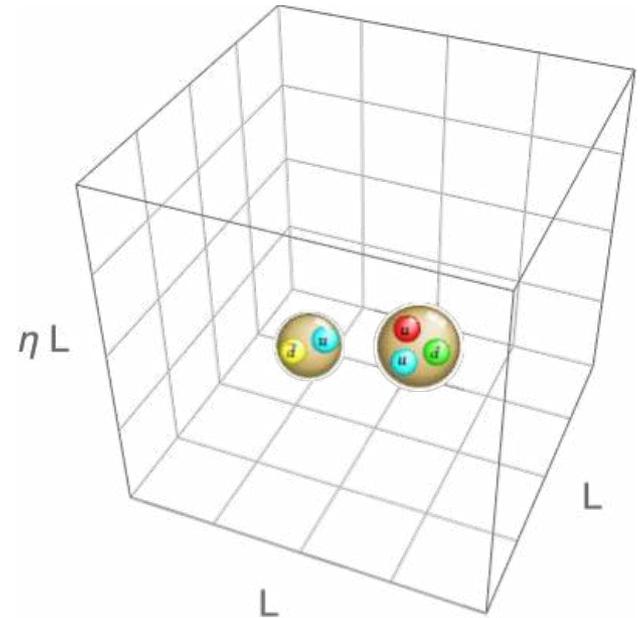


Scattering phaseshift formulas in elongated boxes

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- Physics motivation
- Cubic vs. Elongated
- Boosting
- Conclusion



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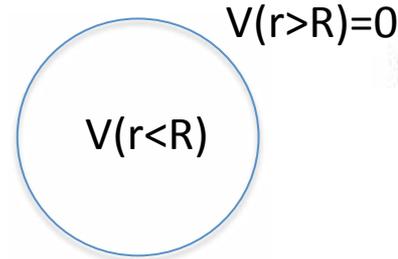
Thanks to U.S. Department of Energy and National Science Foundation

Elastic scattering phaseshift

$$\psi(\vec{r}) = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}$$

Partial-wave expansion of scattering amplitude:

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta)$$



Phaseshift, S-matrix, and T-matrix

$$f_l = \frac{S_l - 1}{2ik} = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l} \sin \delta_l}{k} = \frac{1}{k(\cot \delta_l - i)} = \frac{T_l}{k}$$

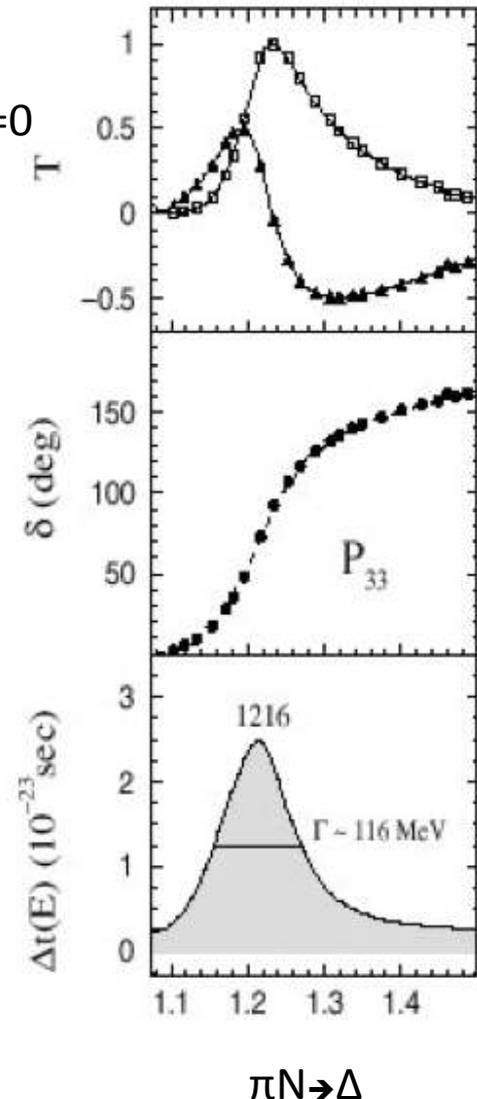
$$\text{Cross section: } \sigma = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

By matching up the interior wavefunction with the exterior one:

$$\psi \propto \alpha_l j_l(kr) + \beta_l n_l(kr)$$

The phaseshift can be determined by $e^{2i\delta_l(k)} = \frac{\alpha_l(k) + i\beta_l(k)}{\alpha_l(k) - i\beta_l(k)}$

Experimentally



Physics significance:

Attractive interaction ($\delta < 0$); repulsive interaction ($\delta > 0$);

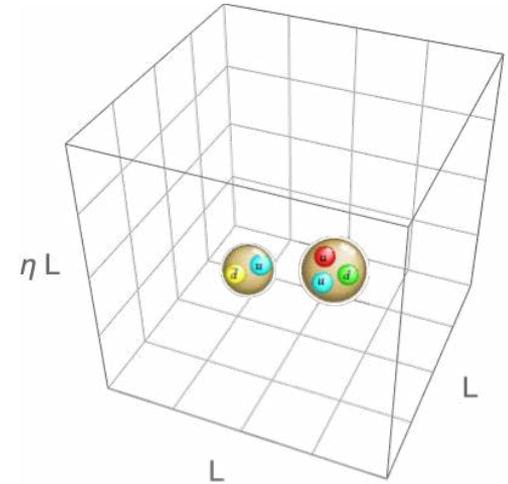
formation of resonances; scattering length from effective range expansion, etc

Lüscher method

Lüscher established a relation between the discrete energy spectrum in a box of size L^3 and elastic scattering phaseshift in the infinite volume:

$$\det \left[e^{2i\delta(k,L)} - \frac{\mathcal{M}(k,L) + i}{\mathcal{M}(k,L) - i} \right] = 0$$

Instead of Bessel functions, $\mathcal{M}(k,L)$ consists of **zeta functions** which are solutions to the Helmholtz equation under **periodic boundary conditions** of the box.



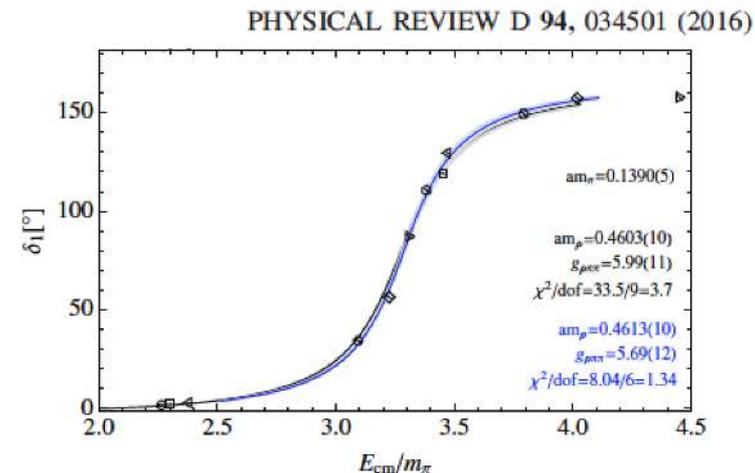
Once the interaction energy is determined, the phaseshift can be extracted from the Lüscher formula.

Two comments:

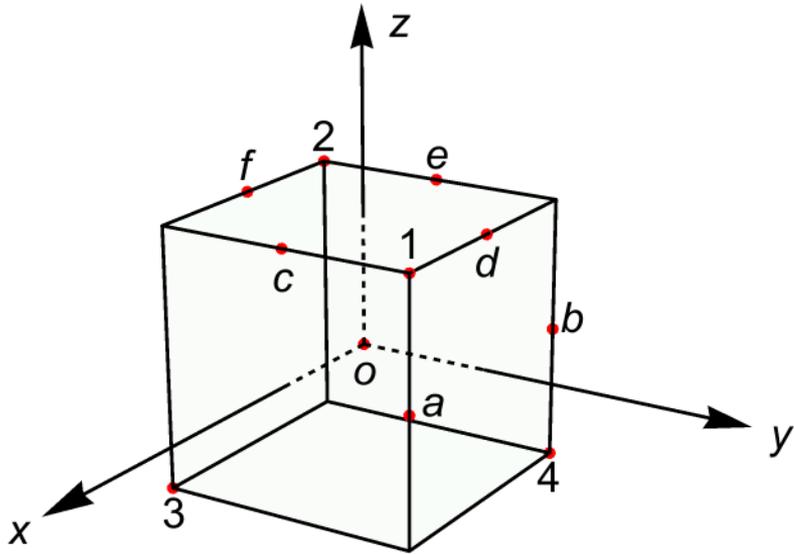
- 1) The energy can be computed in lattice QCD, thus allowing a first-principles understanding of the phaseshift in terms of quark-gluon dynamics.
- 2) Even though the energy is computed in imaginary time in a finite box in lattice QCD, the phaseshift is the same as that defined in the infinite volume and real time. Once the pion mass is brought to the physical point, the phaseshift can be directly compared with experiment.

