Westfälische Wilhelms-Universität Münster
institut für theoretische physik
ALPHA Collaboration

Non-perturbative determination of c_V, Z_V and Z_S/Z_P in $N_f = 3$ lattice QCD

Jochen Heitger, Fabian Joswig, Tassos Vladikas, Christian Wittemeier

The XXXV International Symposium on Lattice Field Theory
University of Granada, Spain, June 18 – 24, 2017

Lattice2017

Motivation

Inherent O(a) discretization errors of Wilson fermions are systematically removable via *Symanzik improvement*, by adding ...

Motivation

Inherent O(a) discretization errors of Wilson fermions are systematically removable via *Symanzik improvement*, by adding ...

- ... a dimension–5 (Sheikoleslami-Wohlert) term to the action, $\propto c_{SW}$

Motivation

Inherent $O(a)$ discretization errors of Wilson fermions are systematically removable via *Symanzik improvement*, by adding ...

- ... a dimension–5 (Sheikoleslami-Wohlert) term to the action, $\propto c_{SW}$
- ... dimension–4 terms to quark bilinear composite operators

Motivation

Inherent $O(a)$ discretization errors of Wilson fermions are systematically removable via *Symanzik improvement*, by adding ...

- ... a dimension−5 (Sheikoleslami-Wohlert) term to the action, $\propto c_{SW}$
- ... dimension−4 terms to quark bilinear composite operators

Improvement & Renormalization patterns of axial & vector currents:

$$\left[\,(A_I)^a_\mu\,\right]_R(x) \;=\; Z_A\left[A^a_\mu(x) + a\,c_A\,\frac{1}{2}\left(\partial_\mu + \partial^*_\mu\right)P^a(x)\right]$$

$$A^a_\mu(x) = \bar\psi(x)\frac{\tau^a}{2}\gamma_\mu\gamma_5\psi(x)\;,\;\; P^a(x) = \bar\psi(x)\frac{\tau^a}{2}\gamma_5\psi(x)$$

$$\left[\,(V_I)^a_\mu\,\right]_R(x) \;=\; Z_V\left[V^a_\mu(x) + a\,c_V\,\frac{1}{2}\left(\partial_\mu + \partial^*_\mu\right)T^a_{\mu\nu}(x)\right]$$

$$V^a_\mu(x) = \bar\psi(x)\frac{\tau^a}{2}\gamma_\mu\psi(x)\;,\;\; T^a_{\mu\nu}(x) = \mathrm{i}\bar\psi(x)\frac{\tau^a}{2}\sigma_{\mu\nu}\psi(x)$$

Motivation

Here:

- Lattice QCD with $N_f = 3$ $O(a)$ improved Wilson fermions and tree-level Symanzik improved ("Lüscher-Weisz") gauge action

Motivation

Here:

- Lattice QCD with $N_f = 3$ $O(a)$ improved Wilson fermions and tree-level Symanzik improved ("Lüscher-Weisz") gauge action
  - ✓ non-pert. c_{sw} [Bulava & Schaefer, NPB 874 (2013) 188, arXiv:1304.7093]

Motivation

Here:

- Lattice QCD with $N_f = 3$ $O(a)$ improved Wilson fermions and tree-level Symanzik improved ("Lüscher-Weisz") gauge action
  - ✓ non-pert. c_{sw} [Bulava & Schaefer, NPB 874 (2013) 188, arXiv:1304.7093]
  - ✓ non-pert. c_A, Z_A [ALPHA Collaboration, Bulava, Della Morte, H. & Wittemeier]

Motivation

Here:

- Lattice QCD with $N_f = 3$ $O(a)$ improved Wilson fermions and tree-level Symanzik improved ("Lüscher-Weisz") gauge action
  - ✓ non-pert. c_{sw} [Bulava & Schaefer, NPB 874 (2013) 188, arXiv:1304.7093]
  - ✓ non-pert. c_A, Z_A [ALPHA Collaboration, Bulava, Della Morte, H. & Wittemeier]
- Coupling / β–range of "CLS" $N_f = 2 + 1$ ensembles:

$$\beta = 3.4 - 3.85 \; (g_0^2 \approx 1.76 - 1.56) \quad \Leftrightarrow \quad a \approx 0.085 - 0.039 \, \text{fm}$$

Motivation

Here:

- Lattice QCD with $N_f = 3$ $O(a)$ improved Wilson fermions and tree-level Symanzik improved ("Lüscher-Weisz") gauge action
  - ✓ non-pert. c_{sw} [Bulava & Schaefer, NPB 874 (2013) 188, arXiv:1304.7093]
  - ✓ non-pert. c_A, Z_A [ALPHA Collaboration, Bulava, Della Morte, H. & Wittemeier]
- Coupling / β–range of "CLS" $N_f = 2 + 1$ ensembles:

$$\beta = 3.4 - 3.85 \; (g_0^2 \approx 1.76 - 1.56) \quad \Leftrightarrow \quad a \approx 0.085 - 0.039 \, \text{fm}$$

The vector channel was not investigated so far for the three-flavour theory, despite interesting applications involving c_V and/or Z_V, e.g.:

Motivation

Here:

- Lattice QCD with $N_f = 3$ $O(a)$ improved Wilson fermions and tree-level Symanzik improved ("Lüscher-Weisz") gauge action
  - ✓ non-pert. c_{sw} [Bulava & Schaefer, NPB 874 (2013) 188, arXiv:1304.7093]
  - ✓ non-pert. c_A, Z_A [ALPHA Collaboration, Bulava, Della Morte, H. & Wittemeier]
- Coupling / β–range of "CLS" $N_f = 2 + 1$ ensembles:

$$\beta = 3.4 - 3.85 \; (g_0^2 \approx 1.76 - 1.56) \quad \Leftrightarrow \quad a \approx 0.085 - 0.039 \,\mathrm{fm}$$

The vector channel was not investigated so far for the three-flavour theory, despite interesting applications involving c_V and/or Z_V, e.g.:

- Vector current matrix elements / semi-lept. decay form factors ($\rightarrow$ also relevant for the NP HQET-QCD matching in finite volume)
- Computation of the timelike pion form factor $\rightarrow$ talk by John Bulava
- Contributions to the anomalous magnetic moment of the muon
- Thermal correlators related to di-lepton production rate in QGP

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

- Define c_X, Z_X on a *line of constant physics (LCP)*

  $\Rightarrow$ O(a) ambiguities vanish smoothly towards the continuum limit

ALPHA Collaboration

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

- Define c_X, Z_X on a *line of constant physics (LCP)*

  ALPHA Collaboration

  $\Rightarrow$ $O(a)$ ambiguities vanish smoothly towards the continuum limit

Realized in the QCD Schrödinger Functional, *finite-volume* scheme

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

- Define c_X, Z_X on a *line of constant physics (LCP)* ALPHA Collaboration
  $\Rightarrow$ O(a) ambiguities vanish smoothly towards the continuum limit

Realized in the QCD Schrödinger Functional, *finite-volume* scheme

- Employ $N_f = 3$ SF gauge configuration ensembles, generated in the context of earlier determinations of c_A, Z_A

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

- Define c_X, Z_X on a *line of constant physics (LCP)* — ALPHA Collaboration
  $\Rightarrow$ $O(a)$ ambiguities vanish smoothly towards the continuum limit

