A Lattice Calculation of the Hadronic Vacuum Polarisation Contribution to $(g-2)_\mu$

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PRISMA

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

Lattice QCD approach to HVP

* Convolution integral over Euclidean momenta: *[Lautrup & de Rafael; Blum]*

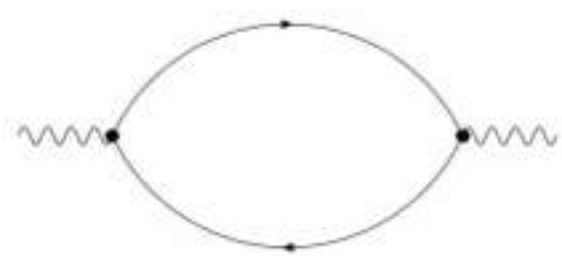
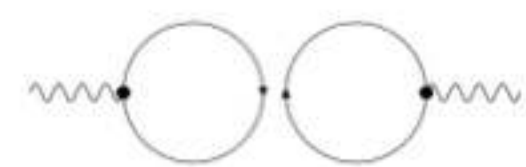
$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, f(Q^2)\hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2\left(\Pi(Q^2) - \Pi(0)\right)$$

* Weight function $f(Q^2)$ strongly peaked near muon mass

* **Direct method**: determine $\Pi(Q^2)$ from VP tensor

$$\Pi_{\mu\nu}(Q) = \int d^4x\, {\rm e}^{iQ\cdot x}\left\langle J_\mu(x)J_\nu(0)\right\rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2)$$

$$J_\mu = \tfrac{2}{3}\overline{u}\gamma_\mu u - \tfrac{1}{3}\overline{d}\gamma_\mu d - \tfrac{1}{3}\overline{s}\gamma_\mu s + \ldots$$

- Determine $\Pi(0)$ and Padé representation of $\Pi(Q^2)$ from fits
  $Q^2 \leq Q^2_{\rm cut} \approx 0.1 - 0.5\,{\rm GeV}^2$

Lattice QCD approach to HVP

- **Time-momentum representation:** *[Bernecker & Meyer]*

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{f}(x_0)\, G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$

$$\tilde{f}(x_0) = 4\pi^2 \int_0^\infty dQ^2\, f(Q^2) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\tfrac{1}{2} Q x_0\right)\right]$$

- Kernel $\tilde{f}(x_0)$ admits expansion in $x_0 m_\mu$ which is accurate to $\mathrm{O}(10^{-6})$

[arXiv:1705.01775]

- Control long-distance behaviour of $G(x_0)$

$$G(x_0) = \begin{cases} G(x_0)_{\rm data}, & x_0 \le x_{0,\rm cut} \\ G(x_0)_{\rm ext}, & x_0 > x_{0,\rm cut} \end{cases}$$

- $G(x_0)$ dominated by two-pion state for $x_0 \to \infty$

Lattice QCD approach to HVP

* **Time moments:** [Chakraborty et al.]

$$G_{2n} \equiv a \sum_{x_0} x_0^{2n} G(x_0) = (-1)^n \frac{\partial^{2n}}{\partial \omega^{2n}} \left\{ \omega^2 \Pi(\omega^2) \right\}_{\omega^2=0}$$

* Expansion of VPF at low Q^2: $\Pi(Q^2) = \Pi_0 + \sum_{j=1}^{\infty} Q^{2j} \Pi_j$

* Coefficients: $\Pi(0) \equiv \Pi_0 = -\frac{1}{2} G_2, \quad \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}$

* Construct low-energy representation of $\Pi(Q^2)$ from time moments

* Control large-x_0 regime (c.f. TMR): $G_{2n} \equiv a \sum_{x_0} x_0^{2n} G(x_0)$

The Mainz $(g-2)_\mu$ project

Collaborators:

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Topics:

* Hadronic vacuum polarisation
* Hadronic light-by-light scattering
* Running of $\alpha_{\rm em}$ and $\sin^2\theta_W$

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Current data sets

CLS consortium — "Coordinated Lattice Simulations"

* $N_\mathrm{f} = 2$ flavours of $\mathrm{O}(a)$ improved Wilson fermions

* Three values of the lattice spacing: $a = 0.076, 0.066, 0.049\,\mathrm{fm}$

* Pion masses and volumes: $m_\pi^{\mathrm{min}} = 185\,\mathrm{MeV}, \quad m_\pi L > 4$

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* $N_\mathrm{f} = 2{+}1$ flavours of $\mathrm{O}(a)$ improved Wilson fermions; tree-level Symanzik gauge action; open and periodic boundary conditions

* Three values of the lattice spacing: $a = 0.085, 0.065, 0.050\,\mathrm{fm}$

* Pion masses and volumes: $m_\pi^{\mathrm{min}} = 200\,\mathrm{MeV}, \quad m_\pi L > 4$

* To be included: two more lattice spacings; physical pion mass

Current data sets: $N_f = 2$

arXiv:1705.1775

The hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD

M. Della Morte[a], A. Francis[b], V. Gülpers[c], G. Herdoíza[d], G. von Hippel[e], H. Horch[e], B. Jäger[f], H.B. Meyer[e,g], A. Nyffeler[e], H. Wittig[e,g]

Run	L/a	β	κ	$m_\pi L$	a [fm]	m_π [MeV]	$N_{\rm cfg}$	$N_{\rm meas}$
A3	32	5.20	0.13580	6.0	0.0755(9)(7)	495	251	1004
A4	32	5.20	0.13590	4.7	0.0755(9)(7)	381	400	1600
A5	32	5.20	0.13594	4.0	0.0755(9)(7)	331	251	1004
B6	48	5.20	0.13597	5.0	0.0755(9)(7)	281	306	1224
E5	32	5.30	0.13625	4.7	0.0658(7)(7)	437	1000	4000
F6	48	5.30	0.13635	5.0	0.0658(7)(7)	311	300	1200
F7	48	5.30	0.13638	4.2	0.0658(7)(7)	265	250	1000
G8	64	5.30	0.13642	4.0	0.0658(7)(7)	185	325	4588
N5	48	5.50	0.13660	5.2	0.0486(4)(5)	441	347	1388
N6	48	5.50	0.13667	4.0	0.0486(4)(5)	340	559	2236
O7	64	5.50	0.13671	4.2	0.0486(4)(5)	268	149	2384