Symmetry reduction

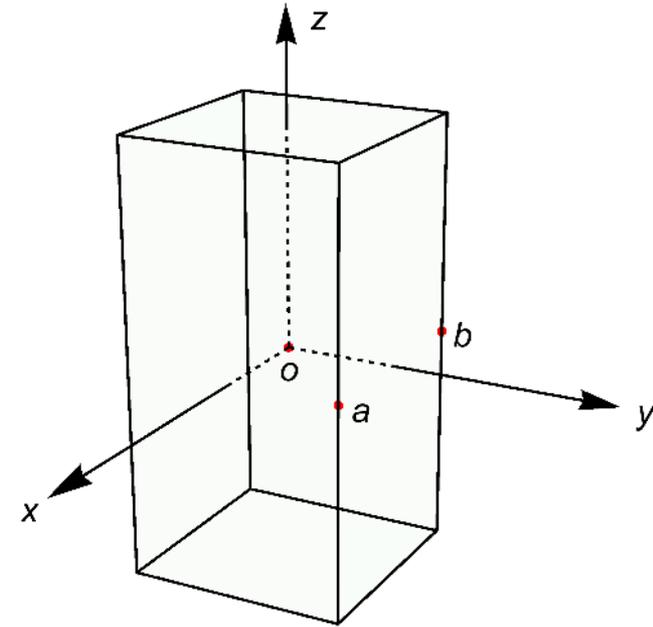
- Since physics is extracted from a finite box, we must adapt to the symmetry of the box.
- Internal symmetries such as color, flavor, isospin are not affected; but angular momentum, which is a measure of rotational symmetry, is greatly affected. Specifically,
 - How is angular momentum resolved in a finite box?
 - How are Lüscher phaseshift formulas reduced?
 - How to construct operators that transform according to the symmetry of the box?
- Our goal: perform such an reduction in the **elongated box**.
 - Elongated box provides a cost-effective way of varying the box size, thus relative momentum k .
 - Successfully used in $\pi\pi\rightarrow\rho$
 - Need formulas for $\pi N\rightarrow\Delta$



Group theory: Cubic vs. Elongated



- O group
 - 24 elements
 - 5 irreps (A_1, A_2, T_1, T_2, E)
- 2O group (half-integer)
 - 48 elements
 - 8 irreps ($A_1, A_2, T_1, T_2, E, G_1, G_2, H$) of dimensions (1,1,3,3,2,2,2,4)
- 2O_h group (parity)
 - 32 elements
 - 16 irreps ($A_1^\pm, A_2^\pm, T_1^\pm, T_2^\pm, E^\pm, G_1^\pm, G_2^\pm, H^\pm$)



- D_4 group
 - 8 elements
 - 5 irreps (A_1, A_2, B_1, B_2, E)
- 2D_4 group (half-integer)
 - 16 elements
 - 7 irreps ($A_1, A_2, B_1, B_2, E, G_1, G_2$) of dimensions (1,1,1,1,2,2,2)
- ${}^2D_{4h}$ group (parity)
 - 32 elements
 - 14 irreps ($A_1^\pm, A_2^\pm, B_1^\pm, B_2^\pm, E^\pm, G_1^\pm, G_2^\pm$)

Group symmetry in elongated box

Elements and irreps

Class\Irrep	k	n	ω	(α, β, γ)	A_1	A_2	B_1	B_2	E	G_1	G_2
I	1	{0,0,1}	4π	{0,0,0}	1	1	1	1	1	1	1
\tilde{I}	2	{0,0,1}	2π	{0,0,2 π }	1	1	1	1	1	-1	-1
C_{4z}^+	3	{0,0,1}	$\frac{\pi}{2}$	{0,0, $\frac{\pi}{2}$ }	1	1	-1	-1	$-i\sigma_2$	$\frac{1-i\sigma_2}{\sqrt{2}}$	$\frac{-i\sigma_2-1}{\sqrt{2}}$
C_{4z}^-	4	{0,0,1}	$\frac{7\pi}{2}$	{0,0, $\frac{7\pi}{2}$ }	1	1	-1	-1	$i\sigma_2$	$\frac{i\sigma_2+1}{\sqrt{2}}$	$\frac{i(\sigma_2+i)}{\sqrt{2}}$
\tilde{C}_{4z}^+	5	{0,0,1}	$\frac{5\pi}{2}$	{0,0, $\frac{5\pi}{2}$ }	1	1	-1	-1	$-i\sigma_2$	$\frac{i(\sigma_2+i)}{\sqrt{2}}$	$\frac{i\sigma_2+1}{\sqrt{2}}$
\tilde{C}_{4z}^-	6	{0,0,1}	$\frac{3\pi}{2}$	{0,0, $\frac{3\pi}{2}$ }	1	1	-1	-1	$i\sigma_2$	$\frac{-i\sigma_2-1}{\sqrt{2}}$	$\frac{1-i\sigma_2}{\sqrt{2}}$
C_{2z}	7	{0,0,1}	π	{0,0, π }	1	1	1	1	-1	$-i\sigma_2$	$i\sigma_2$
\tilde{C}_{2z}	8	{0,0,1}	3π	{0,0,3 π }	1	1	1	1	-1	$i\sigma_2$	$-i\sigma_2$
C_{2x}	9	{1,0,0}	π	{0, π , π }	1	-1	1	-1	σ_3	$i\sigma_3$	$i\sigma_3$
C_{2y}	10	{0,1,0}	π	{0, π ,0}	1	-1	1	-1	$-\sigma_3$	$i\sigma_1$	$-i\sigma_1$
\tilde{C}_{2x}	11	{1,0,0}	3π	{0, π ,3 π }	1	-1	1	-1	σ_3	$-i\sigma_3$	$-i\sigma_3$
\tilde{C}_{2y}	12	{0,1,0}	3π	{0, π ,2 π }	1	-1	1	-1	$-\sigma_3$	$-i\sigma_1$	$i\sigma_1$
C_{2a}	13	{1,1,0}	π	{0, π , $\frac{\pi}{2}$ }	1	-1	-1	1	σ_1	$\frac{i(\sigma_1+\sigma_3)}{\sqrt{2}}$	$\frac{i(\sigma_1-\sigma_3)}{\sqrt{2}}$
C_{2b}	14	{-1,1,0}	π	{0, π , $\frac{7\pi}{2}$ }	1	-1	-1	1	$-\sigma_1$	$\frac{i(\sigma_1-\sigma_3)}{\sqrt{2}}$	$\frac{i(\sigma_1+\sigma_3)}{\sqrt{2}}$
\tilde{C}_{2a}	15	{1,1,0}	3π	{0, π , $\frac{5\pi}{2}$ }	1	-1	-1	1	σ_1	$\frac{-i(\sigma_1+\sigma_3)}{\sqrt{2}}$	$\frac{-i(\sigma_1-\sigma_3)}{\sqrt{2}}$
\tilde{C}_{2b}	16	{-1,1,0}	3π	{0, π , $\frac{3\pi}{2}$ }	1	-1	-1	1	$-\sigma_1$	$\frac{-i(\sigma_1-\sigma_3)}{\sqrt{2}}$	$\frac{-i(\sigma_1+\sigma_3)}{\sqrt{2}}$

character table

2D_4	I	\tilde{I}	$2C_4$	$2\tilde{C}_4$	$C_2 + \tilde{C}_2$	$2C_2' + 2\tilde{C}_2'$	$2C_2'' + 2\tilde{C}_2''$
A_1	1	1	1	1	1	1	1
A_2	1	1	1	1	1	-1	-1
B_1	1	1	-1	-1	1	1	-1
B_2	1	1	-1	-1	1	-1	1
E	2	2	0	0	-2	0	0
G_1	2	-2	$\sqrt{2}$	$-\sqrt{2}$	0	0	0
G_2	2	-2	$-\sqrt{2}$	$\sqrt{2}$	0	0	0
ω	4π	2π	$\pi/2$	$5\pi/2$	π	π	π

Angular momentum decomposition

J	${}^2D_{4h}$	${}^2D_{4h}$	J
0	A_1^+	A_1^+	0, 2, 4(2), ...
1	$A_2^- \oplus E^-$	A_1^-	5, 7, 9(2), ...
2	$A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+$	A_2^-	1, 3, 5(2), ...
3	$A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^-$	A_2^+	4, 6, 8(2), ...
4	$2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+$	B_1^+	2, 4, 6(2), ...
5	$A_1^- \oplus 2A_2^- \oplus B_1^- \oplus B_2^- \oplus 3E^-$	B_1^-	3, 5, 7(2), ...
6	$2A_1^+ \oplus A_2^+ \oplus 2B_1^+ \oplus 2B_2^+ \oplus 3E^+$	B_2^+	2, 4, 6(2), ...
...	...	B_2^-	3, 5, 7(2), ...
...	...	E^-	1, 3(2), 5(3), ...
...	...	E^+	2, 4(2), 6(3), ...
1/2	G_1^\pm	G_1^\pm	1/2, 3/2, 5/2, ...
3/2	$G_1^\pm \oplus G_2^\pm$	G_2^\pm	3/2, 5/2(2), 7/2(2), ...
5/2	$G_1^\pm \oplus 2G_2^\pm$		
7/2	$2G_1^\pm \oplus 2G_2^\pm$		
...	...		