Realized in the QCD Schrödinger Functional, *finite-volume* scheme

- Employ $N_f = 3$ SF gauge configuration ensembles, generated in the context of earlier determinations of c_A, Z_A
- LCP condition $L = L_{phys} \stackrel{!}{=}$ const., using universal parts in:

$$a(g_0^2)/a\big((g_0')^2\big) \;=\; e^{-[g_0^{-2}-(g_0')^{-2}]/(2b_0)}\left[\,g_0^2/(g_0')^2\,\right]^{-b_1/(2b_0^2)}$$

living.knowledge WWU Münster

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

- Define c_X, Z_X on a *line of constant physics (LCP)*
  $\Rightarrow$ $O(a)$ ambiguities vanish smoothly towards the continuum limit

ALPHA Collaboration

Realized in the QCD Schrödinger Functional, *finite-volume* scheme

- Employ $N_f = 3$ SF gauge configuration ensembles, generated in the context of earlier determinations of c_A, Z_A
- LCP condition $L = L_{phys} \stackrel{!}{=}$ const., using universal parts in:

$$a(g_0^2)/a\big((g_0')^2\big) \;=\; e^{-[g_0^{-2}-(g_0')^{-2}]/(2b_0)}\left[g_0^2/(g_0')^2\right]^{-b_1/(2b_0^2)}$$

- Final results on $c_X(g_0^2), Z_X(g_0^2)$ can be described by smooth interpolating curves based on functional forms (or Padé fits), asymptotically constrained to 1–loop perturbation theory

Strategy & Previous work for this discretization

Basic framework:
Impose improvement and renormalization conditions such that all physical scales kept fixed

- Define c_X, Z_X on a *line of constant physics (LCP)* ALPHA Collaboration
  $\Rightarrow$ O(a) ambiguities vanish smoothly towards the continuum limit

Realized in the QCD Schrödinger Functional, *finite-volume* scheme

- Employ $N_f = 3$ SF gauge configuration ensembles, generated in the context of earlier determinations of c_A, Z_A
- LCP condition $L = L_{phys} \stackrel{!}{=}$ const., using universal parts in:

$$a(g_0^2)/a\left((g_0')^2\right) = e^{-[g_0^{-2}-(g_0')^{-2}]/(2b_0)}\left[g_0^2/(g_0')^2\right]^{-b_1/(2b_0^2)}$$

- Final results on $c_X(g_0^2), Z_X(g_0^2)$ can be described by smooth interpolating curves based on functional forms (or Padé fits), asymptotically constrained to 1–loop perturbation theory

Note: ∃ an alternative approach via the "χSF" (chirally rotated SF) [Dalla Brida & Sint]

$N_f = 3$ SF ensembles in the CLS β–range

$L^3 \times T/a^4$	β	κ	#REP	#MDU	ID
$12^3 \times 17$	3.3	0.13652	10	10240	A1k1
		0.13660	10	12620	A1k2
$14^3 \times 21$	3.414	0.13690	32	10360	E1k1
		0.13695	48	13984	E1k2
$16^3 \times 23$	3.512	0.13700	2	20480	B1k1
		0.13703	1	8192	B1k2
		0.13710	3	24560	B1k3
$16^3 \times 23$	3.47	0.13700	3	29584	B2k1
$20^3 \times 29$	3.676	0.13700	4	15232	C1k2
		0.13719	4	15472	C1k3
$24^3 \times 35$	3.810	0.13712	5	10240	D1k1

ID	$am_{\rm PCAC}$	$am_{\rm PCAC,0}$
A1k1	−0.00234(85)	−0.00367(92)
A1k2	−0.01085(71)	−0.01206(81)
	0.0	0.0
E1k1	0.00275(38)	0.00262(49)
E1k2	0.00004(33)	−0.00072(41)
	0.0	0.0
B1k1	0.00565(16)	0.00554(23)
B1k2	0.00494(25)	0.00423(36)
B1k3	0.00160(18)	0.00109(21)
	0.0	0.0
B2k1	0.00349(17)	0.00306(23)
C1k2	0.00618(14)	0.00610(25)
C1k3	−0.00082(12)	−0.00099(13)
	0.0	0.0
D1k1	n.q.	−0.002909(72)

- Standard SF setup with: $T = (3/2)L - a$, $\theta = 0$, vanishing BF
  (gauge field configurations were generated using `openQCD-1.2` with SF BCs)

$N_f = 3$ SF ensembles in the CLS β–range

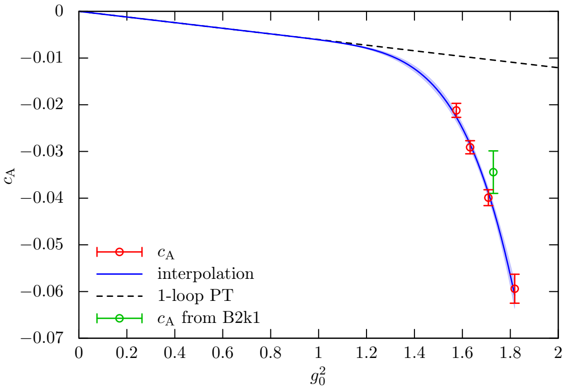
$L^3 \times T/a^4$	β	κ	#REP	#MDU	ID
$12^3 \times 17$	3.3	0.13652	10	10240	A1k1
		0.13660	10	12620	A1k2
$14^3 \times 21$	3.414	0.13690	32	10360	E1k1
		0.13695	48	13984	E1k2
$16^3 \times 23$	3.512	0.13700	2	20480	B1k1
		0.13703	1	8192	B1k2
		0.13710	3	24560	B1k3
$16^3 \times 73$	3.47	0.13700	3	29584	B2k1
$20^3 \times 29$	3.676	0.13700	4	15232	C1k2
		0.13719	4	15472	C1k3
$24^3 \times 35$	3.810	0.13712	5	10240	D1k1

ID	am_{PCAC}	$am_{PCAC,0}$
A1k1	−0.00234(85)	−0.00367(92)
A1k2	−0.01085(71)	−0.01206(81)
	0.0	0.0
E1k1	0.00275(38)	0.00262(49)
E1k2	0.00004(33)	−0.00072(41)
	0.0	0.0
B1k1	0.00565(16)	0.00554(23)
B1k2	0.00494(25)	0.00423(36)
B1k3	0.00160(18)	0.00109(21)
	0.0	0.0
B2k1	0.00349(17)	0.00306(23)
C1k2	0.00618(14)	0.00610(25)
C1k3	−0.00082(12)	−0.00099(13)
	0.0	0.0
D1k1	n.q.	−0.002909(72)

- Standard SF setup with: $T = (3/2)L - a$, $\theta = 0$, vanishing BF
  (gauge field configurations were generated using `openQCD-1.2` with SF BCs)
- LCP condition: $L = L_{phys} \stackrel{!}{=}$ const. (≈ 1.2 fm)
- Mass-degen. & indep. scheme: sea quarks with $m_{PCAC} \sim 0$

c_A

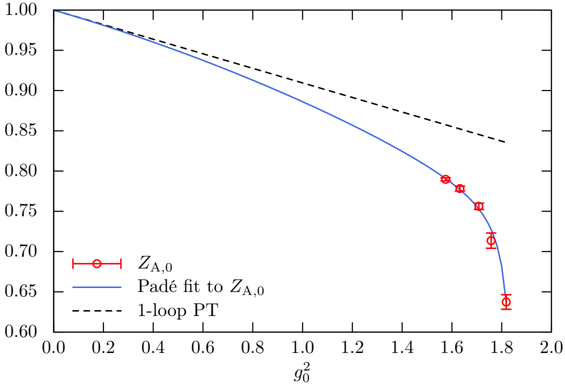
[ALPHA Collaboration, Bulava, Della Morte, H. & Wittemeier, NPB 896 (2015) 555, arXiv:1502.04999]



$$c_A(g_0^2) = -0.006033\, g_0^2 \times \left[1 + \exp\left(9.2056 - \frac{13.9847}{g_0^2}\right)\right]$$