* Focus on methodology and systematics

Standard Method

$$a^4 \sum_f q_f^2 Z_V \sum_x \left(e^{iQ(x+a\hat{\mu}/2)} - 1\right) \left\langle V_{\mu,f}^{\rm con}(x) V_{\nu,f}^{\rm loc}(0) \right\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(\hat{Q})$$

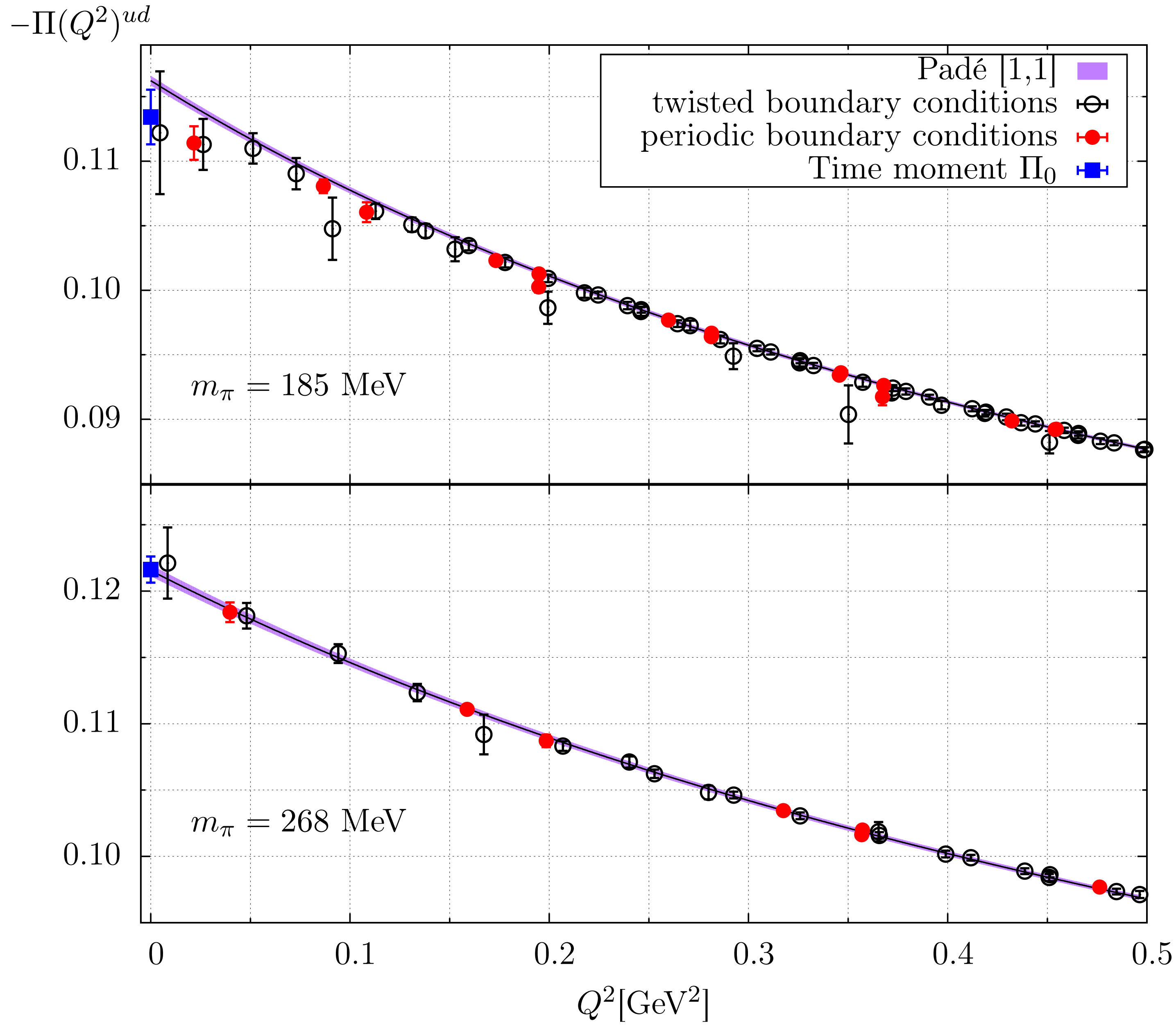
* Use twisted boundary conditions *[Della Morte et al. 2011]*

$$\psi(x + Le_\mu) = e^{i\theta_\mu}\psi(x)$$

$$\Rightarrow \quad Q_\mu = \frac{2\pi}{L} + \frac{\theta_\mu}{L}$$

* Fit $\Pi(Q^2)$ to low-order Padé for

$$Q^2 \le Q_{\rm cut}^2 \approx 0.5\,{\rm GeV}^2$$



Time-momentum representation

* Lattice observable: $G^f(x_0) = -\frac{a^3}{3}\sum_k\sum_{\vec{x}} q_f^2\, Z_V\, \langle V_{k,f}^{\rm con}(x_0,\vec{x})V_{k,f}^{\rm loc}(0)\rangle$

$$(a_\mu^{\rm hvp})^f = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{f}(x_0)\, G^f(x_0)$$