$$J = \bigoplus_i n(\Gamma_i, J) \Gamma_i$$

$$n(\Gamma_i, J) = \frac{1}{g} \sum_k n_k \chi(k, \Gamma_i) \chi(\omega_k, J)$$

$$\chi(\omega, J) = \frac{\sin[(J + 1/2)\omega]}{\sin(\omega/2)}$$

Group symmetry in cubic box

character table

Elements and irreps

k	Elem	n	ω	$\{\alpha, \beta, \gamma\}$	A_1	A_2	E	T_1	T_2	G_1	G_2	H
1	I	{0,0,1}	4π	{0,0,0}	1	1	$\mathbb{1}$	t_1	t_1	g_1	g_1	h_1
2	\tilde{I}	{0,0,1}	2π	{0,0,2 π }	1	1	$\mathbb{1}$	t_1	t_1	$-g_1$	$-g_1$	$-h_1$
3	C_{2x}	{1,0,0}	π	{0, π , π }	1	1	$\mathbb{1}$	t_2	t_2	g_2	g_2	h_2
4	C_{2y}	{0,1,0}	π	{0, π ,0}	1	1	$\mathbb{1}$	t_3	t_3	g_3	g_3	h_3
5	C_{2z}	{0,0,1}	π	{0,0, π }	1	1	$\mathbb{1}$	t_4	t_4	g_4	g_4	h_4
6	\tilde{C}_{2x}	{1,0,0}	3π	{0, π , 3π }	1	1	$\mathbb{1}$	t_2	t_2	$-g_2$	$-g_2$	$-h_2$
7	\tilde{C}_{2y}	{0,1,0}	3π	{0, π , 2π }	1	1	$\mathbb{1}$	t_3	t_3	$-g_3$	$-g_3$	$-h_3$
8	\tilde{C}_{2z}	{0,0,1}	3π	{0,0, 3π }	1	1	$\mathbb{1}$	t_4	t_4	$-g_4$	$-g_4$	$-h_4$
9	C_{31}^+	{1,1,1}	$\frac{2\pi}{3}$	{0, $\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_5	t_5	g_5	g_5	h_5
10	C_{32}^+	{-1,-1,1}	$\frac{2\pi}{3}$	{ $\pi,\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_6	t_6	g_6	g_6	h_6
11	C_{33}^+	{1,-1,-1}	$\frac{2\pi}{3}$	{ $\pi,\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_7	t_7	g_7	g_7	h_7
12	C_{34}^+	{-1,-1,-1}	$\frac{2\pi}{3}$	{0, $\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_8	t_8	g_8	g_8	h_8
13	C_{31}^-	{1,1,1}	$\frac{10\pi}{3}$	{ $\frac{\pi}{2},\frac{\pi}{2},3\pi$ }	1	1	e_2	t_9	t_9	g_9	g_9	h_9
14	C_{32}^-	{-1,-1,1}	$\frac{10\pi}{3}$	{ $\frac{3\pi}{2},\frac{\pi}{2},2\pi$ }	1	1	e_2	t_{10}	t_{10}	g_{10}	g_{10}	h_{10}
15	C_{33}^-	{1,-1,-1}	$\frac{10\pi}{3}$	{ $\frac{\pi}{2},\frac{\pi}{2},0$ }	1	1	e_2	t_{11}	t_{11}	g_{11}	g_{11}	h_{11}
16	C_{34}^-	{-1,1,-1}	$\frac{10\pi}{3}$	{ $\frac{3\pi}{2},\frac{\pi}{2},3\pi$ }	1	1	e_2	t_{12}	t_{12}	g_{12}	g_{12}	h_{12}
17	C_{31}^+	{1,1,1}	$\frac{8\pi}{3}$	{0, $\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_5	t_5	$-g_5$	$-g_5$	$-h_5$
18	C_{32}^+	{-1,-1,1}	$\frac{8\pi}{3}$	{ $\pi,\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_6	t_6	$-g_6$	$-g_6$	$-h_6$
19	C_{33}^+	{1,-1,-1}	$\frac{8\pi}{3}$	{ $\pi,\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_7	t_7	$-g_7$	$-g_7$	$-h_7$
20	C_{34}^+	{-1,-1,-1}	$\frac{8\pi}{3}$	{0, $\frac{\pi}{2},\frac{\pi}{2}$ }	1	1	e_3	t_8	t_8	$-g_8$	$-g_8$	$-h_8$
21	C_{31}^-	{1,1,1}	$\frac{4\pi}{3}$	{ $\frac{\pi}{2},\frac{\pi}{2},\pi$ }	1	1	e_2	t_9	t_9	$-g_9$	$-g_9$	$-h_9$
22	C_{32}^-	{-1,-1,1}	$\frac{4\pi}{3}$	{ $\frac{3\pi}{2},\frac{\pi}{2},0$ }	1	1	e_2	t_{10}	t_{10}	$-g_{10}$	$-g_{10}$	$-h_{10}$
23	C_{33}^-	{1,-1,-1}	$\frac{4\pi}{3}$	{ $\frac{\pi}{2},\frac{\pi}{2},2\pi$ }	1	1	e_2	t_{11}	t_{11}	$-g_{11}$	$-g_{11}$	$-h_{11}$
24	C_{34}^-	{-1,1,-1}	$\frac{4\pi}{3}$	{ $\frac{3\pi}{2},\frac{\pi}{2},\pi$ }	1	1	e_2	t_{12}	t_{12}	$-g_{12}$	$-g_{12}$	$-h_{12}$
25	C_{4x}^+	{1,0,0}	$\frac{\pi}{2}$	{ $\frac{3\pi}{2},\frac{\pi}{2},\frac{\pi}{2}$ }	1	-1	e_4	t_{13}	$-t_{13}$	g_{13}	$-g_{13}$	h_{13}
26	C_{4y}^+	{0,1,0}	$\frac{\pi}{2}$	{0, $\frac{\pi}{2},0$ }	1	-1	e_5	t_{14}	$-t_{14}$	g_{14}	$-g_{14}$	h_{14}
27	C_{4z}^+	{0,0,1}	$\frac{\pi}{2}$	{0,0, $\frac{\pi}{2}$ }	1	-1	e_1	t_{15}	$-t_{15}$	g_{15}	$-g_{15}$	h_{15}
28	C_{4x}^-	{1,0,0}	$\frac{7\pi}{2}$	{ $\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}$ }	1	-1	e_4	t_{16}	$-t_{16}$	g_{16}	$-g_{16}$	h_{16}
29	C_{4y}^-	{0,1,0}	$\frac{7\pi}{2}$	{ $\pi,\frac{\pi}{2},3\pi$ }	1	-1	e_5	t_{17}	$-t_{17}$	g_{17}	$-g_{17}$	h_{17}
30	C_{4z}^-	{0,0,1}	$\frac{7\pi}{2}$	{0,0, $\frac{7\pi}{2}$ }	1	-1	e_1	t_{18}	$-t_{18}$	g_{18}	$-g_{18}$	h_{18}
31	\tilde{C}_{4x}^+	{1,0,0}	$\frac{5\pi}{2}$	{ $\frac{3\pi}{2},\frac{\pi}{2},\frac{\pi}{2}$ }	1	-1	e_4	t_{13}	$-t_{13}$	$-g_{13}$	g_{13}	$-h_{13}$
32	\tilde{C}_{4y}^+	{0,1,0}	$\frac{5\pi}{2}$	{0, $\frac{\pi}{2},2\pi$ }	1	-1	e_5	t_{14}	$-t_{14}$	$-g_{14}$	g_{14}	$-h_{14}$
33	\tilde{C}_{4z}^+	{0,0,1}	$\frac{5\pi}{2}$	{0,0, $\frac{5\pi}{2}$ }	1	-1	e_1	t_{15}	$-t_{15}$	$-g_{15}$	g_{15}	$-h_{15}$
34	\tilde{C}_{4x}^-	{1,0,0}	$\frac{3\pi}{2}$	{ $\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$ }	1	-1	e_4	t_{16}	$-t_{16}$	$-g_{16}$	g_{16}	$-h_{16}$
35	\tilde{C}_{4y}^-	{0,1,0}	$\frac{3\pi}{2}$	{ $\pi,\frac{\pi}{2},\pi$ }	1	-1	e_5	t_{17}	$-t_{17}$	$-g_{17}$	g_{17}	$-h_{17}$
36	\tilde{C}_{4z}^-	{0,0,1}	$\frac{3\pi}{2}$	{0,0, $\frac{3\pi}{2}$ }	1	-1	e_1	t_{18}	$-t_{18}$	$-g_{18}$	g_{18}	$-h_{18}$
37	C_{2a}	{1,1,0}	π	{0, π , $\frac{\pi}{2}$ }	1	-1	e_1	t_{19}	$-t_{19}$	g_{19}	$-g_{19}$	h_{19}
38	C_{2b}	{-1,1,0}	π	{0, $\pi,\frac{3\pi}{2}$ }	1	-1	e_1	t_{20}	$-t_{20}$	g_{20}	$-g_{20}$	h_{20}
39	C_{2c}	{1,0,1}	π	{0, $\frac{\pi}{2},\pi$ }	1	-1	e_5	t_{21}	$-t_{21}$	g_{21}	$-g_{21}$	h_{21}
40	C_{2d}	{0,1,1}	π	{ $\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}$ }	1	-1	e_4	t_{22}	$-t_{22}$	g_{22}	$-g_{22}$	h_{22}
41	C_{2e}	{-1,0,1}	π	{ $\pi,\frac{\pi}{2},0$ }	1	-1	e_5	t_{23}	$-t_{23}$	g_{23}	$-g_{23}$	h_{23}
42	C_{2f}	{0,-1,1}	π	{ $\frac{3\pi}{2},\frac{\pi}{2},\frac{\pi}{2}$ }	1	-1	e_4	t_{24}	$-t_{24}$	g_{24}	$-g_{24}$	h_{24}
43	\tilde{C}_{2a}	{1,1,0}	3π	{0, $\pi,\frac{3\pi}{2}$ }	1	-1	e_1	t_{19}	$-t_{19}$	$-g_{19}$	g_{19}	$-h_{19}$
44	\tilde{C}_{2b}	{-1,1,0}	3π	{0, $\pi,\frac{\pi}{2}$ }	1	-1	e_1	t_{20}	$-t_{20}$	$-g_{20}$	g_{20}	$-h_{20}$
45	\tilde{C}_{2c}	{1,0,1}	3π	{0, $\frac{\pi}{2},3\pi$ }	1	-1	e_5	t_{21}	$-t_{21}$	$-g_{21}$	g_{21}	$-h_{21}$
46	\tilde{C}_{2d}	{0,1,1}	3π	{ $\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$ }	1	-1	e_4	t_{22}	$-t_{22}$	$-g_{22}$	g_{22}	$-h_{22}$
47	\tilde{C}_{2e}	{-1,0,1}	3π	{ $\pi,\frac{\pi}{2},2\pi$ }	1	-1	e_5	t_{23}	$-t_{23}$	$-g_{23}$	g_{23}	$-h_{23}$
48	\tilde{C}_{2f}	{0,-1,1}	3π	{ $\frac{3\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$ }	1	-1	e_4	t_{24}	$-t_{24}$	$-g_{24}$	g_{24}	$-h_{24}$

2O	I	\tilde{I}	$3C_2 + 3\tilde{C}_2$	$8C_3$	$8\tilde{C}_3$	$6C_4$	$6\tilde{C}_4$	$6C_2' + 6\tilde{C}_2'$
A_1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	1	-1	-1	-1
E	2	2	2	-1	-1	0	0	0
T_1	3	3	-1	0	0	1	1	-1
T_2	3	3	-1	0	0	-1	-1	1
G_1	2	-2	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$	0
G_2	2	-2	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$	0
H	4	-4	0	-1	1	0	0	0
ω	4π	2π	π	$2\pi/3$	$4\pi/3$	$\pi/2$	$3\pi/2$	π

Angular momentum decomposition

J	2O_h	2O_h	J
0	A_1^+	A_1^+	0, 4, 6, ...
1	T_1^-	A_1^-	9, 13, 15, ...
2	$T_2^+ \oplus E^+$	T_1^-	1, 3, 5(2), ...
3	$A_2^- \oplus T_1^- \oplus T_2^-$	T_1^+	4, 6, 8(2), ...
4	$A_1^+ \oplus E^+ \oplus T_1^+ \oplus T_2^+$	T_2^+	2, 4, 6(2), ...
5	$E^- \oplus 2T_1^- \oplus T_2^-$	T_2^-	3, 5, 7(2), ...
6	$A_1^+ \oplus A_2^+ \oplus E^+ \oplus T_1^+ \oplus 2T_2^+$	E^+	2, 4, 6, ...
		E^-	5, 7, 9, ...
		A_2^-	3, 7, 9, ...
		A_2^+	6, 10, 12, ...
1/2	G_1^\pm	G_1^\pm	1/2, 7/2, 9/2, ...
3/2	H^\pm	H^\pm	3/2, 5/2, 7/2, ...
5/2	$G_2^\pm \oplus H^\pm$	G_2^\pm	5/2, 7/2, 11/2, ...
7/2	$G_1^\pm \oplus G_2^\pm \oplus H^\pm$		
...	...		