Z_A

[ALPHA Collaboration, Bulava, Della Morte, H. & Wittemeier, PRD 93 (2016) 114513, arXiv:1604.05827]



$$Z_A(g_0^2) = 1 - 0.090488\, g_0^2 \times \frac{1 - 0.29026\, g_0^2 - 0.12881\, g_0^4}{1 - 0.53843\, g_0^2}$$

Schrödinger Functional (SF) correlation functions

- QCD, volume $T \times L^3$, Dirichlet boundary conditions in time, periodic ones in space (up to periodicity phase θ for quarks)

Schrödinger Functional (SF) correlation functions

- QCD, volume $T \times L^3$, Dirichlet boundary conditions in time, periodic ones in space (up to periodicity phase θ for quarks)
- Boundary source fields $\zeta, \bar{\zeta}$ to build correlators (incl. optimized spatial boundary wave functions)

Schrödinger Functional (SF) correlation functions

- QCD, volume $T \times L^3$, Dirichlet boundary conditions in time, periodic ones in space (up to periodicity phase θ for quarks)
- Boundary source fields $\zeta, \bar{\zeta}$ to build correlators (incl. optimized spatial boundary wave functions)
- Well known renormalization properties

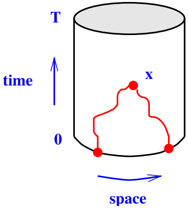
$$k_V(x_0) = -\frac{a^3}{9L^6} \sum_{\mathbf{x}} \langle\, V_k^a(x)\, \mathcal{Q}_k^a \,\rangle$$

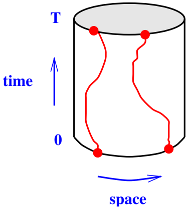
$$\propto \quad a^9 \sum_{\mathbf{u},\mathbf{v},\mathbf{x}} \langle \bar{\psi}(x)\gamma_k \tau^a \psi(x)\, \bar{\zeta}(\mathbf{u})\gamma_k \tau^a \zeta(\mathbf{v}) \rangle$$

[analogously: $f_A(x_0)$, $f_P(x_0)$, $f_V(x_0)$, $k_T(x_0)$, ...]

$$F_1 = -\frac{1}{3L^6} \langle\, \mathcal{O}'^a\, \mathcal{O}^a \,\rangle$$

$$\propto \quad a^{12} \sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'} \langle \bar{\zeta}'(\mathbf{u}')\gamma_5 \tau^a \zeta'(\mathbf{v}')\, \bar{\zeta}(\mathbf{u})\gamma_5 \tau^a \zeta(\mathbf{v}) \rangle$$





Z_V

Determination of Z_V for $N_f = 3$ in the CLS β–range

Z_V

Determination of Z_V for $N_f = 3$ in the CLS β–range

- The renormalization of the vector current is based on the vector Ward identity (implemented within the standard SF)

$$\int_{\partial R} d\sigma_\mu(x) \left\langle V_\mu^a(x)\, \mathcal{O}_{\rm int}^b(y)\, \mathcal{O}_{\rm ext}^c(z) \right\rangle = - \left\langle \left[\delta_V^a \mathcal{O}_{\rm int}^b(y) \right] \mathcal{O}_{\rm ext}^c(z) \right\rangle$$

Z_V

Determination of Z_V for $N_f = 3$ in the CLS β–range

- The renormalization of the vector current is based on the vector Ward identity (implemented within the standard SF)

$$\int_{\partial R} d\sigma_\mu(x) \left\langle V_\mu^a(x)\, \mathcal{O}_{\rm int}^b(y)\, \mathcal{O}_{\rm ext}^c(z) \right\rangle = -\left\langle \left[\delta_V^a \mathcal{O}_{\rm int}^b(y) \right] \mathcal{O}_{\rm ext}^c(z) \right\rangle$$

- Identifying $\mathcal{O}_{\rm int}^b(y)$, $\mathcal{O}_{\rm ext}^c(z)$ with pseudoscalar boundary operators from above, and expressing everything by SF correlation functions, yields normalization condition to fix Z_V:

$$Z_V\,(1 + b_V\, am_q)\, f_V(x_0) = F_1 + \mathrm{O}(a^2) \qquad \text{e.g.: for } x_0 = T/2$$

$$f_V(x_0) = \frac{a^3}{6L^6} \sum_{\mathbf{x}} i\, \epsilon^{abc} \left\langle \mathcal{O}'^a\, V_0^b(x)\, \mathcal{O}^c \right\rangle$$

Z_V

Determination of Z_V for $N_f = 3$ in the CLS β–range

- The renormalization of the vector current is based on the vector Ward identity (implemented within the standard SF)

$$\int_{\partial R} d\sigma_\mu(x) \left\langle V^a_\mu(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle = - \left\langle [\, \delta^a_V \mathcal{O}^b_{\rm int}(y) \,]\, \mathcal{O}^c_{\rm ext}(z) \right\rangle$$

- Identifying $\mathcal{O}^b_{\rm int}(y)$, $\mathcal{O}^c_{\rm ext}(z)$ with pseudoscalar boundary operators from above, and expressing everything by SF correlation functions, yields normalization condition to fix Z_V:

$$Z_V\,(1 + b_V\, am_q)\, f_V(x_0) = F_1 + O(a^2) \qquad \text{e.g.: for } x_0 = T/2$$

$$f_V(x_0) = \frac{a^3}{6L^6} \sum_{\mathbf{x}} i\, \epsilon^{abc} \left\langle \mathcal{O}'^a\, V^b_0(x)\, \mathcal{O}^c \right\rangle$$

  - With spatially periodic BCs, term $\propto c_V$ doesn't contribute to f_V

Z_V

Determination of Z_V for $N_f = 3$ in the CLS β–range

- The renormalization of the vector current is based on the vector Ward identity (implemented within the standard SF)

$$\int_{\partial R} d\sigma_\mu(x) \left\langle V_\mu^a(x)\, \mathcal{O}_{\rm int}^b(y)\, \mathcal{O}_{\rm ext}^c(z) \right\rangle = -\left\langle [\, \delta_V^a \mathcal{O}_{\rm int}^b(y)\,]\, \mathcal{O}_{\rm ext}^c(z) \right\rangle$$

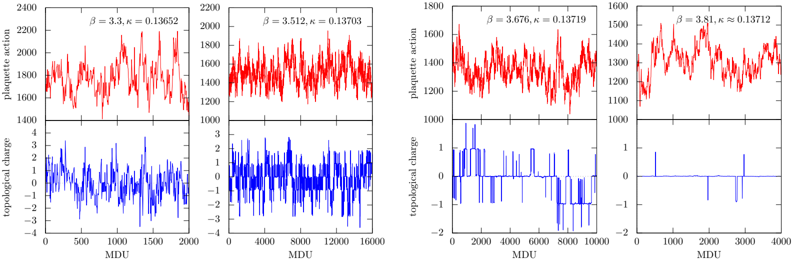
- Identifying $\mathcal{O}_{\rm int}^b(y)$, $\mathcal{O}_{\rm ext}^c(z)$ with pseudoscalar boundary operators from above, and expressing everything by SF correlation functions, yields normalization condition to fix Z_V:

$$Z_V\,(1 + b_V\, am_q)\, f_V(x_0) = F_1 + O(a^2) \qquad \text{e.g.: for } x_0 = T/2$$
$$f_V(x_0) = \frac{a^3}{6L^6} \sum_{\mathbf{x}} i\, \epsilon^{abc} \left\langle \mathcal{O}'^a\, V_0^b(x)\, \mathcal{O}^c \right\rangle$$