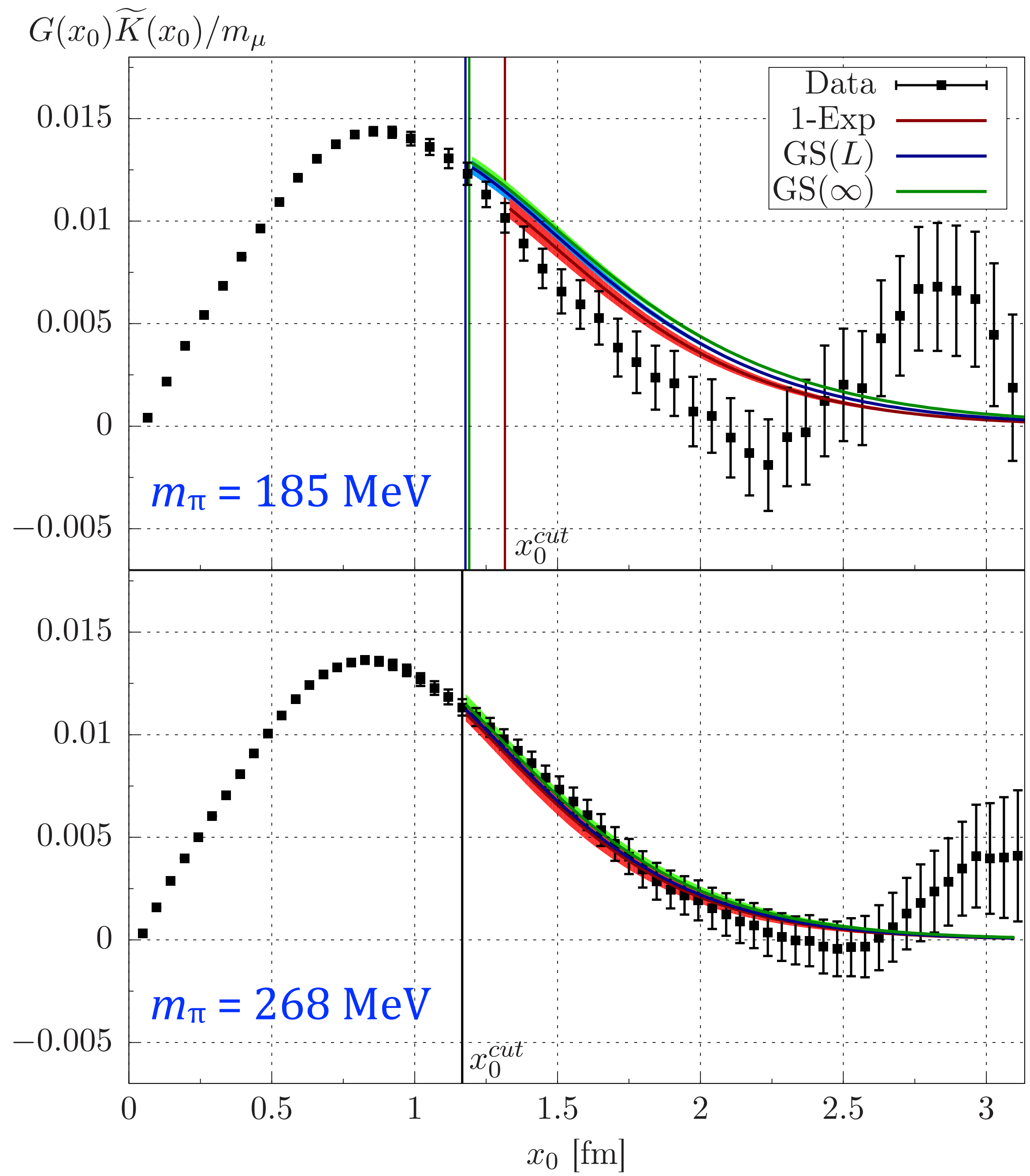
* Control tail of integrand:
  - Naive single exponential:

    $$G(x_0)_{\rm ext} = A\, e^{-m_\rho x_0}$$

  - Single exponential plus 2-pion state:

    $$G(x_0)_{\rm ext} = A\, e^{-m_\rho x_0} + B\, e^{-E_{2\pi}(\vec{p})x_0}$$

  - Gounaris-Sakurai parameterisation of timeline pion form factor



TMR analysis of finite-volume effects

- Large-x_0 behaviour of $G(x_0, L)$ and $G(x_0, \infty)$:
  - $G(x_0, \infty)$ contains a continuum of states with $E \geq 2m_\pi$
  - $G(x_0, L)$: discrete energy levels: $E \geq 2\sqrt{m_\pi^2 + (2\pi/L)^2}$

⇒ Long-distance regime related to finite-volume effects

- Isospin decomposition:

$$G(x_0) = G^{\rho\rho}(x_0) + G^{I=0}(x_0), \quad G^{\rho\rho}(x_0) = \frac{9}{10}\,G^{ud}(x_0)$$

- Long-distance behaviour of $G^{\rho\rho}(x_0)$ constrained by time-like pion form factor $F_\pi(\omega)$

TMR analysis of finite-volume effects

* Iso-vector correlator in infinite volume:

$$G^{\rho\rho}(x_0) = \int_0^\infty d\omega\,\omega^2\rho(\omega^2)\mathrm{e}^{-\omega|x_0|} = \frac{1}{48\pi^2}\int_0^\infty d\omega\,\omega^2(1-4m_\pi^2/\omega^2)^{3/2}|F_\pi(\omega)|^2\mathrm{e}^{-\omega x_0}$$

* Approximate $F_\pi(\omega)$ by Gounaris-Sakurai parameterisation: (m_ρ, Γ_ρ)

* **Finite volume**: $G^{\rho\rho}(x_0,L) = \sum_n |A_n|^2\,\mathrm{e}^{-\omega_n x_0}, \quad \omega_n = 2\sqrt{m_\pi^2+k_n^2}$

  - Fix m_ρ from fits to smeared correlation function
  - Determine Γ_ρ from correlator $G(x_0, L)$ using m_ρ as input

$$|F_\pi(\omega)|^2 = \left\{(z\phi'(z))_{z=kL/2\pi} + k\frac{\partial\delta_1(k)}{\partial k}\right\}\frac{3\pi\omega^2}{2k^2}\,|A|^2$$

  - Determine energy levels ω_n and amplitudes A_n via Lüscher formalism and GS

$$\delta_{11}(k) + \phi\Big(\frac{kL}{2\pi}\Big) = n\pi, \quad n = 1,2,\ldots$$

TMR analysis of finite-volume effects

* **Infinite volume**:
  Compute $G^{\rho\rho}(x_0)$ beyond $x_{0,\text{cut}}$ using GS parameterisation

$$G^{\rho\rho}(x_0) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2) \mathrm{e}^{-\omega|x_0|} = \frac{1}{48\pi^2} \int_0^\infty d\omega\, \omega^2 (1 - 4m_\pi^2/\omega^2)^{3/2} |F_\pi(\omega)|^2 \mathrm{e}^{-\omega x_0}$$