See arXiv:1706.00262

Phaseshift reduction

$$\det \left[e^{2i\delta(k,L)} - \frac{\mathcal{M}(k,L) + i}{\mathcal{M}(k,L) - i} \right] = 0$$

Start with Lüscher's formula for two equal-mass, spinless mesons in elongated box (Feng et al):

$$\mathcal{M}_{lm,l'm'}(q,\eta) = \sum_{j=|l-l'|}^{l+l'} \sum_{s=-j}^j \frac{(-1)^l i^{l+l'}}{\pi^{3/2} \eta q^{j+1}} Z_{js}(1, q^2, \eta) \times \langle l0j0 | l'0 \rangle \langle lmjs | l'm' \rangle \sqrt{\frac{(2l+1)(2j+1)}{(2l'+1)}}$$

It can be projected to spin-1/2 baryon with total angular momentum J and partial wave l:

$$\mathcal{M}_{JlM, J'l'M'} = \sum_{mm'm_s m'_s} \langle lm, \frac{1}{2}m_s | JM \rangle \langle l'm', \frac{1}{2}m'_s | J'M' \rangle \mathcal{M}_{lm,l'm'}$$

The phaseshift is determined from the quantization condition:

$$\det [\mathcal{M}_{JlM, J'l'M'} - \delta_{JJ'} \delta_{ll'} \delta_{MM'} \cot \delta_{Jl}] = 0$$

$$\mathcal{Z}_{lm}(s, q^2; \eta) = \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\mathcal{Y}_{lm}(\tilde{\mathbf{n}})}{(\tilde{\mathbf{n}}^2 - q^2)^s} \quad \tilde{\mathbf{n}} = (n_x, n_y, n_z/\eta)$$

$$\mathcal{W}_{lm}(1, q^2; \eta) = \frac{\mathcal{Z}_{lm}(1, q^2; \eta)}{\pi^{3/2} \eta q^{l+1}}$$

They still must be reduced according to the symmetry of the box.

Phaseshift reduction continued

In infinite volume, we can rotate the spherical harmonics $|JM\rangle$ in any way we like.

In a box, only certain discrete rotations are permitted, generating unique linear combinations of $|JM\rangle$ in each irrep, called **basis vectors**:

$$P_\alpha^\Gamma = \sum_{J^k} (R_k^\Gamma)_{\alpha\alpha}^* O_k$$

$$O_k |JM\rangle = \sum_{M'=-J}^J D_{MM'}^J(\alpha_k, \beta_k, \gamma_k) |JM'\rangle$$

$$|\Gamma\alpha Jln\rangle = \sum_M C_{JlM}^{\Gamma\alpha n} |JlM\rangle$$

Another change of basis according to these combinations

$$\langle \Gamma\alpha Jln | \mathcal{M} | \Gamma'\alpha' J'l'n' \rangle = \sum_{MM'} (C_{JlM}^{\Gamma\alpha n})^* C_{J'l'M'}^{\Gamma'\alpha'n'} \mathcal{M}_{JlM, J'l'M'} = \delta_{\Gamma\Gamma'} \delta_{\alpha\alpha'} \mathcal{M}_{Jln, J'l'n'}^\Gamma$$

leads to block-diagonal form by irreps:

$$\prod_{\Gamma} \det [\mathcal{M}_{Jln, J'l'n'}^\Gamma - \delta_{JJ'} \delta_{ll'} \delta_{nn'} \cot \delta_{Jl}] = 0$$

2D_4 group basis vectors

Γ	J	Basis in terms of $ JM\rangle$
A_1	0	$ 0, 0\rangle$
	2	$ 2, 0\rangle$
	4	$ 4, 0\rangle; \frac{1}{\sqrt{2}}(4, 4\rangle + 4, -4\rangle)$
A_2	1	$ 1, 0\rangle$
	3	$ 3, 0\rangle$
	4	$\frac{1}{\sqrt{2}}(4, 4\rangle - 4, -4\rangle)$
B_1	2	$\frac{1}{\sqrt{2}}(2, 2\rangle + 2, -2\rangle)$
	3	$\frac{1}{\sqrt{2}}(3, 2\rangle - 3, -2\rangle)$
	4	$\frac{1}{\sqrt{2}}(4, 2\rangle + 4, -2\rangle)$
B_2	2	$\frac{1}{\sqrt{2}}(2, 2\rangle - 2, -2\rangle)$
	3	$\frac{1}{\sqrt{2}}(3, 2\rangle + 3, -2\rangle)$
	4	$\frac{1}{\sqrt{2}}(4, 2\rangle - 4, -2\rangle)$
E	1	$\frac{1}{\sqrt{2}}(1, 1\rangle \mp 1, -1\rangle)$
	2	$\frac{1}{\sqrt{2}}(2, 1\rangle \pm 2, -1\rangle)$
	3	$\frac{1}{\sqrt{2}}(3, 1\rangle \mp 3, -1\rangle); \frac{1}{\sqrt{2}}(\mp 3, 3\rangle + 3, -3\rangle)$
	4	$\frac{1}{\sqrt{2}}(4, 1\rangle \pm 4, -1\rangle); \frac{1}{\sqrt{2}}(\pm 4, 3\rangle + 4, -3\rangle)$
G_1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\frac{1}{2}, \frac{1}{2}\rangle \mp \frac{1}{2}, -\frac{1}{2}\rangle)$
	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}(\frac{3}{2}, \frac{1}{2}\rangle \pm \frac{3}{2}, -\frac{1}{2}\rangle)$
	$\frac{5}{2}$	$\frac{1}{\sqrt{2}}(\frac{5}{2}, \frac{1}{2}\rangle \mp \frac{5}{2}, -\frac{1}{2}\rangle)$
	$\frac{7}{2}$	$\frac{1}{\sqrt{2}}(\frac{7}{2}, \frac{1}{2}\rangle \pm \frac{7}{2}, -\frac{1}{2}\rangle); \frac{1}{\sqrt{2}}(\pm \frac{7}{2}, \frac{7}{2}\rangle + \frac{7}{2}, -\frac{7}{2}\rangle)$
G_2	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}(\frac{3}{2}, \frac{3}{2}\rangle \pm \frac{3}{2}, -\frac{3}{2}\rangle)$
	$\frac{5}{2}$	$\frac{1}{\sqrt{2}}(\frac{5}{2}, \frac{3}{2}\rangle \mp \frac{5}{2}, -\frac{3}{2}\rangle); \frac{1}{\sqrt{2}}(\mp \frac{5}{2}, \frac{5}{2}\rangle + \frac{5}{2}, -\frac{5}{2}\rangle)$
	$\frac{7}{2}$	$\frac{1}{\sqrt{2}}(\frac{7}{2}, \frac{3}{2}\rangle \pm \frac{7}{2}, -\frac{3}{2}\rangle); \frac{1}{\sqrt{2}}(\pm \frac{7}{2}, \frac{5}{2}\rangle + \frac{7}{2}, -\frac{5}{2}\rangle)$

Elongated box and integer J

Key feature of box quantization:
mixing of different partial waves in
each irrep in the finite box.

$$A_2^- \text{ sector : } \cot \delta_1 = \mathcal{W}_{00} + \frac{2}{\sqrt{5}} \mathcal{W}_{20}$$

$$E^- \text{ sector : } \cot \delta_1 = \mathcal{W}_{00} - \frac{1}{\sqrt{5}} \mathcal{W}_{20}$$

if mixing with higher partial waves can be ignored.