  - With spatially periodic BCs, term $\propto c_V$ doesn't contribute to f_V
  - As $m_{\rm valence} = m_{\rm sea} \sim 0$, effect from term $\propto b_V$ numerically very small and disappears upon extrapolation to the chiral limit

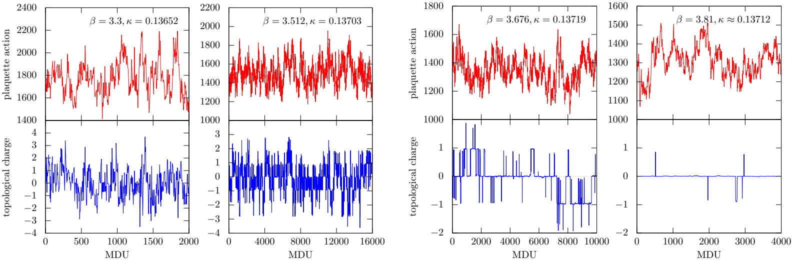
Z_V

"Topology freezing": Simulations of finest lattice ($\beta = 3.81$, $\frac{L}{a} = 24$) sample the non-trivial topological sectors insufficiently



Z_V

"Topology freezing": Simulations of finest lattice ($\beta = 3.81$, $\frac{L}{a} = 24$) sample the non-trivial topological sectors insufficiently

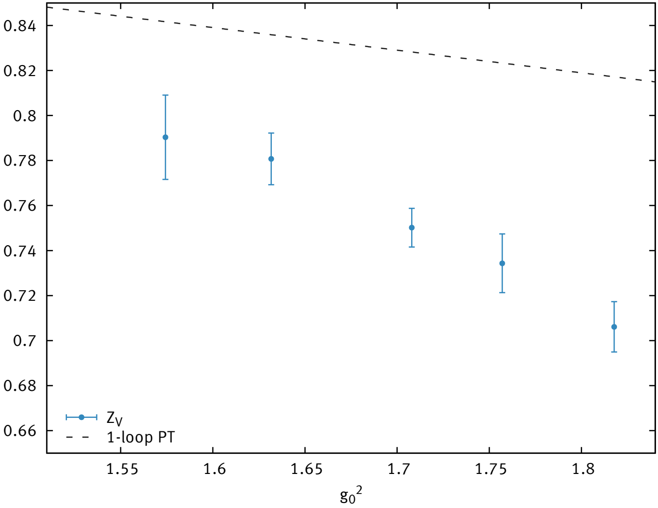


Analysis including projection to the zero-topology ($Q_{top} = 0$) sector:

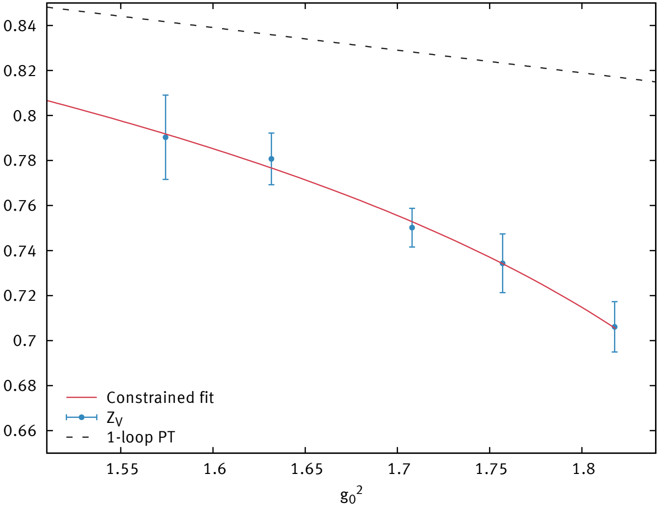
$$\langle O \rangle \Big|_{Q_{top}=0} = \frac{\langle O \cdot \delta_{Q_{top},0} \rangle}{\langle \delta_{Q_{top},0} \rangle} \qquad \delta_{Q_{top},0} \equiv \Theta(Q_{top} + 0.5)\,\Theta(0.5 - Q_{top})$$

(Note: Ward identities should hold in any topological sector)

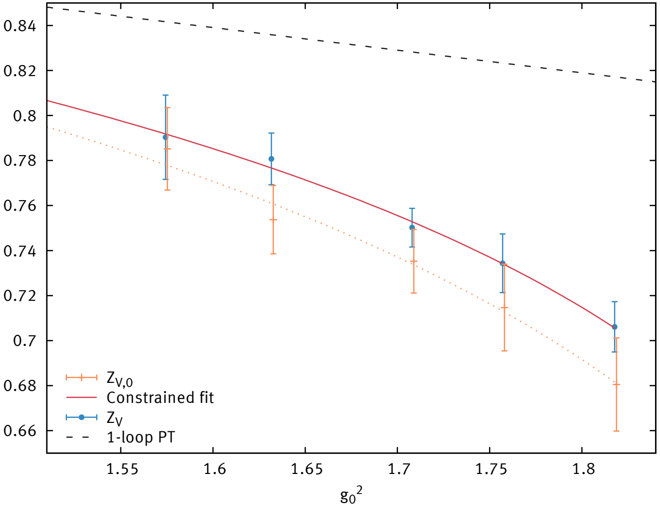
Z_V — Preliminary result



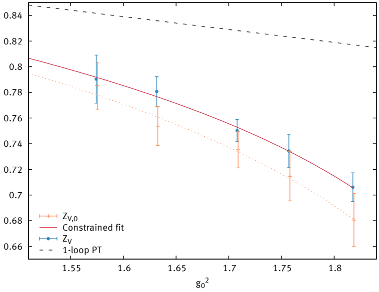
Z_V — Preliminary result



Z_V — Preliminary result



Z_V — Preliminary result

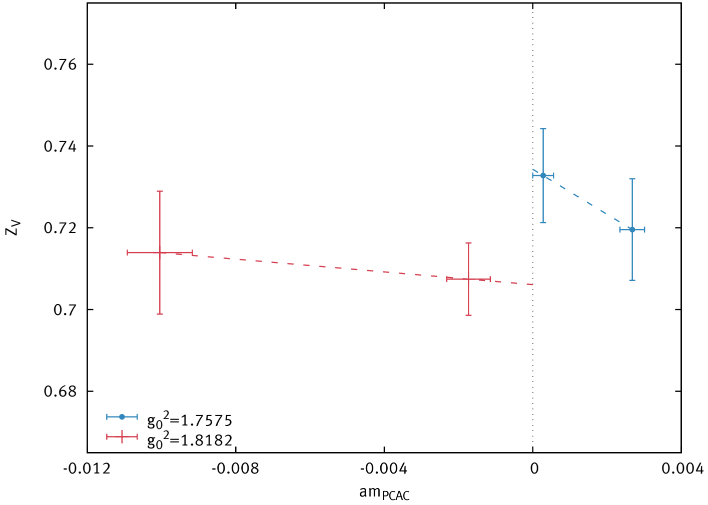


- Interpolating formula constrained to 1–loop PT:

$$Z_V(g_0^2) = 1 - 0.10057\, g_0^2 \times \begin{cases} \dfrac{1 - 0.404\, g_0^2}{1 - 0.459 g_0^2} & (\text{all } Q_{\text{top}}\text{–sectors}) \\ \dfrac{1 - 0.375\, g_0^2}{1 - 0.450 g_0^2} & (\text{projection to } Q_{\text{top}} = 0) \end{cases}$$

- Part of statistics and a few ensembles not yet (fully) analyzed
  $\rightarrow$ chiral extrapolation still to be finalized, even though it is mild ...