* Determine finite-volume shift from $G(x_0, \infty) - G(x_0, L)$

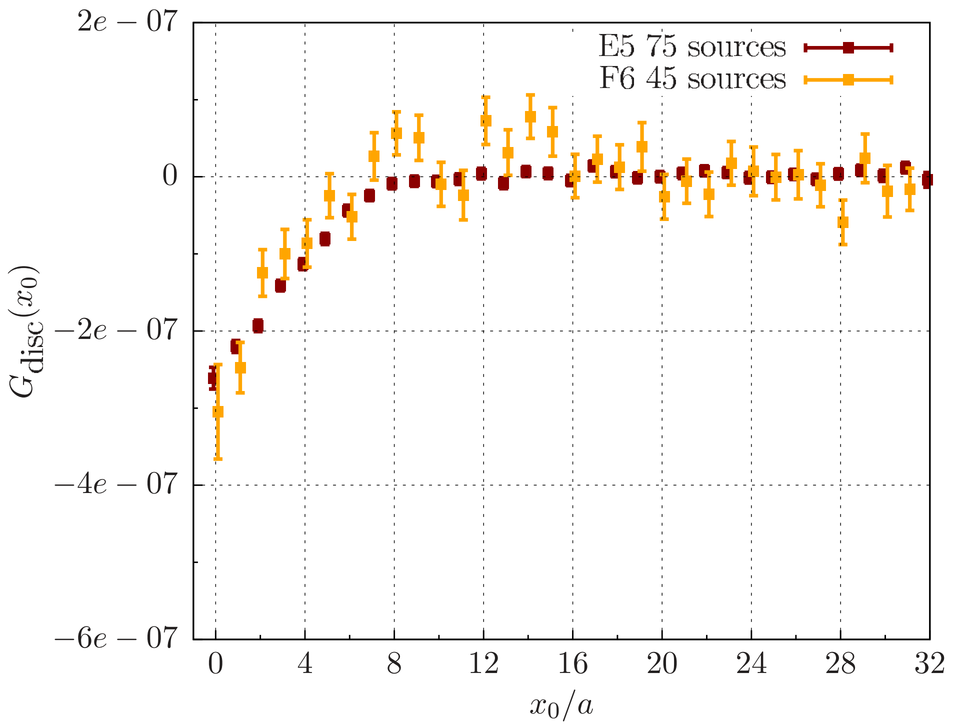
* At $m_\pi = 185$ MeV, $L = 4.0$ fm, $m_\pi L = 4.0$:

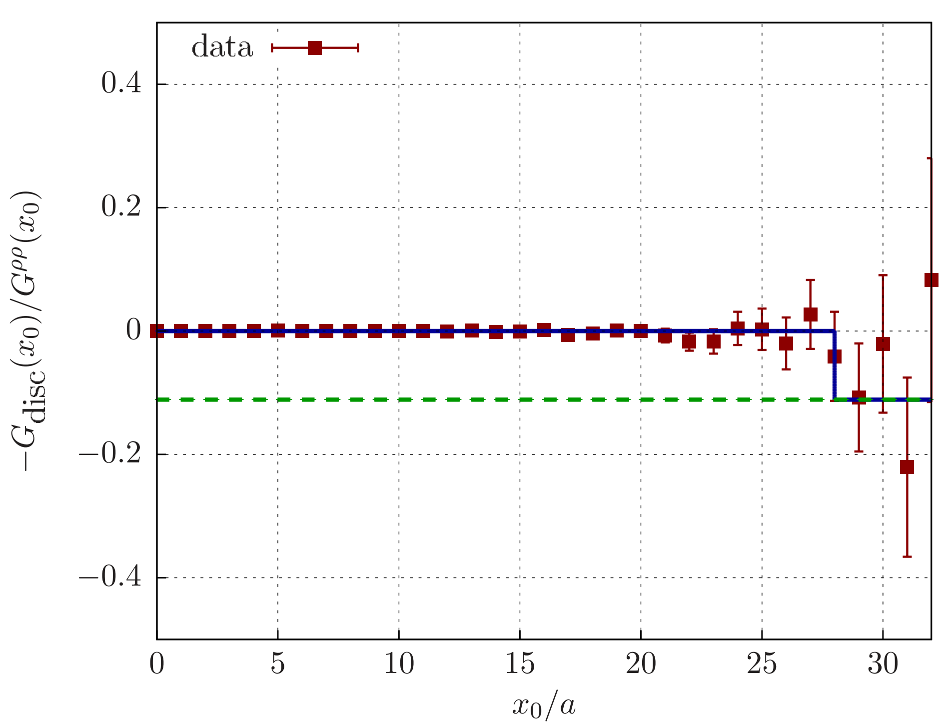
$$a_\mu^{\text{hvp}}(\infty) - a_\mu^{\text{hvp}}(L) \approx 3.5\%$$

* Assign 20% uncertainty to determination of volume shift

Disconnected Contributions

* Exploit stochastic noise cancellation between (ud) and s quarks
  [Gülpers et al., arXiv:1411.7592; V. Gülpers, PhD Thesis 2015]





* Asymptotic behaviour: $$-\frac{G_{\rm disc}(x_0)}{G^{\rho\rho}(x_0)} \xrightarrow{x_0\to\infty} -\frac{1}{9}, \qquad G^{\rho\rho}(x_0) = \frac{9}{10}G^{ud}(x_0)$$

* Data compatible with: $$-\frac{G_{\rm disc}(x_0)}{G^{\rho\rho}(x_0)} = \begin{cases} 0, & x_0 \leq x_0^*, \\ -1/9, & x_0 > x_0^* \end{cases}$$

Disconnected Contributions

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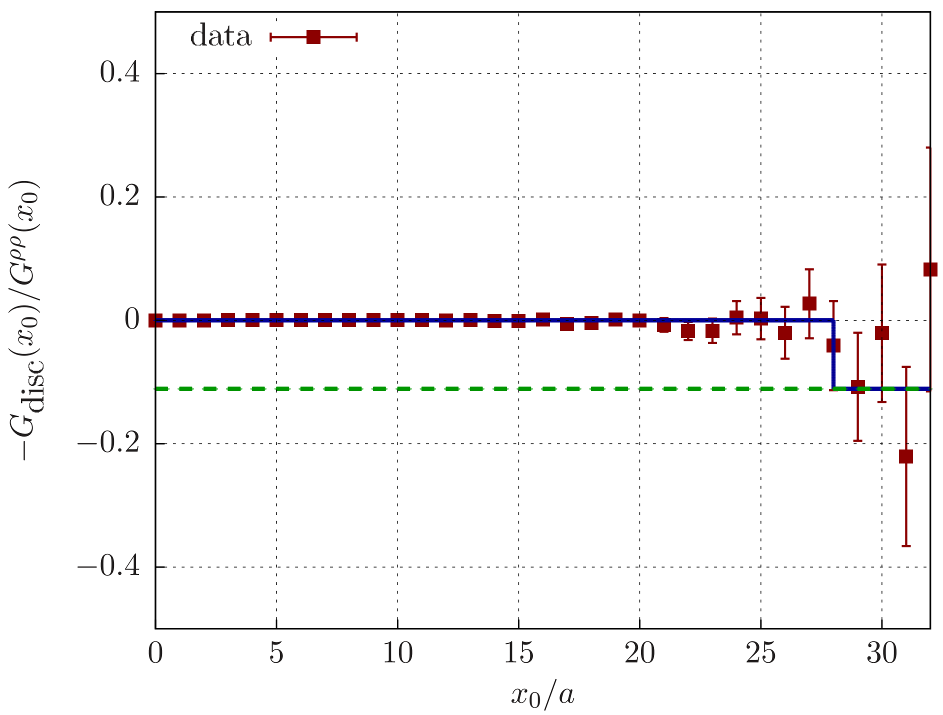
⇒ Upper bound on disconnected contribution:

$$(a_\mu^{\rm hvp})_{\rm disc} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{f}(x_0)(-G_{\rm disc}(x_0)) = -\left(\frac{\alpha}{\pi}\right)^2 \int_{x_0^*}^\infty dx_0\, \tilde{f}(x_0)\, \frac{1}{9} G^{\rho\rho}(x_0)$$

$$\Rightarrow \quad \Delta a_\mu^{\rm hvp} \equiv -\frac{(a_\mu^{\rm hvp})_{\rm disc}}{(a_\mu^{\rm hvp})_{\rm con}} \leq 2\%$$

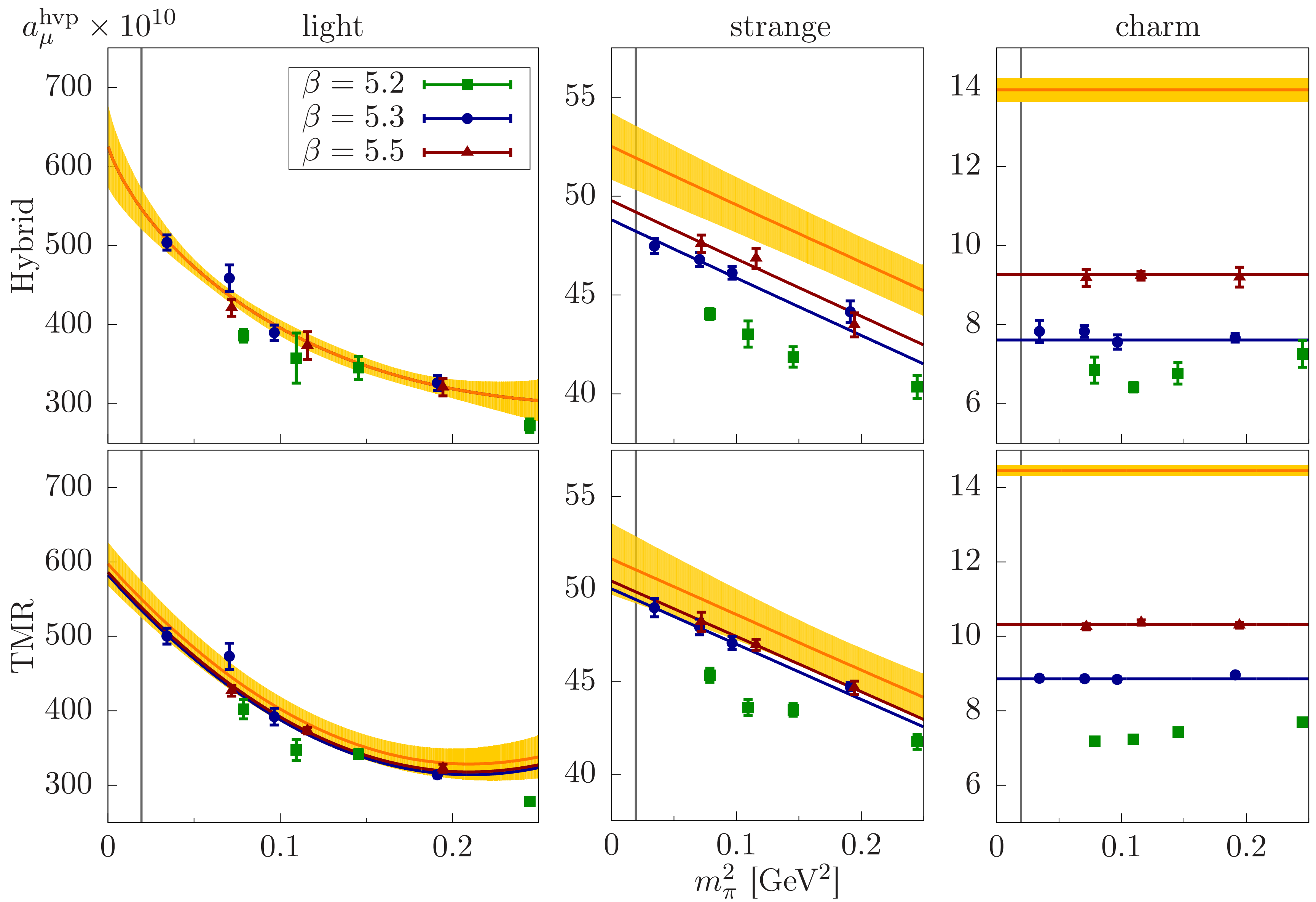
Run	$N_{\rm cfg}$	$N_{\rm r}$	T/a	x_0^*	$\Delta a_\mu^{\rm hvp}$
E5	1000	75	64	25	0.7%
				28	0.3%
F6	300	45	96	22	1.8%
				23	1.5%