Γ	J	n	J'	n'	$\mathcal{M}_{Jn,J'n'}^\Gamma$
A_1^+	0	1	0	1	\mathcal{W}_{00}
	0	1	2	1	$-\mathcal{W}_{20}$
	0	1	4	1	\mathcal{W}_{40}
	0	1	4	2	$\sqrt{2}\mathcal{W}_{44}$
	2	1	2	1	$\mathcal{W}_{00} + \frac{2\sqrt{5}}{7}\mathcal{W}_{20} + \frac{6}{7}\mathcal{W}_{40}$
	2	1	4	1	$-\frac{6}{7}\mathcal{W}_{20} - \frac{20\sqrt{5}}{77}\mathcal{W}_{40} - \frac{15}{11}\sqrt{\frac{5}{13}}\mathcal{W}_{60}$
	2	1	4	2	$\frac{4\sqrt{10}}{11}\mathcal{W}_{44} - \frac{15}{11}\sqrt{\frac{2}{13}}\mathcal{W}_{64}$
	4	1	4	1	$\mathcal{W}_{00} + \frac{20\sqrt{5}}{77}\mathcal{W}_{20} + \frac{486}{1001}\mathcal{W}_{40} + \frac{20}{11\sqrt{13}}\mathcal{W}_{60} + \frac{490}{143\sqrt{13}}\mathcal{W}_{80}$
	4	1	4	2	$\frac{54\sqrt{2}}{143}\mathcal{W}_{44} - \frac{12}{11}\sqrt{\frac{10}{13}}\mathcal{W}_{64} + \frac{21}{13}\sqrt{\frac{10}{187}}\mathcal{W}_{84}$
4	2	4	2	$\mathcal{W}_{00} - \frac{4\sqrt{5}}{11}\mathcal{W}_{20} + \frac{54}{143}\mathcal{W}_{40} - \frac{4}{11\sqrt{13}}\mathcal{W}_{60} + \frac{7}{143\sqrt{17}}\mathcal{W}_{80} + 21\sqrt{\frac{10}{2431}}\mathcal{W}_{88}$	
A_2^-	1	1	1	1	$\mathcal{W}_{00} + \frac{2}{\sqrt{5}}\mathcal{W}_{20}$
	1	1	3	1	$-3\sqrt{\frac{3}{35}}\mathcal{W}_{20} - \frac{4}{\sqrt{21}}\mathcal{W}_{40}$
	3	1	3	1	$\mathcal{W}_{00} + \frac{4}{3\sqrt{5}}\mathcal{W}_{20} + \frac{6}{11}\mathcal{W}_{40} + \frac{100}{33\sqrt{13}}\mathcal{W}_{60}$
A_2^+	4	1	4	1	$\mathcal{W}_{00} - \frac{4\sqrt{5}}{11}\mathcal{W}_{20} + \frac{54}{143}\mathcal{W}_{40} - \frac{4}{11\sqrt{13}}\mathcal{W}_{60} + \frac{7}{143\sqrt{17}}\mathcal{W}_{80} - 21\sqrt{\frac{10}{2431}}\mathcal{W}_{88}$
$B_{1/2}^+$	2	1	2	1	$\mathcal{W}_{00} - \frac{2\sqrt{5}}{7}\mathcal{W}_{20} + \frac{1}{7}\mathcal{W}_{40} \pm \sqrt{\frac{10}{7}}\mathcal{W}_{44}$
	2	1	4	1	$-\frac{\sqrt{15}}{7}\mathcal{W}_{20} + \frac{30\sqrt{3}}{77}\mathcal{W}_{40} - \frac{5}{11}\sqrt{\frac{3}{13}}\mathcal{W}_{60} \pm \frac{2}{11}\sqrt{\frac{30}{7}}\mathcal{W}_{44} \mp \frac{5}{11}\sqrt{\frac{42}{13}}\mathcal{W}_{64}$
	4	1	4	1	$\mathcal{W}_{00} + \frac{8\sqrt{5}}{77}\mathcal{W}_{20} - \frac{27}{91}\mathcal{W}_{40} - \frac{2}{\sqrt{13}}\mathcal{W}_{60} + \frac{196}{143\sqrt{17}}\mathcal{W}_{80}$ $\pm \frac{81}{143}\sqrt{\frac{10}{7}}\mathcal{W}_{44} \pm \frac{6}{11}\sqrt{\frac{14}{13}}\mathcal{W}_{64} \pm \frac{42}{13}\sqrt{\frac{14}{187}}\mathcal{W}_{84}$
$B_{1/2}^-$	3	1	3	1	$\mathcal{W}_{00} - \frac{7}{11}\mathcal{W}_{40} + \frac{10}{11\sqrt{13}}\mathcal{W}_{60} \mp \frac{\sqrt{70}}{11}\mathcal{W}_{44} \mp \frac{10}{11}\sqrt{\frac{14}{13}}\mathcal{W}_{64}$
E^-	1	1	1	1	$\mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20}$
	1	1	3	1	$-3\sqrt{\frac{2}{35}}\mathcal{W}_{20} + \sqrt{\frac{2}{7}}\mathcal{W}_{40}$
	3	1	3	1	$\mathcal{W}_{00} + \frac{1}{\sqrt{5}}\mathcal{W}_{20} + \frac{1}{11}\mathcal{W}_{40} - \frac{25}{11\sqrt{13}}\mathcal{W}_{60}$
	3	2	3	2	$\mathcal{W}_{00} - \frac{\sqrt{5}}{3}\mathcal{W}_{20} + \frac{3}{11}\mathcal{W}_{40} - \frac{5}{33\sqrt{13}}\mathcal{W}_{60}$
E^+	2	1	2	1	$\mathcal{W}_{00} + \frac{\sqrt{5}}{7}\mathcal{W}_{20} - \frac{4}{7}\mathcal{W}_{40}$
	2	1	4	1	$-\frac{\sqrt{30}}{7}\mathcal{W}_{20} - \frac{5\sqrt{6}}{77}\mathcal{W}_{40} + \frac{10}{11}\sqrt{\frac{6}{13}}\mathcal{W}_{60}$
	4	1	4	1	$\mathcal{W}_{00} + \frac{17\sqrt{5}}{77}\mathcal{W}_{20} + \frac{243}{1001}\mathcal{W}_{40} - \frac{1}{11\sqrt{13}}\mathcal{W}_{60} - \frac{392}{143\sqrt{17}}\mathcal{W}_{80}$
	4	2	4	2	$\mathcal{W}_{00} - \frac{4\sqrt{5}}{11}\mathcal{W}_{20} + \frac{54}{143}\mathcal{W}_{40} - \frac{4}{11\sqrt{13}}\mathcal{W}_{60} + \frac{7}{143\sqrt{17}}\mathcal{W}_{80}$

Elongated box and half-integer J

Γ	J	n	J'	n'	$\mathcal{M}_{J'n, J'V'n'}^{\Gamma}$
G_1^{\pm}	$\frac{1}{2}$	1	$\frac{1}{2}$	1	\mathcal{W}_{00}
	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\pm\sqrt{\frac{2}{5}}\mathcal{W}_{20}$
	$\frac{1}{2}$	1	$\frac{5}{2}$	1	$-\sqrt{\frac{3}{5}}\mathcal{W}_{20}$
	$\frac{1}{2}$	1	$\frac{7}{2}$	1	$-\frac{2}{3}\mathcal{W}_{40}$
	$\frac{1}{2}$	1	$\frac{7}{2}$	2	$\mp\frac{2}{3}\sqrt{2}\mathcal{W}_{44}$
	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\mathcal{W}_{00} + \frac{1}{\sqrt{5}}\mathcal{W}_{20}$
	$\frac{3}{2}$	1	$\frac{5}{2}$	1	$\mp\frac{1}{7}\sqrt{\frac{6}{5}}\mathcal{W}_{20} \mp \frac{2\sqrt{6}}{7}\mathcal{W}_{40}$
	$\frac{3}{2}$	1	$\frac{7}{2}$	1	$-\frac{9}{7}\sqrt{\frac{2}{5}}\mathcal{W}_{20} - \frac{5\sqrt{2}}{21}\mathcal{W}_{40}$
	$\frac{3}{2}$	1	$\frac{7}{2}$	2	$\frac{2}{3}\mathcal{W}_{44}$
	$\frac{5}{2}$	1	$\frac{5}{2}$	1	$\mathcal{W}_{00} + \frac{8}{7\sqrt{5}}\mathcal{W}_{20} + \frac{2}{7}\mathcal{W}_{40}$
	$\frac{5}{2}$	1	$\frac{7}{2}$	1	$\pm\frac{2}{7\sqrt{15}}\mathcal{W}_{20} \pm \frac{10\sqrt{3}}{77}\mathcal{W}_{40} \pm \frac{50}{11\sqrt{39}}\mathcal{W}_{60}$
	$\frac{5}{2}$	1	$\frac{7}{2}$	2	$\mp\frac{2\sqrt{6}}{11}\mathcal{W}_{44} \pm \frac{10}{11}\sqrt{\frac{10}{39}}\mathcal{W}_{64}$
	$\frac{7}{2}$	1	$\frac{7}{2}$	1	$\mathcal{W}_{00} + \frac{5}{21}\sqrt{5}\mathcal{W}_{20} + \frac{27}{77}\mathcal{W}_{40} + \frac{25\mathcal{W}_{60}}{33\sqrt{13}}$
	$\frac{7}{2}$	1	$\frac{7}{2}$	2	$\frac{3\sqrt{2}}{11}\mathcal{W}_{44} - \frac{5}{11}\sqrt{\frac{10}{13}}\mathcal{W}_{64}$
$\frac{7}{2}$	2	$\frac{7}{2}$	2	$\mathcal{W}_{00} - \frac{\sqrt{5}}{3}\mathcal{W}_{20} + \frac{3}{11}\mathcal{W}_{40} - \frac{5}{33\sqrt{13}}\mathcal{W}_{60}$	
G_2^{\pm}	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20}$
	$\frac{3}{2}$	1	$\frac{5}{2}$	1	$\pm\frac{6}{7\sqrt{5}}\mathcal{W}_{20} \mp \frac{2}{7}\mathcal{W}_{40}$
	$\frac{3}{2}$	1	$\frac{5}{2}$	2	$\mp 2\sqrt{\frac{2}{7}}\mathcal{W}_{44}$
	$\frac{3}{2}$	1	$\frac{7}{2}$	1	$-\frac{3\sqrt{2}}{7}\mathcal{W}_{20} + \frac{\sqrt{10}}{7}\mathcal{W}_{40}$
	$\frac{3}{2}$	1	$\frac{7}{2}$	2	$\frac{2}{\sqrt{21}}\mathcal{W}_{44}$
	$\frac{5}{2}$	1	$\frac{5}{2}$	1	$\mathcal{W}_{00} + \frac{2}{7\sqrt{5}}\mathcal{W}_{20} - \frac{3}{7}\mathcal{W}_{40}$
	$\frac{5}{2}$	1	$\frac{5}{2}$	2	$\sqrt{\frac{2}{7}}\mathcal{W}_{44}$
	$\frac{5}{2}$	2	$\frac{5}{2}$	2	$\mathcal{W}_{00} - \frac{2\sqrt{5}}{7}\mathcal{W}_{20} + \frac{1}{7}\mathcal{W}_{40}$
	$\frac{5}{2}$	1	$\frac{7}{2}$	1	$\pm\frac{\sqrt{2}}{7}\mathcal{W}_{20} \pm \frac{8\sqrt{10}}{77}\mathcal{W}_{40} \mp \frac{3}{11}\sqrt{\frac{10}{13}}\mathcal{W}_{60}$
	$\frac{5}{2}$	1	$\frac{7}{2}$	2	$\mp\frac{8}{11}\sqrt{\frac{2}{7}}\mathcal{W}_{44} \mp \frac{10}{11}\sqrt{\frac{35}{39}}\mathcal{W}_{64}$
	$\frac{5}{2}$	2	$\frac{7}{2}$	1	$\pm\frac{4}{11}\sqrt{\frac{5}{7}}\mathcal{W}_{44} \mp \frac{10}{11}\sqrt{\frac{7}{13}}\mathcal{W}_{64}$
	$\frac{5}{2}$	2	$\frac{7}{2}$	2	$\mp\frac{1}{7}\sqrt{\frac{10}{3}}\mathcal{W}_{20} \pm \frac{10}{77}\sqrt{6}\mathcal{W}_{40} \mp \frac{5}{11}\sqrt{\frac{2}{39}}\mathcal{W}_{60}$
	$\frac{7}{2}$	1	$\frac{7}{2}$	1	$\mathcal{W}_{00} + \frac{\sqrt{5}}{7}\mathcal{W}_{20} - \frac{9}{77}\mathcal{W}_{40} - \frac{15}{11\sqrt{13}}\mathcal{W}_{60}$
	$\frac{7}{2}$	1	$\frac{7}{2}$	2	$\frac{3}{11}\sqrt{\frac{30}{7}}\mathcal{W}_{44} + \frac{5}{11}\sqrt{\frac{14}{39}}\mathcal{W}_{64}$
$\frac{7}{2}$	2	$\frac{7}{2}$	2	$\mathcal{W}_{00} - \frac{\sqrt{5}}{21}\mathcal{W}_{20} - \frac{39}{77}\mathcal{W}_{40} + \frac{25}{33\sqrt{13}}\mathcal{W}_{60}$	