Z_V — Examples of chiral extrapolations



c_V

Determination of c_V for $N_f = 3$ in the CLS β–range

c_V

Determination of c_V for $N_f = 3$ in the CLS β–range

- The improvement condition adopted for c_V originates from the (massive) axial Ward identity

$$\int_{\partial R} d\sigma_\mu(x) \left\langle A^a_\mu(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle$$
$$= - \left\langle \left[\delta^a_A \mathcal{O}^b_{\rm int}(y) \right] \mathcal{O}^c_{\rm ext}(z) \right\rangle$$

c_V

Determination of c_V for $N_f = 3$ in the CLS β–range

- The improvement condition adopted for c_V originates from the (massive) axial Ward identity

$$\int_{\partial R} d\sigma_\mu(x) \left\langle A^a_\mu(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle = - \left\langle \left[\, \delta^a_A \mathcal{O}^b_{\rm int}(y)\, \right] \mathcal{O}^c_{\rm ext}(z) \right\rangle$$

- For $\mathcal{O}_{\rm int}$ we insert another axial vector current and use its variation to arrive at

$$\int_{\partial R} d\sigma_\mu(x) \left\langle A^a_\mu(x)\, A^b_\nu(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x)\, A^b_\nu(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle = i f^{abd} \left\langle V^d_\nu(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle$$

c_V

Determination of c_V for $N_f = 3$ in the CLS β–range

- The improvement condition adopted for c_V originates from the (massive) axial Ward identity

$$\int_{\partial R} d\sigma_\mu(x) \left\langle A^a_\mu(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x)\, \mathcal{O}^b_{\rm int}(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle = - \left\langle \left[\delta^a_A \mathcal{O}^b_{\rm int}(y) \right] \mathcal{O}^c_{\rm ext}(z) \right\rangle$$

- For $\mathcal{O}_{\rm int}$ we insert another axial vector current and use its variation to arrive at

$$\int_{\partial R} d\sigma_\mu(x) \left\langle A^a_\mu(x)\, A^b_\nu(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x)\, A^b_\nu(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle = i f^{abd} \left\langle V^d_\nu(y)\, \mathcal{O}^c_{\rm ext}(z) \right\rangle$$

- Specify R as the space-time region $t_1 < y_0 < t_2$, $\nu = k$ and an external vector source term as $\mathcal{Q}^c_k = a^6 \sum_{\mathbf{u},\mathbf{v}} \bar\zeta(\mathbf{u})\, \gamma_k \frac{\tau^c}{2}\, \zeta(\mathbf{v})$

c_V

Translation to the SF scheme:

Implement this improvement condition in terms of SF correlators

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

c_V

Translation to the SF scheme:

Implement this improvement condition in terms of SF correlators

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- 1st term → diff. of $k_{A_0A_k}(x_0, y_0) \propto \sum_{\mathbf{x}} \epsilon^{abc} \langle A_0^a(x) A_k^b(y) \mathcal{Q}_k^c \rangle$ at $x_0 = t_2, t_1$, the sum is called $\tilde{k}_{PA_k}(t_1, t_2, y_0)$ and the r.h.s. $k_V^I(y_0)$

living.knowledge WWU Münster

c_V

Translation to the SF scheme:

Implement this improvement condition in terms of SF correlators

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- 1st term $\rightarrow$ diff. of $k_{A_0A_k}(x_0, y_0) \propto \sum_{\mathbf{x}} \epsilon^{abc} \langle A_0^a(x) A_k^b(y) \mathcal{Q}_k^c \rangle$ at $x_0 = t_2, t_1$, the sum is called $\tilde{k}_{PA_k}(t_1, t_2, y_0)$ and the r.h.s. $k_V^I(y_0)$
- Improvement & Renormalization $\Rightarrow$ Expression to solve for c_V:

$$Z_A^2 \left[k_{A_0A_k}(t_2, y_0) - k_{A_0A_k}(t_1, y_0) \right] - 2m\, Z_A^2\, \tilde{k}_{PA_k}(t_1, t_2, y_0)$$
$$= Z_V \left[k_V(y_0) + a\, c_V\, \tilde{\partial}_0\, k_T(y_0) \right] + O(a^2)$$

living.knowledge WWU Münster

c_V

Translation to the SF scheme:

Implement this improvement condition in terms of SF correlators

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- 1st term $\rightarrow$ diff. of $k_{A_0 A_k}(x_0, y_0) \propto \sum_{\mathbf{x}} \epsilon^{abc} \langle A_0^a(x) A_k^b(y) \mathcal{Q}_k^c \rangle$ at $x_0 = t_2, t_1$, the sum is called $\tilde{k}_{PA_k}(t_1, t_2, y_0)$ and the r.h.s. $k_V^I(y_0)$
- Improvement & Renormalization $\Rightarrow$ Expression to solve for c_V:

$$Z_A^2 \left[k_{A_0 A_k}(t_2, y_0) - k_{A_0 A_k}(t_1, y_0) \right] - 2m\, Z_A^2\, \tilde{k}_{PA_k}(t_1, t_2, y_0)$$
$$= Z_V \left[k_V(y_0) + a\, c_V\, \tilde{\partial}_0\, k_T(y_0) \right] + \mathrm{O}(a^2)$$

- Use of the *massive* axial Ward identity gives rise to the mass term above and is expected to be important, because the $\mathrm{O}(m)$ error may dominate over remaining $\mathrm{O}(a^2)$ and $\mathrm{O}(am)$ ones

c_V

Translation to the SF scheme:

Implement this improvement condition in terms of SF correlators

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- 1st term → diff. of $k_{A_0 A_k}(x_0, y_0) \propto \sum_{\mathbf{x}} \epsilon^{abc} \langle A_0^a(x) A_k^b(y) \mathcal{Q}_k^c \rangle$ at $x_0 = t_2, t_1$, the sum is called $\tilde{k}_{PA_k}(t_1, t_2, y_0)$ and the r.h.s. $k_V^I(y_0)$
- Improvement & Renormalization ⇒ Expression to solve for c_V:

$$
\begin{aligned}
& Z_A^2 \left[k_{A_0 A_k}(t_2, y_0) - k_{A_0 A_k}(t_1, y_0) \right] - 2m\, Z_A^2\, \tilde{k}_{PA_k}(t_1, t_2, y_0) \\
&= Z_V \left[k_V(y_0) + a\, c_V\, \tilde{\partial}_0\, k_T(y_0) \right] + O(a^2)
\end{aligned}
$$

- Use of the *massive* axial Ward identity gives rise to the mass term above and is expected to be important, because the $O(m)$ error may dominate over remaining $O(a^2)$ and $O(am)$ ones
- Evaluated at insertion times $t_1 = T/4$ and $t_2 = 3T/4$

c_V

Translation to the SF scheme:

Implement this improvement condition in terms of SF correlators

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- 1st term → diff. of $k_{A_0A_k}(x_0, y_0) \propto \sum_{\mathbf{x}} \epsilon^{abc} \langle A_0^a(x) A_k^b(y) \mathcal{Q}_k^c \rangle$ at $x_0 = t_2, t_1$, the sum is called $\tilde{k}_{PA_k}(t_1, t_2, y_0)$ and the r.h.s. $k_V^I(y_0)$
- Improvement & Renormalization ⇒ Expression to solve for c_V:

$$Z_A^2 \left[k_{A_0A_k}(t_2, y_0) - k_{A_0A_k}(t_1, y_0) \right] - 2m\, Z_A^2\, \tilde{k}_{PA_k}(t_1, t_2, y_0)$$
$$= Z_V \left[k_V(y_0) + a\, c_V\, \tilde{\partial}_0\, k_T(y_0) \right] + O(a^2)$$

- Use of the *massive* axial Ward identity gives rise to the mass term above and is expected to be important, because the $O(m)$ error may dominate over remaining $O(a^2)$ and $O(am)$ ones
- Evaluated at insertion times $t_1 = T/4$ and $t_2 = 3T/4$
- Prior knowledge of Z_A and Z_V is required with this method

c_V

Variants/Refinements w.r.t. the early $N_f = 0$ study

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

c_V

Variants/Refinements w.r.t. the early $N_f = 0$ study

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- Line of constant physics

c_V

Variants/Refinements w.r.t. the early $N_f = 0$ study

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- Line of constant physics
- SF correlation functions incorporate optimized boundary wave-functions (once calculated in the context of the determination of c_A) to suppress excited state effects and thus to maximize the overlap with the ground state

c_V

Variants/Refinements w.r.t. the early $N_f = 0$ study

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

- Line of constant physics
- SF correlation functions incorporate optimized boundary wave-functions (once calculated in the context of the determination of c_A) to suppress excited state effects and thus to maximize the overlap with the ground state
- Accounting for the mass term induced by the massive AWI:

$$Z_A^2 \left[k_{A_0 A_k}(t_2, y_0) - k_{A_0 A_k}(t_1, y_0) \right] - 2m\, Z_A^2\, \tilde{k}_{PA_k}(t_1, t_2, y_0)$$
$$= Z_V \left[k_V(y_0) + a\, c_V\, \tilde{\partial}_0\, k_T(y_0) \right] + O(a^2)$$

c_V

Variants/Refinements w.r.t. the early $N_f = 0$ study

[Guagnelli & Sommer, NPB PS 63 (1998) 886, hep-lat/9709088]

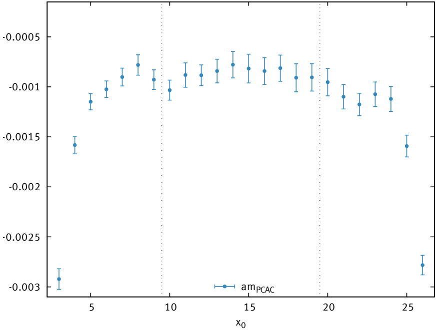
- Line of constant physics
- SF correlation functions incorporate optimized boundary wave-functions (once calculated in the context of the determination of c_A) to suppress excited state effects and thus to maximize the overlap with the ground state
- Accounting for the mass term induced by the massive AWI:

$$Z_A^2 \left[k_{A_0 A_k}(t_2, y_0) - k_{A_0 A_k}(t_1, y_0) \right] - 2m\, Z_A^2\, \tilde{k}_{PA_k}(t_1, t_2, y_0)$$
$$= Z_V \left[k_V(y_0) + a\, c_V\, \tilde{\partial}_0\, k_T(y_0) \right] + O(a^2)$$

- Another set of insertion times $t_1 = T/3$ and $t_2 = 2T/3$

  (where inclusion of the mass term turns out to be beneficial again, as it drastically reduces the systematics from different choices on t_1, t_2)

PCAC current quark mass for $\beta = 3.676, L/a = 20$

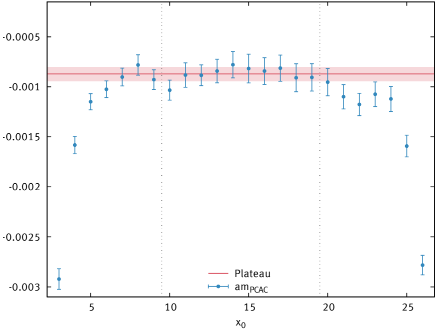


$$am_{PCAC}(x_0) = \frac{\partial_0 \left[f_A(x_0) + a\, c_A\, \partial_0 f_P(x_0) \right]}{2\, f_P(x_0)}$$

Calculated as plateau average over the central $L/2$ timeslices:

$$am \equiv am_{PCAC} = \frac{1}{T-L+1} \sum_{x_0=L/2}^{T-L/2} am_{PCAC}(x_0)$$

PCAC current quark mass for $\beta = 3.676$, $L/a = 20$

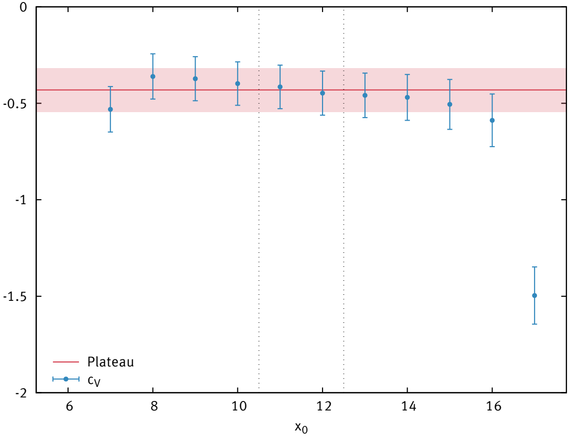


$$am_{\mathrm{PCAC}}(x_0) = \frac{\partial_0\left[f_A(x_0) + a\,c_A\,\partial_0 f_P(x_0)\right]}{2\,f_P(x_0)}$$

Calculated as plateau average over the central $L/2$ timeslices:

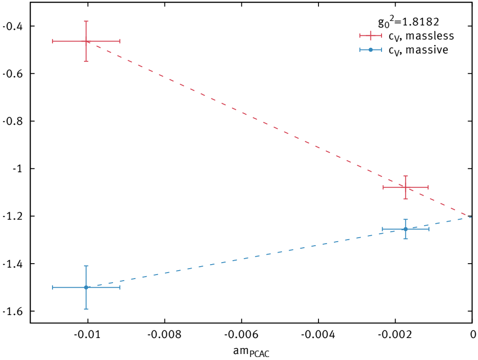
$$am \equiv am_{\mathrm{PCAC}} = \frac{1}{T-L+1}\sum_{x_0=L/2}^{T-L/2} am_{\mathrm{PCAC}}(x_0)$$

c_V — Example of plateau for $\beta = 3.512, L/a = 16$

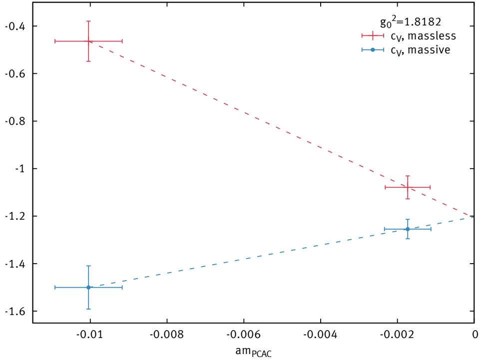


Choice of insertion times $t_1 = T/3, t_2 = 2T/3$ seems superior, as it stabilizes the plateaux, weakens the mass term's influence and leads to smaller statistical errors

