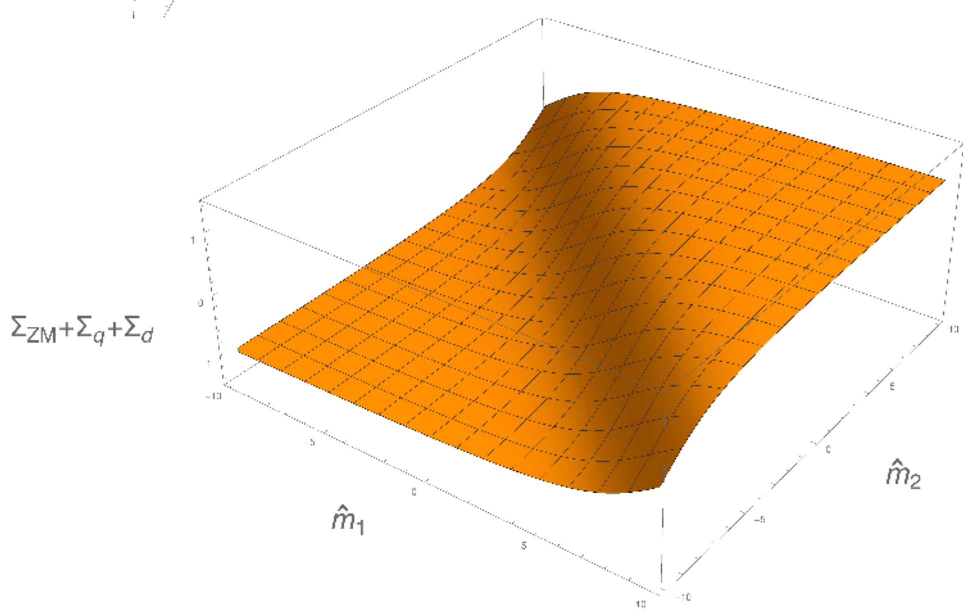




Adding the Contributions







V. Conclusions

- One and two-flavor QCD show a Silver Blaze phenomenon when the chiral condensate remains constant while the quark mass crosses a line of eigenvalues.

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- In the ϵ domain of QCD we have obtained simple exact analytical expressions for the eigenvalue density of the Dirac operator at $\theta \neq 0$ both for one and two flavors. However, in this case the oscillating part of the spectral density gives an exponentially increasing contribution to the chiral condensate.

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- The zero modes are essential for the continuity of the chiral condensate. Their exponentially increasing contribution is canceled exactly against the contribution from the nonzero modes.
- Contributions of zero models and nonzero modes have to be perfectly balanced. Lattice simulations at nonzero θ angle can only be trusted if this balance is preserved by the algorithm.