G_1^{\pm} sector

$$l = J \pm \frac{1}{2}$$

$$\text{Parity} = (-1)^l$$

$$\begin{vmatrix} \mathcal{M}_{\frac{1}{2}\frac{1}{2}} - \cot \delta_{\frac{1}{2}} & \mathcal{M}_{\frac{1}{2}\frac{3}{2}} & \mathcal{M}_{\frac{1}{2}\frac{5}{2}} \\ \mathcal{M}_{\frac{3}{2}\frac{1}{2}} & \mathcal{M}_{\frac{3}{2}\frac{3}{2}} - \cot \delta_{\frac{3}{2}} & \mathcal{M}_{\frac{3}{2}\frac{5}{2}} \\ \mathcal{M}_{\frac{5}{2}\frac{1}{2}} & \mathcal{M}_{\frac{5}{2}\frac{3}{2}} & \mathcal{M}_{\frac{5}{2}\frac{5}{2}} - \cot \delta_{\frac{5}{2}} \end{vmatrix} = 0,$$

$$\cot \delta_{\frac{1}{2}} = \mathcal{W}_{00} + \frac{\frac{2}{5}\mathcal{W}_{20}}{\cot \delta_{\frac{3}{2}} - (\mathcal{W}_{00} + \frac{1}{\sqrt{5}}\mathcal{W}_{20})}$$

G_2^{\pm} sector

$$\cot \delta_{\frac{3}{2}} = \mathcal{M}_{\frac{3}{2}1, \frac{3}{2}1} + \frac{\mathcal{M}_{\frac{3}{2}1, \frac{5}{2}1}\mathcal{M}_{\frac{3}{2}1, \frac{5}{2}2}}{\cot \delta_{\frac{5}{2}} - \mathcal{M}_{\frac{5}{2}1, \frac{5}{2}1}} + \frac{\mathcal{M}_{\frac{5}{2}2, \frac{5}{2}2}}{\cot \delta_{\frac{5}{2}} - \mathcal{M}_{\frac{5}{2}2, \frac{5}{2}2}}$$

$$\cot \delta_{\frac{3}{2}} = \mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20}$$

Cubic box and integer J

Γ	J	J'	$\mathcal{M}_{J,J'}^\Gamma$
A_1^+	0	0	\mathcal{W}_{00}
	0	4	$\frac{36}{\sqrt{21}}\mathcal{W}_{40}$
	4	4	$\mathcal{W}_{00} + \frac{108}{143}\mathcal{W}_{40} + \frac{80}{11\sqrt{13}}\mathcal{W}_{60} + \frac{560}{143\sqrt{17}}\mathcal{W}_{80}$
A_2^-	3	3	$\mathcal{W}_{00} - \frac{12}{11}\mathcal{W}_{40} + \frac{80}{11\sqrt{13}}\mathcal{W}_{60}$
E^+	2	2	$\mathcal{W}_{00} + \frac{6}{7}\mathcal{W}_{40}$
	2	4	$-\frac{40\sqrt{3}}{77}\mathcal{W}_{40} - \frac{30}{11}\sqrt{\frac{3}{13}}\mathcal{W}_{60}$
	4	4	$\mathcal{W}_{00} + \frac{108}{1001}\mathcal{W}_{40} - \frac{64}{11\sqrt{13}}\mathcal{W}_{60} + \frac{392}{143\sqrt{17}}\mathcal{W}_{80}$
T_1^-	1	1	\mathcal{W}_{00}
	1	3	$-\frac{4}{\sqrt{21}}\mathcal{W}_{40}$
	3	3	$\mathcal{W}_{00} + \frac{6}{11}\mathcal{W}_{40} + \frac{100}{33\sqrt{13}}\mathcal{W}_{60}$
T_1^+	4	4	$\mathcal{W}_{00} + \frac{54}{143}\mathcal{W}_{40} - \frac{4}{11\sqrt{13}}\mathcal{W}_{60} - \frac{448}{143\sqrt{17}}\mathcal{W}_{80}$
T_2^+	2	2	$\mathcal{W}_{00} - \frac{4}{7}\mathcal{W}_{40}$
	2	4	$-\frac{20\sqrt{3}}{77}\mathcal{W}_{40} + \frac{40}{11}\sqrt{\frac{3}{13}}\mathcal{W}_{60}$
	4	4	$\mathcal{W}_{00} - \frac{54}{77}\mathcal{W}_{40} + \frac{20}{11\sqrt{13}}\mathcal{W}_{60}$
T_2^-	3	3	$\mathcal{W}_{00} - \frac{2}{11}\mathcal{W}_{40} - \frac{60}{11\sqrt{13}}\mathcal{W}_{60}$

$$\mathbf{A}_1^+ \text{ sector : } \cot \delta_0 = \mathcal{W}_{00}$$

$$\mathbf{T}_1^- \text{ sector : } \cot \delta_1 = \mathcal{W}_{00}$$

$$\mathbf{T}_2^+ \text{ sector : } \cot \delta_2 = \mathcal{W}_{00} - \frac{4}{7}\mathcal{W}_{40}$$

$$\mathbf{E}^+ \text{ sector : } \cot \delta_2 = \mathcal{W}_{00} + \frac{6}{7}\mathcal{W}_{40}$$

Cubic box and half-integer J

Γ	J	J'	$\mathcal{M}_{Jl,J'l'}^\Gamma$
G_1^\pm	$\frac{1}{2}$	$\frac{1}{2}$	\mathcal{W}_{00}
	$\frac{1}{2}$	$\frac{7}{2}$	$\mp \frac{4}{\sqrt{21}} \mathcal{W}_{40}$
	$\frac{7}{2}$	$\frac{7}{2}$	$\mathcal{W}_{00} + \frac{6}{11} \mathcal{W}_{40} + \frac{100}{33\sqrt{13}} \mathcal{W}_{60}$
G_2^\pm	$\frac{5}{2}$	$\frac{5}{2}$	$\mathcal{W}_{00} - \frac{4}{7} \mathcal{W}_{40}$
	$\frac{5}{2}$	$\frac{7}{2}$	$\pm \frac{20\sqrt{3}}{77} \mathcal{W}_{40} \mp \frac{40}{11} \sqrt{\frac{3}{13}} \mathcal{W}_{60}$
	$\frac{7}{2}$	$\frac{7}{2}$	$\mathcal{W}_{00} - \frac{54}{77} \mathcal{W}_{40} + \frac{20}{11\sqrt{13}} \mathcal{W}_{60}$
H^\pm	$\frac{3}{2}$	$\frac{3}{2}$	\mathcal{W}_{00}
	$\frac{3}{2}$	$\frac{5}{2}$	$\mp \frac{2\sqrt{6}}{7} \mathcal{W}_{40}$
	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{2}{7} \sqrt{\frac{10}{3}} \mathcal{W}_{40}$
	$\frac{5}{2}$	$\frac{5}{2}$	$\mathcal{W}_{00} + \frac{2}{7} \mathcal{W}_{40}$
	$\frac{5}{2}$	$\frac{7}{2}$	$\mp \frac{12\sqrt{5}}{77} \mathcal{W}_{40} \mp \frac{20}{11} \sqrt{\frac{5}{13}} \mathcal{W}_{60}$
	$\frac{7}{2}$	$\frac{7}{2}$	$\mathcal{W}_{00} + \frac{6}{77} \mathcal{W}_{40} - \frac{80}{33\sqrt{13}} \mathcal{W}_{60}$

$$G_1^\pm \text{ sector : } \cot \delta_{\frac{1}{2}} = \mathcal{W}_{00}$$

$$G_2^\pm \text{ sector : } \cot \delta_{\frac{5}{2}} = \mathcal{W}_{00} - \frac{4}{7} \mathcal{W}_{40}$$

$$H^\pm \text{ sector : } \cot \delta_{\frac{3}{2}} = \mathcal{W}_{00}$$

Cubic vs. Elongated

Subduction rules

2O_h	A_1^+	A_2^+	E^+	T_1^+	T_2^+	G_1^+	G_2^+	H^+	A_1^-	A_2^-	E^-	T_1^-
${}^2D_{4h}$	A_1^+	B_1^+	$A_1^+ \oplus B_1^+$	$A_2^+ \oplus E^+$	$B_2^+ \oplus E^+$	G_1^+	G_2^+	$G_1^+ \oplus G_2^+$	A_1^-	B_1^-	$A_1^- \oplus B_1^-$	$A_2^- \oplus E^-$
${}^2C_{4v}$	A_1	B_1	$A_1 \oplus B_1$	$A_2 \oplus E$	$B_2 \oplus E$	G_1	G_2	$G_1 \oplus G_2$	A_2	B_2	$A_2 \oplus B_2$	$A_1 \oplus E$

$$\mathcal{M}_{11}^{T_1^-} = \frac{1}{3} \mathcal{M}_{11}^{A_2^-} + \frac{2}{3} \mathcal{M}_{11}^{E^-}$$

$$\cot \delta_1(\mathcal{T}_1^-) = \frac{1}{3} \cot \delta_1(A_2^-) + \frac{2}{3} \cot \delta_1(E^-)$$

$$\mathcal{M}_{\frac{3}{2} \frac{3}{2}}^{H^\pm} = \frac{1}{2} \left(\mathcal{M}_{\frac{3}{2} \frac{3}{2}}^{G_1^\pm} + \mathcal{M}_{\frac{3}{2} \frac{3}{2}}^{G_2^\pm} \right)$$