c_V — Examples of chiral extrapolations

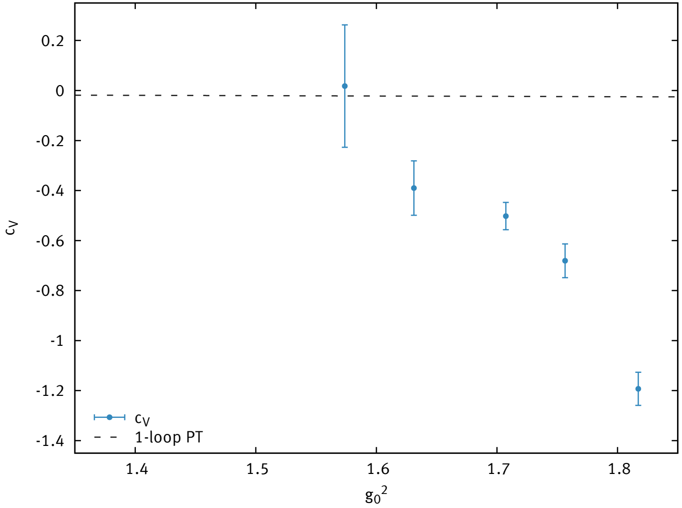


c_V — Examples of chiral extrapolations

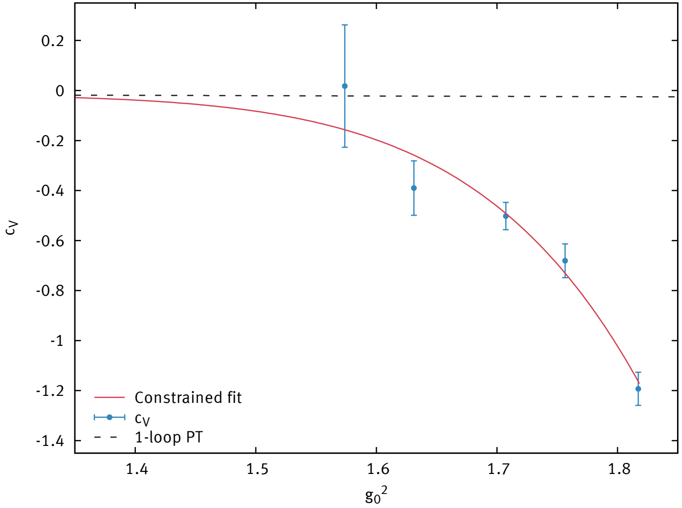


- Inclusion of the mass term (from massive AWI normalization condition) tames the quite significant quark mass dependence
- Note agreement of the unconstrained fits in the chiral limit ...

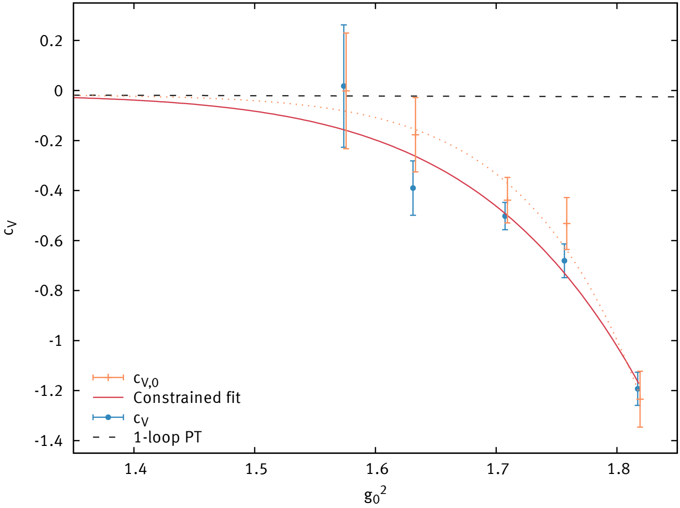
c_V — Preliminary result



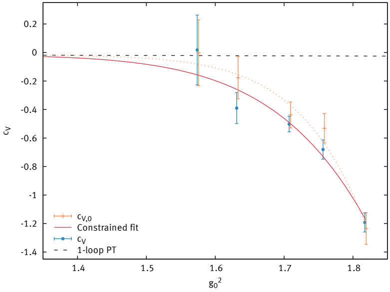
c_V — Preliminary result



c_V — Preliminary result



c_V — Preliminary result



- Interpolating formula constrained to 1–loop PT:

$$c_V(g_0^2) = -0.013733\, g_0^2 \times \begin{cases} \left[1 + \exp\left(16.66 - \frac{23.34}{g_0^2}\right)\right] & (\text{all } Q_{\rm top}) \\ \left[1 + \exp\left(22.14 - \frac{33.25}{g_0^2}\right)\right] & (Q_{\rm top} = 0) \end{cases}$$

- Part of statistics and a few ensembles not yet (fully) analyzed
(→ final chiral extrapolation may still have some impact ...)

Z_S/Z_P

Determination of Z_S/Z_P for $N_f = 3$ in the CLS β–range

Z_S/Z_P

Determination of Z_S/Z_P for $N_f = 3$ in the CLS β–range

- Exploits a renormalization condition, again derived from the massive axial Ward identity (closely following ALPHA Collaboration's method to compute Z_A for $N_f = 2, 3$), but now relying on a *pseudoscalar* insertion as internal operator, with behaviour under variation

$$\delta_A^a P^b(x) = d^{abc} S^c(x) + \frac{\delta^{ab}}{N_f} \bar{\psi}(x)\psi(x)$$

Z_S/Z_P

Determination of Z_S/Z_P for $N_f = 3$ in the CLS β–range

- Exploits a renormalization condition, again derived from the massive axial Ward identity (closely following ALPHA Collaboration's method to compute Z_A for $N_f = 2, 3$), but now relying on a *pseudoscalar* insertion as internal operator, with behaviour under variation

$$\delta_A^a P^b(x) = d^{abc} S^c(x) + \frac{\delta^{ab}}{N_f} \bar{\psi}(x)\psi(x)$$

- Work with $SU(N_f)$ algebra in the valence sector, where $N_f \geq 3$; here: $SU(3)$ and assume $a \neq b$

Z_S/Z_P

Determination of Z_S/Z_P for $N_f = 3$ in the CLS β–range

- Exploits a renormalization condition, again derived from the massive axial Ward identity (closely following ALPHA Collaboration's method to compute Z_A for $N_f = 2, 3$), but now relying on a *pseudoscalar* insertion as internal operator, with behaviour under variation

$$\delta_A^a P^b(x) = d^{abc} S^c(x) + \frac{\delta^{ab}}{N_f} \bar{\psi}(x)\psi(x)$$

- Work with $SU(N_f)$ algebra in the valence sector, where $N_f \geq 3$; here: $SU(3)$ and assume $a \neq b$
- Use product of 2 PS boundary sources as external operator in

$$\int_{\partial R} d\sigma_\mu(x) \left\langle A_\mu^a(x) P^b(y) \mathcal{O}_{\text{ext}}^c(z) \right\rangle - 2m \int_R d^4x \left\langle P^a(x) P^b(y) \mathcal{O}_{\text{ext}}^c(z) \right\rangle$$

$$= -d^{abc} \langle S^c(y) \mathcal{O}_{\text{ext}}^c(z) \rangle \qquad \mathcal{O}_{\text{ext}}^{ba} = \frac{1}{(N_f^2 - 1) L^6} \mathcal{O}'^b \mathcal{O}^a$$

Z_S/Z_P

Identifying each piece with an SF correlator finally gives ...