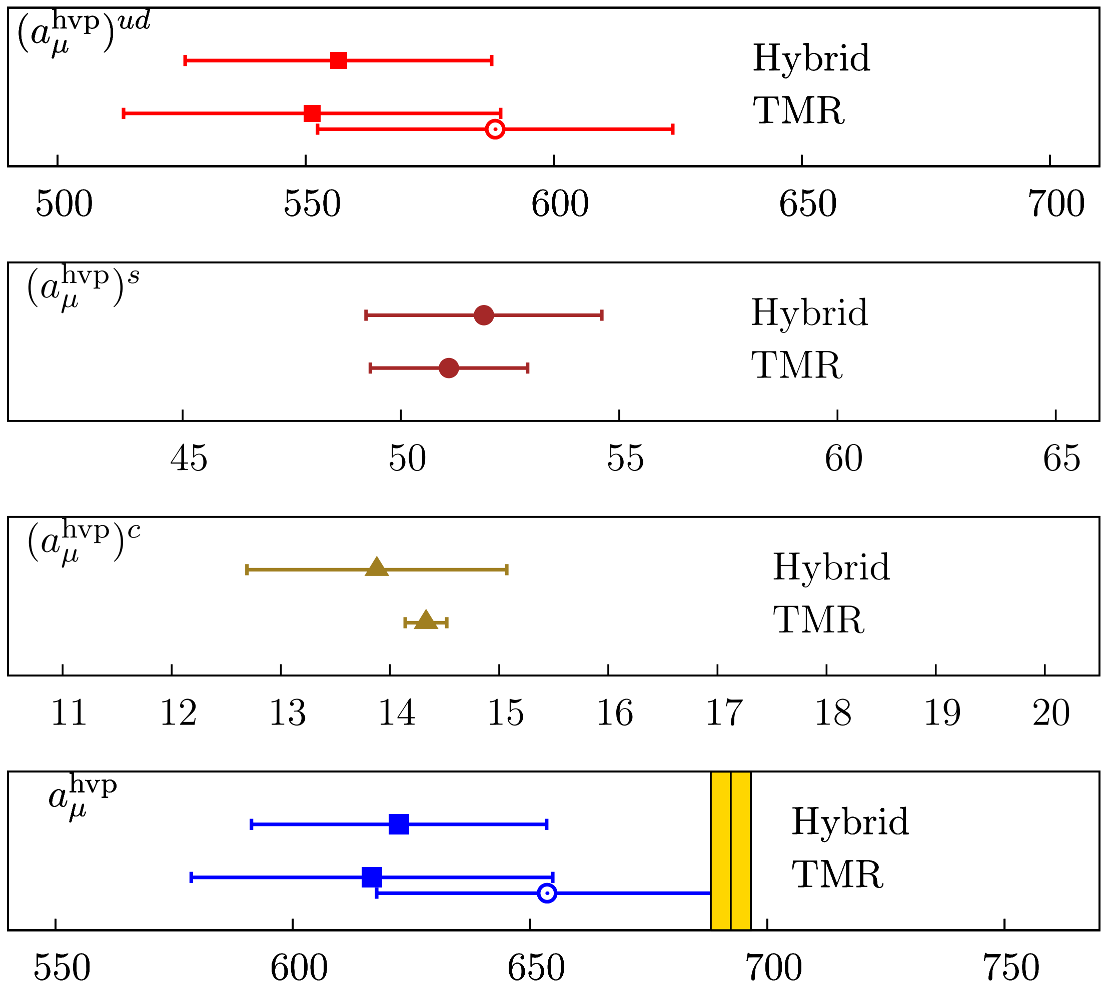
Chiral and continuum extrapolations

* Employ variety of phenomenological *ansätze* to fit a_μ^{hvp} e.g.

$$\alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a \qquad \beta_1 + \beta_2 m_\pi^2 + \beta_3 m_\pi^4 + \beta_4 a \qquad \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a$$



Results at the physical point



* Systematic errors estimated from distribution of fit variations
* Good consistency between "hybrid" method and TMR
* Finite-volume corrections sizeable

Scale setting error

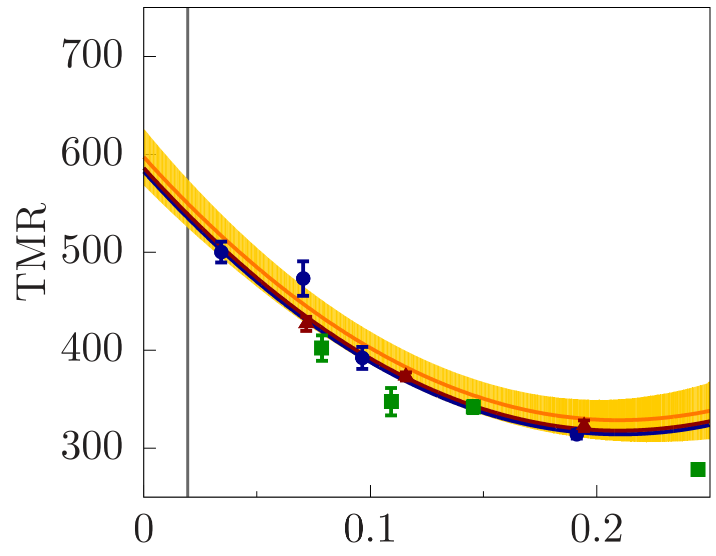
* Evaluate $$\delta a_\mu^{\rm hvp} = \left| a \frac{da_\mu^{\rm hvp}}{da} \right| \frac{\delta a}{a} = \left| m_\mu \frac{\partial a_\mu^{\rm hvp}}{\partial m_\mu} + m_\pi \frac{\partial a_\mu^{\rm hvp}}{\partial m_\pi} \right| \frac{\delta a}{a}$$

* TMR: $$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{f}(x_0)\, G(x_0) \qquad x_0 \tilde{f}'(x_0) - \tilde{f}(x_0) = J(x_0)$$

$$\Rightarrow \quad m_\mu \frac{\partial a_\mu^{\rm hvp}}{\partial m_\mu} = -a_\mu^{\rm hvp} + \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, G(x_0)\, J(x_0) = 1.22 \cdot 10^{-7}$$

* Determine $m_\pi\,(\partial a_\mu^{\rm hvp}/\partial m_\pi)$ from slope of chiral fit

$$\Rightarrow \quad \frac{\delta a_\mu^{\rm hvp}}{a_\mu^{\rm hvp}} = \left| \underbrace{\frac{m_\mu}{a_\mu^{\rm hvp}} \frac{\partial a_\mu^{\rm hvp}}{\partial m_\mu}}_{\approx 1.8} + \underbrace{\frac{m_\pi}{a_\mu^{\rm hvp}} \frac{\partial a_\mu^{\rm hvp}}{\partial m_\pi}}_{\approx -0.2} \right| \frac{\delta a}{a}$$