Boosting in the cubic box: kinematics

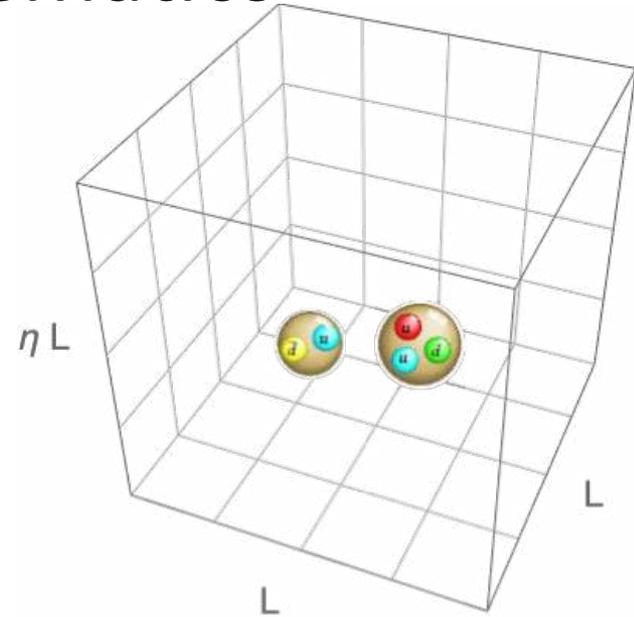
$$W = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2}$$

$$E_{lab} = \sqrt{m_1^2 + \mathbf{p}_1^2} + \sqrt{m_2^2 + \mathbf{p}_2^2}$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \frac{2\pi}{L} \mathbf{d}$$

$$W = \sqrt{E_{lab}^2 - \mathbf{P}^2}$$

$$W = E_{lab}/\gamma, \text{ where } \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2}} \text{ with } \mathbf{v} = \frac{\mathbf{P}}{E_{lab}}$$



We only consider boosting in z-direction:
 $\mathbf{d}=(0,0,1)/\eta$, $\mathbf{d}=(0,0,2)/\eta$, etc

Boosting leads to lower CM energies.

Boosting in the cubic box: zeta functions

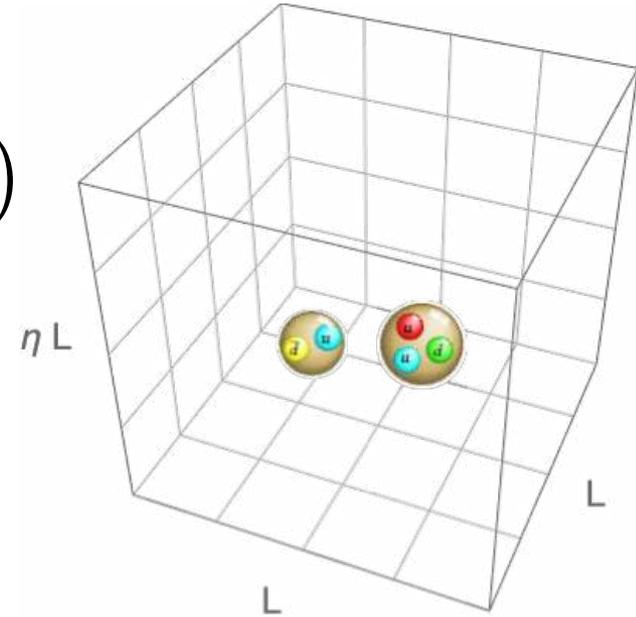
Viewed from the moving frame, the box is distorted and boundary condition is modified from

$$\psi(\mathbf{r} + \mathbf{n}L) = \psi(\mathbf{r})$$

to

$$\psi(\mathbf{r} + \hat{\gamma}\mathbf{n}L) = e^{i\pi A\mathbf{n}\cdot\mathbf{d}}\psi(\mathbf{r})$$

$$A = 1 + \frac{m_1^2 - m_2^2}{W^2}$$



The zeta function is modified

$$\mathcal{Z}_{lm}^{\mathbf{d}}(s, q^2, \gamma) = \sum_{\tilde{\mathbf{n}} \in P_{\mathbf{d}}(\gamma)} \frac{\mathcal{Y}_{lm}(\tilde{\mathbf{n}})}{(\tilde{\mathbf{n}}^2 - q^2)^s}$$

with a new summation region:

$$P_{\mathbf{d}}(\gamma) = \left\{ \tilde{\mathbf{n}} \in R^3 \mid \tilde{\mathbf{n}} = \hat{\gamma}^{-1}(\mathbf{m} + \frac{1}{2}A\mathbf{d}), \mathbf{m} \in Z^3 \right\}$$

Boosting in the cubic box: basis vectors

From 2O_h group to ${}^2C_{4v}$ group.

Loss of parity leads to mixing of even and odd states

The two sets for G1 and G2 have different l values: $l = J \pm \frac{1}{2}$.

Γ	J	Basis $ JM\rangle$ (even branch)	Basis $ JM\rangle$ (odd branch)
A_1	0	$ 0, 0\rangle$	
	1		$ 1, 0\rangle$
	2	$ 2, 0\rangle$	
	3		$ 3, 0\rangle$
	4	$ 4, 0\rangle; \frac{1}{\sqrt{2}}(4, 4\rangle + 4, -4\rangle)$	
A_2	4	$\frac{1}{\sqrt{2}}(4, 4\rangle - 4, -4\rangle)$	
B_1	2	$\frac{1}{\sqrt{2}}(2, 2\rangle + 2, -2\rangle)$	
	3		$\frac{1}{\sqrt{2}}(3, 2\rangle + 3, -2\rangle)$
	4	$\frac{1}{\sqrt{2}}(4, 2\rangle + 4, -2\rangle)$	
B_2	2	$\frac{1}{\sqrt{2}}(2, 2\rangle - 2, -2\rangle)$	
	3		$\frac{1}{\sqrt{2}}(3, 2\rangle - 3, -2\rangle)$
	4	$\frac{1}{\sqrt{2}}(4, 2\rangle - 4, -2\rangle)$	
E	1		$\frac{1}{\sqrt{2}}(1, 1\rangle \pm 1, -1\rangle)$
	2	$\frac{1}{\sqrt{2}}(2, 1\rangle \pm 2, -1\rangle)$	
	3		$\frac{1}{\sqrt{2}}(3, 1\rangle \pm 3, -1\rangle); \frac{1}{\sqrt{2}}(\pm 3, 3\rangle + 3, -3\rangle)$
	4	$\frac{1}{\sqrt{2}}(4, 1\rangle \pm 4, -1\rangle); \frac{1}{\sqrt{2}}(\pm 4, 3\rangle + 4, -3\rangle)$	
G_1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\frac{1}{2}, \frac{1}{2}\rangle \mp \frac{1}{2}, -\frac{1}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{1}{2}, \frac{1}{2}\rangle \pm \frac{1}{2}, -\frac{1}{2}\rangle)$
	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}(\frac{3}{2}, \frac{1}{2}\rangle \pm \frac{3}{2}, -\frac{1}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{3}{2}, \frac{1}{2}\rangle \mp \frac{3}{2}, -\frac{1}{2}\rangle)$
	$\frac{5}{2}$	$\frac{1}{\sqrt{2}}(\frac{5}{2}, \frac{1}{2}\rangle \mp \frac{5}{2}, -\frac{1}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{5}{2}, \frac{1}{2}\rangle \pm \frac{5}{2}, -\frac{1}{2}\rangle)$
	$\frac{7}{2}$	$\frac{1}{\sqrt{2}}(\frac{7}{2}, \frac{1}{2}\rangle \pm \frac{7}{2}, -\frac{1}{2}\rangle); \frac{1}{\sqrt{2}}(\pm \frac{7}{2}, \frac{3}{2}\rangle + \frac{7}{2}, -\frac{3}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{7}{2}, \frac{1}{2}\rangle \mp \frac{7}{2}, -\frac{1}{2}\rangle); \frac{1}{\sqrt{2}}(\mp \frac{7}{2}, \frac{3}{2}\rangle + \frac{7}{2}, -\frac{3}{2}\rangle)$
G_2	$\frac{3}{2}$	$\frac{1}{\sqrt{2}}(\frac{3}{2}, \frac{3}{2}\rangle \pm \frac{3}{2}, -\frac{3}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{3}{2}, \frac{3}{2}\rangle \mp \frac{3}{2}, -\frac{3}{2}\rangle)$
	$\frac{5}{2}$	$\frac{1}{\sqrt{2}}(\frac{5}{2}, \frac{3}{2}\rangle \mp \frac{5}{2}, -\frac{3}{2}\rangle); \frac{1}{\sqrt{2}}(\mp \frac{5}{2}, \frac{5}{2}\rangle + \frac{5}{2}, -\frac{5}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{5}{2}, \frac{3}{2}\rangle \pm \frac{5}{2}, -\frac{3}{2}\rangle); \frac{1}{\sqrt{2}}(\pm \frac{5}{2}, \frac{5}{2}\rangle + \frac{5}{2}, -\frac{5}{2}\rangle)$
	$\frac{7}{2}$	$\frac{1}{\sqrt{2}}(\frac{7}{2}, \frac{3}{2}\rangle \pm \frac{7}{2}, -\frac{3}{2}\rangle); \frac{1}{\sqrt{2}}(\pm \frac{7}{2}, \frac{5}{2}\rangle + \frac{7}{2}, -\frac{5}{2}\rangle)$	$\frac{1}{\sqrt{2}}(\frac{7}{2}, \frac{3}{2}\rangle \mp \frac{7}{2}, -\frac{3}{2}\rangle); \frac{1}{\sqrt{2}}(\mp \frac{7}{2}, \frac{5}{2}\rangle + \frac{7}{2}, -\frac{5}{2}\rangle)$

Boosting in cubic box and integer J

Parity is lost for unequal-mass systems.