Z_S/Z_P

Identifying each piece with an SF correlator finally gives ...

- ... a formula that can be solved for Z_P/Z_S:

$$Z_A Z_P \left[(f_{AP}^{ba})^{\text{I}}(t_2, y_0) - (f_{AP}^{ba})^{\text{I}}(t_1, y_0) - 2m\, \tilde{f}_{PP}^{ba}(t_2, t_1, y_0) \right] = -Z_S\, f_S^{ba}(y_0)$$

Z_S/Z_P

Identifying each piece with an SF correlator finally gives ...

- ... a formula that can be solved for Z_P/Z_S:

$$Z_A Z_P \left[(f_{AP}^{ba})^{\prime}(t_2, y_0) - (f_{AP}^{ba})^{\prime}(t_1, y_0) - 2m\, \tilde{f}_{PP}^{ba}(t_2, t_1, y_0) \right] = -Z_S\, f_S^{ba}(y_0)$$

- Z_A must be known; insertions at $y_0 = T/2$, $t_1 = T/3$, $t_2 = 2T/3$

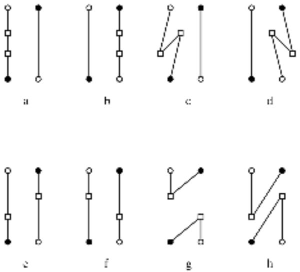
Z_S/Z_P

Identifying each piece with an SF correlator finally gives ...

- ... a formula that can be solved for Z_P/Z_S:

$$Z_A\, Z_P \left[\left(f_{AP}^{ba}\right)^{I}(t_2, y_0) \; - \; \left(f_{AP}^{ba}\right)^{I}(t_1, y_0) \; - \; 2m\, \tilde{f}_{PP}^{ba}(t_2, t_1, y_0) \right] \; = \; -Z_S\, f_S^{ba}(y_0)$$

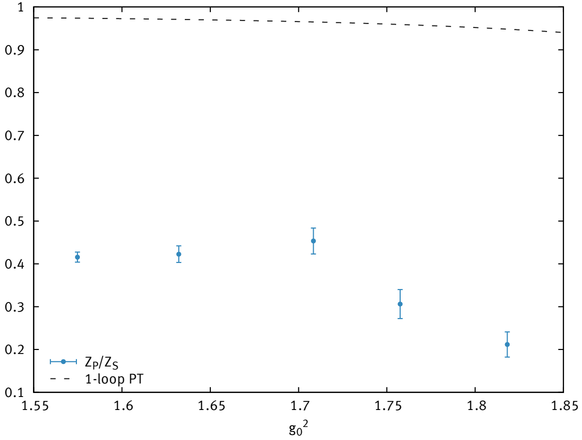
- Z_A must be known; insertions at $y_0 = T/2$, $t_1 = T/3$, $t_2 = 2T/3$
- Graphical representation of the possible Wick contractions for correlation functions of generic form $f_{\Gamma\tilde{\Gamma}}(x_0, y_0)$:



$Z_{\mathrm{S}}/Z_{\mathrm{P}}$ — Example of contributing Wick contractions

$$
\begin{aligned}
F^{abcd}_{\Gamma\tilde{\Gamma}}(x) &= -\langle \mathcal{O}'^{a}\,\bar{\psi}(x)\Gamma T^{b}\psi(x)\,\bar{\psi}(x)\tilde{\Gamma}T^{c}\psi(x)\,\mathcal{O}^{d}\rangle \\
= &-a^{12}\,\mathrm{Tr}\left(T^{a}T^{b}T^{c}T^{d}\right)\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\psi}(x)]\Gamma[\psi(x)\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\zeta}(\mathbf{u})]\gamma_5[\zeta(\mathbf{v})\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\right\rangle \\
&-a^{12}\,\mathrm{Tr}\left(T^{a}T^{b}T^{d}T^{c}\right)\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\psi}(x)]\Gamma[\psi(x)\bar{\zeta}(\mathbf{u})]\gamma_5[\zeta(\mathbf{v})\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\right\rangle \\
&-a^{12}\,\mathrm{Tr}\left(T^{a}T^{c}T^{d}T^{b}\right)\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\zeta}(\mathbf{u})]\gamma_5[\zeta(\mathbf{v})\bar{\psi}(x)]\Gamma[\psi(x)\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\right\rangle \\
&-a^{12}\,\mathrm{Tr}\left(T^{a}T^{c}T^{b}T^{d}\right)\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\psi}(x)]\Gamma[\psi(x)\bar{\zeta}(\mathbf{u})]\gamma_5[\zeta(\mathbf{v})\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\right\rangle \\
&-a^{12}\,\mathrm{Tr}\left(T^{a}T^{d}T^{c}T^{b}\right)\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\zeta}(\mathbf{u})]\gamma_5[\zeta(\mathbf{v})\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\psi}(x)]\Gamma[\psi(x)\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\right\rangle \\
&-a^{12}\,\mathrm{Tr}\left(T^{a}T^{d}T^{b}T^{c}\right)\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\zeta}(\mathbf{u})]\gamma_5[\zeta(\mathbf{v})\bar{\psi}(x)]\Gamma[\psi(x)\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\right\rangle \\
&+a^{12}\mathrm{Tr}(T^{a}T^{b})\mathrm{Tr}(T^{d}T^{c}) \\
&\quad\times\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\psi}(x)]\Gamma[\psi(x)\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\,\mathrm{tr}\left\{[\psi(y)\zeta(\bar{\mathbf{u}})]\gamma_5[\zeta(\mathbf{v})\bar{\psi}(y)]\tilde{\Gamma}\right\}\right\rangle \\
&+a^{12}\mathrm{Tr}(T^{a}T^{c})\mathrm{Tr}(T^{d}T^{b}) \\
&\quad\times\sum_{\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'}\left\langle \mathrm{tr}\left\{[\zeta'(\mathbf{v}')\bar{\psi}(y)]\tilde{\Gamma}[\psi(y)\bar{\zeta}'(\mathbf{u}')]\gamma_5\right\}\,\mathrm{tr}\left\{[\psi(x)\zeta(\bar{\mathbf{u}})]\gamma_5[\zeta(\mathbf{v})\bar{\psi}(x)]\Gamma\right\}\right\rangle
\end{aligned}
$$

Z_P/Z_S — Numerical analysis in progress ...



..., cross-checks under way and, hence, numbers indicative only !

Summary & Outlook

✓ Preliminary results for c_V and Z_V in $N_f = 3$ QCD, based on imposing improvement and renormalization conditions (derived from chiral Ward identities) at constant physics

$\rightarrow$ 2-do's:

- Finalize the analysis
- Investigate possibly large autocorreation times in individual pieces contributing to the c_V–estimator at $L/a = 24, \beta = 3.81$
- Estimate systematic effects, such as the uncertainty due to deviations from the LCP, via including a further ensemble with $L/a = 16, \beta = 3.47$

✓ Strategy for a non-perturb. computation of Z_P/Z_S along similar lines exists, evaluation of correlators and analysis in progress

- Determinations of b_m, $b_A - b_P$ and $Z \equiv \frac{Z_m Z_P}{Z_A}$ for the "CLS" β–range, employing the same $N_f = 3$ SF ensembles

$\rightarrow$ next talk by Giulia Maria de Divitiis