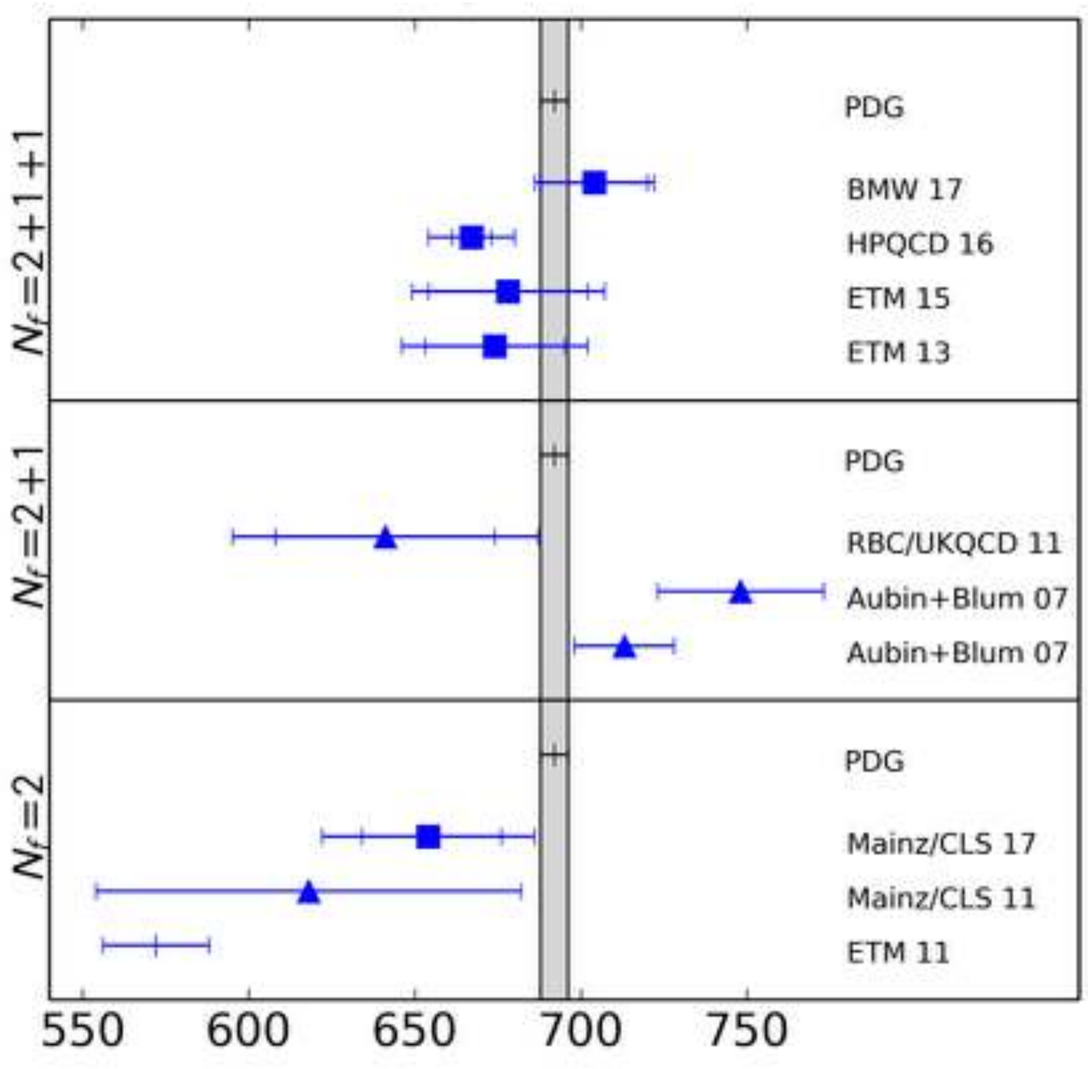


⇒ Rather precise determination of lattice spacing a [fm] required!

Final result for $N_f = 2$

* Estimate from TMR including finite-volume correction

$$a_\mu^{\rm hvp} = (654 \pm 32_{\rm stat} \pm 17_{\rm syst} \pm 10_{\rm scale} \pm 7_{\rm FV}\,{}^{+\;0}_{-10}{}_{\rm disc}) \cdot 10^{-10}$$



Preliminary results for $N_f = 2 + 1$

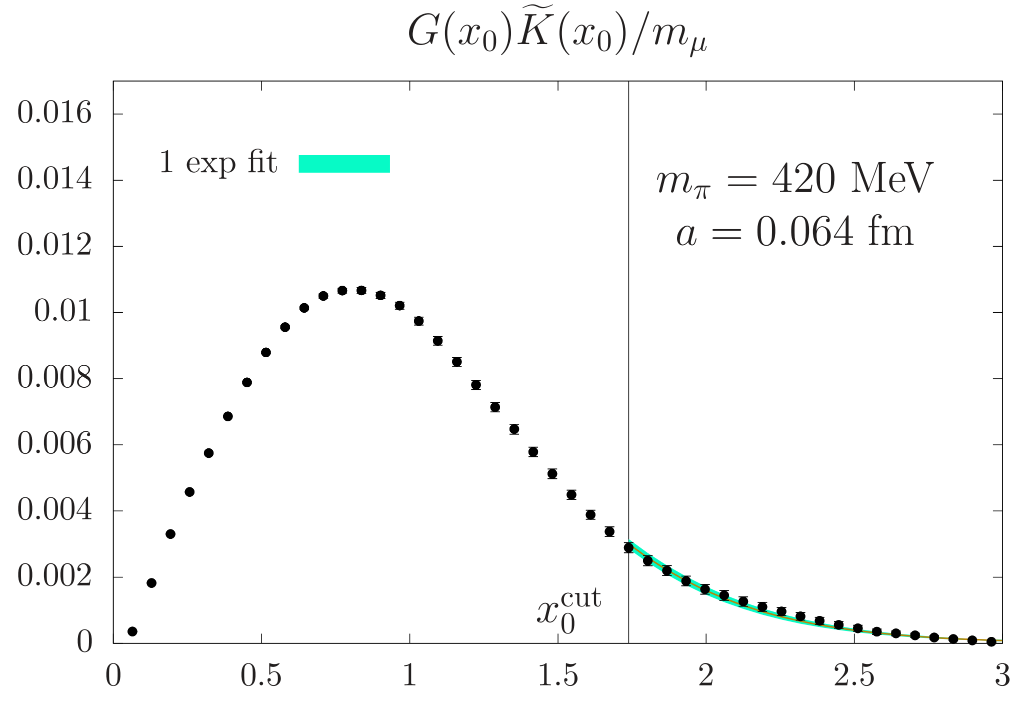
* Run table:

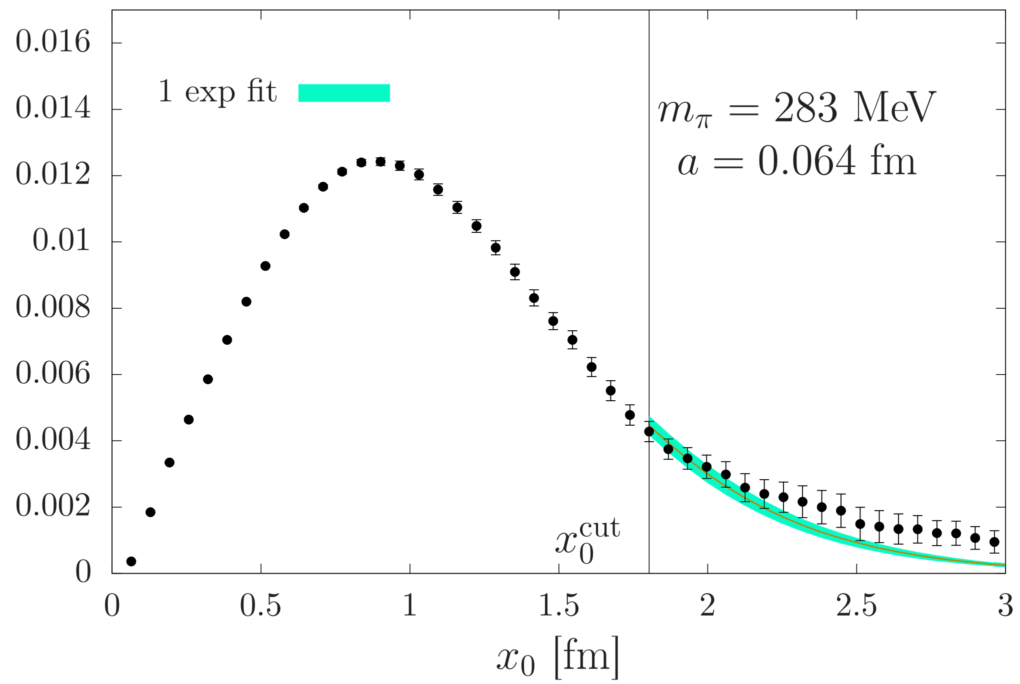
CLS	β	$L^3 \times T$	a [fm]	m_π [MeV]	$m_\pi L$	#confs	#src	$\delta a_\mu / a_\mu$ [%]	$\delta a_\mu / a_\mu$ [%]
H101	3.4	$32^3 \times 96$	0.085	420	5.8	500	15	1.2	1.4
H102		$32^3 \times 96$		350	4.9	1900	5	1.4	1.8
H105		$32^3 \times 96$		280	3.9	2800	5	2.2	4.6
C101		$48^3 \times 96$		220	4.7	1100	5	3.5	3.0
H200	3.55	$32^3 \times 96$	0.065	420	4.3	2000	5	1.4	1.3
N203		$48^3 \times 128$		340	5.4	$\times$	$\times$	$\times$	2.6
N200		$48^3 \times 128$		280	4.4	1700	5	2.2	3.6
D200		$64^3 \times 128$		200	4.2	1100	5	4.2	2.5
N300	3.70	$48^3 \times 128$	0.050	420	5.1	900	5	2.4	1.5
N303		$48^3 \times 128$		?	?	$\times$	$\times$	$\times$	2.2
J303		$64^3 \times 192$		260	4.1	$\times$	$\times$	$\times$	3.1

[A. Gérardin]

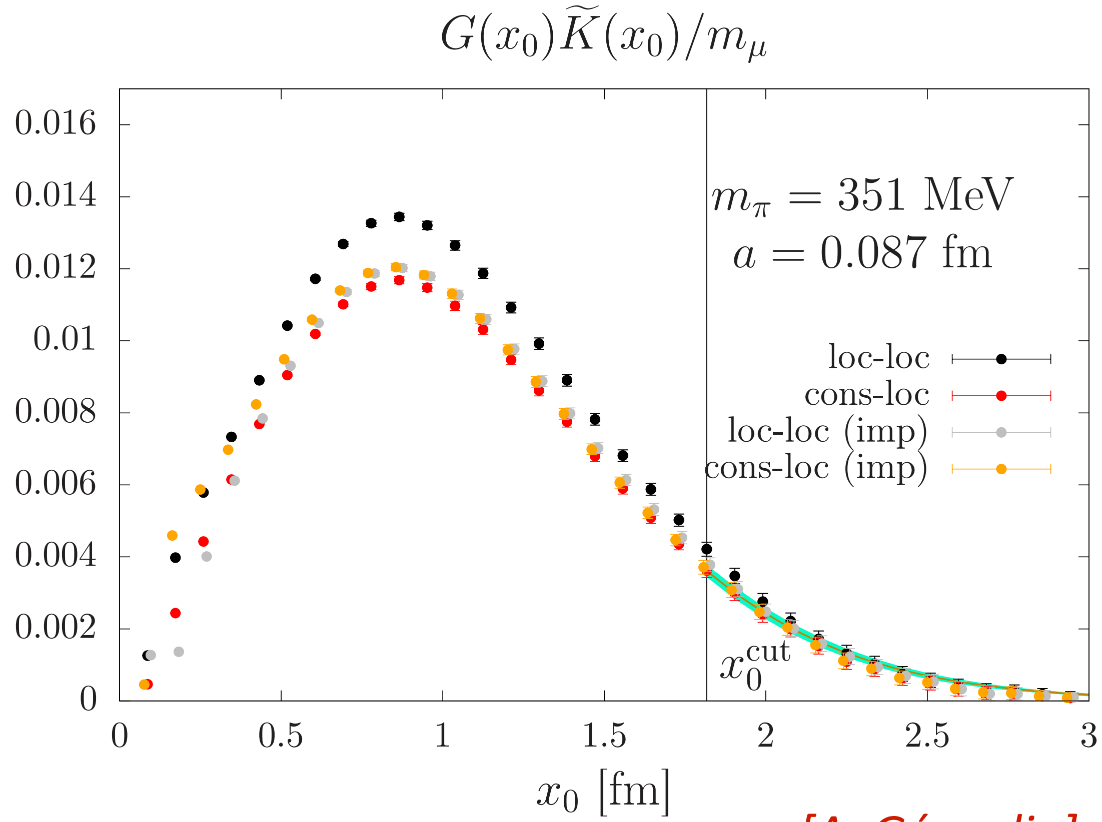
* Current relative error comparable to two-flavour case

Preliminary results for $N_f = 2 + 1$





Improved vs. unimproved vector currents:



[A. Gérardin]

Summary and Outlook

- Consistency among different methods to determine the HVP contribution to $(g-2)$
- Result in the two-flavour case:

$$a_\mu^{\mathrm{hvp}} = (654 \pm 32\,{}^{+21}_{-23}) \cdot 10^{-10}$$

4.8% statistical error
3.3% total systematic error

- Preliminary results for $N_\mathrm{f} = 2+1$
  - Constrain long-distance behaviour of $G(x_0)$ using pion time-like form factor
  - Include physical pion mass
  - Include isospin breaking