Appearance of odd-l zeta functions.

Mixing between even and odd partial waves.

A1 sector:

$$\begin{vmatrix} \mathcal{W}_{00} - \cot \delta_0 & i\mathcal{W}_{10} \\ -i\mathcal{W}_{10} & \mathcal{W}_{00} + \frac{2}{\sqrt{5}}\mathcal{W}_{20} - \cot \delta_1 \end{vmatrix} = 0$$

E sector:

$$\begin{vmatrix} \mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20} - \cot \delta_1 & i\sqrt{\frac{3}{5}}\mathcal{W}_{10} - \frac{3i}{\sqrt{35}}\mathcal{W}_{30} \\ -i\sqrt{\frac{3}{5}}\mathcal{W}_{10} + \frac{3i}{\sqrt{35}}\mathcal{W}_{30} & \mathcal{W}_{00} + \frac{\sqrt{5}}{7}\mathcal{W}_{20} - \frac{4}{7}\mathcal{W}_{40} - \cot \delta_2 \end{vmatrix} = 0$$

Γ	J	n	J'	n'	$\mathcal{M}_{Jn,J'n'}^\Gamma$
A_1	0	1	0	1	\mathcal{W}_{00}
	0	1	1	1	$i\mathcal{W}_{10}$
	0	1	2	1	$-\mathcal{W}_{20}$
	0	1	3	1	$-i\mathcal{W}_{30}$
	1	1	1	1	$\mathcal{W}_{00} + \frac{2}{\sqrt{5}}\mathcal{W}_{20}$
	1	1	2	1	$\frac{2i}{\sqrt{5}}\mathcal{W}_{10} + 3i\sqrt{\frac{3}{35}}\mathcal{W}_{30}$
	1	1	3	1	$-3\sqrt{\frac{3}{35}}\mathcal{W}_{20} - \frac{4}{\sqrt{21}}\mathcal{W}_{40}$
	2	1	2	1	$\mathcal{W}_{00} + \frac{2\sqrt{5}}{7}\mathcal{W}_{20} + \frac{6}{7}\mathcal{W}_{40}$
	2	1	3	1	$3i\sqrt{\frac{3}{35}}\mathcal{W}_{10} + \frac{4i}{3\sqrt{5}}\mathcal{W}_{30} + \frac{10}{3}i\sqrt{\frac{5}{77}}\mathcal{W}_{50}$
	3	1	3	1	$\mathcal{W}_{00} + \frac{4}{3\sqrt{5}}\mathcal{W}_{20} + \frac{6}{11}\mathcal{W}_{40} + \frac{100}{33\sqrt{13}}\mathcal{W}_{60}$
$B_{1/2}$	2	1	2	1	$\mathcal{W}_{00} - \frac{2\sqrt{5}}{7}\mathcal{W}_{20} + \frac{1}{7}\mathcal{W}_{40} \pm \sqrt{\frac{10}{7}}\mathcal{W}_{44}$
	2	1	3	1	$i\sqrt{\frac{3}{7}}\mathcal{W}_{10} - \frac{2i}{3}\mathcal{W}_{30} + \frac{5i}{3\sqrt{77}}\mathcal{W}_{50} \pm i\sqrt{\frac{10}{11}}\mathcal{W}_{54}$
	3	1	3	1	$\mathcal{W}_{00} - \frac{7}{11}\mathcal{W}_{40} + \frac{10}{11\sqrt{13}}\mathcal{W}_{60} \pm \frac{\sqrt{70}}{11}\mathcal{W}_{44} \pm \frac{10}{11}\sqrt{\frac{14}{13}}\mathcal{W}_{64}$
E	1	1	1	1	$\mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20}$
	1	1	2	1	$i\sqrt{\frac{3}{5}}\mathcal{W}_{10} - \frac{3i}{\sqrt{35}}\mathcal{W}_{30}$
	1	1	3	1	$-3\sqrt{\frac{2}{35}}\mathcal{W}_{20} + \sqrt{\frac{2}{7}}\mathcal{W}_{40}$
	1	1	3	2	$\frac{2}{\sqrt{3}}\mathcal{W}_{44}$
	2	1	2	1	$\mathcal{W}_{00} + \frac{\sqrt{5}}{7}\mathcal{W}_{20} - \frac{4}{7}\mathcal{W}_{40}$
	2	1	3	1	$2i\sqrt{\frac{6}{35}}\mathcal{W}_{10} + \frac{1}{3}i\sqrt{\frac{2}{5}}\mathcal{W}_{30} - \frac{5}{3}i\sqrt{\frac{10}{77}}\mathcal{W}_{50}$
	2	1	3	2	$-2i\sqrt{\frac{5}{33}}\mathcal{W}_{54}$
	3	1	3	1	$\mathcal{W}_{00} + \frac{1}{\sqrt{5}}\mathcal{W}_{20} + \frac{1}{11}\mathcal{W}_{40} - \frac{25}{11\sqrt{13}}\mathcal{W}_{60}$
3	2	3	2	$\mathcal{W}_{00} - \frac{\sqrt{5}}{3}\mathcal{W}_{20} + \frac{3}{11}\mathcal{W}_{40} - \frac{5}{33\sqrt{13}}\mathcal{W}_{60}$	

Boosting in cubic box and half-integer J

Γ	J	l	J'	l'	$\mathcal{M}_{Jl,J'l'}^\Gamma$		
G_1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	\mathcal{W}_{00}		
	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$-\frac{i}{\sqrt{3}}\mathcal{W}_{10}$		
	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$i\sqrt{\frac{2}{3}}\mathcal{W}_{10}$		
	$\frac{1}{2}$	0	$\frac{3}{2}$	2	$\sqrt{\frac{2}{5}}\mathcal{W}_{20}$		
	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$-\sqrt{\frac{2}{5}}\mathcal{W}_{20}$		
	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$i\sqrt{\frac{2}{3}}\mathcal{W}_{10}$		
	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\mathcal{W}_{00} + \frac{1}{\sqrt{5}}\mathcal{W}_{20}$		
	$\frac{3}{2}$	1	$\frac{3}{2}$	2	$-i\frac{1}{5\sqrt{3}}\mathcal{W}_{10} - i\frac{9}{5\sqrt{7}}\mathcal{W}_{30}$		
	$\frac{1}{2}$	0	$\frac{5}{2}$	2	$-\sqrt{\frac{3}{5}}\mathcal{W}_{20}$		
	$\frac{1}{2}$	0	$\frac{5}{2}$	3	$i\sqrt{\frac{3}{7}}\mathcal{W}_{30}$		
	$\frac{1}{2}$	1	$\frac{5}{2}$	3	$-i\sqrt{\frac{3}{7}}\mathcal{W}_{30}$		
	$\frac{3}{2}$	1	$\frac{5}{2}$	2	$i\frac{3\sqrt{2}}{5}\mathcal{W}_{10} + \frac{2}{5}i\sqrt{\frac{6}{7}}\mathcal{W}_{30}$		
	$\frac{3}{2}$	1	$\frac{5}{2}$	3	$\frac{1}{7}\sqrt{\frac{6}{5}}\mathcal{W}_{20} + \frac{2\sqrt{6}}{7}\mathcal{W}_{40}$		
	$\frac{3}{2}$	2	$\frac{5}{2}$	2	$-\frac{1}{7}\sqrt{\frac{6}{5}}\mathcal{W}_{20} - \frac{2\sqrt{6}}{7}\mathcal{W}_{40}$		
	$\frac{5}{2}$	2	$\frac{5}{2}$	2	$\mathcal{W}_{00} + \frac{8}{7\sqrt{5}}\mathcal{W}_{20} + \frac{2}{7}\mathcal{W}_{40}$		
$\frac{5}{2}$	2	$\frac{5}{2}$	3	$-i\frac{\sqrt{3}}{35}\mathcal{W}_{10} - i\frac{8}{15\sqrt{7}}\mathcal{W}_{30} - i\frac{50}{21\sqrt{11}}\mathcal{W}_{50}$			
	J	l	n	J'	l'	n'	$\mathcal{M}_{Jln,J'l'n'}^\Gamma$
G_2	$\frac{3}{2}$	1	1	$\frac{3}{2}$	1	1	$\mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20}$
	$\frac{3}{2}$	1	1	$\frac{3}{2}$	2	1	$-i\frac{\sqrt{3}}{5}\mathcal{W}_{10} + i\frac{3}{5\sqrt{7}}\mathcal{W}_{30}$
	$\frac{3}{2}$	1	1	$\frac{5}{2}$	2	1	$i\frac{2\sqrt{3}}{5}\mathcal{W}_{10} - i\frac{6}{5\sqrt{7}}\mathcal{W}_{30}$
	$\frac{3}{2}$	1	1	$\frac{5}{2}$	3	1	$\frac{6}{7\sqrt{5}}\mathcal{W}_{20} - \frac{2}{7}\mathcal{W}_{40}$
	$\frac{3}{2}$	1	1	$\frac{5}{2}$	3	2	$-2\sqrt{\frac{2}{7}}\mathcal{W}_{44}$
	$\frac{3}{2}$	2	1	$\frac{5}{2}$	2	1	$-\frac{6}{7\sqrt{5}}\mathcal{W}_{20} + \frac{2}{7}\mathcal{W}_{40}$
	$\frac{3}{2}$	2	1	$\frac{5}{2}$	2	2	$2\sqrt{\frac{2}{7}}\mathcal{W}_{44}$
	$\frac{5}{2}$	2	1	$\frac{5}{2}$	2	1	$\mathcal{W}_{00} + \frac{2}{7\sqrt{5}}\mathcal{W}_{20} - \frac{3}{7}\mathcal{W}_{40}$
	$\frac{5}{2}$	2	1	$\frac{5}{2}$	2	2	$\sqrt{\frac{2}{7}}\mathcal{W}_{44}$
	$\frac{5}{2}$	2	2	$\frac{5}{2}$	2	2	$\mathcal{W}_{00} - \frac{2\sqrt{5}}{7}\mathcal{W}_{20} + \frac{1}{7}\mathcal{W}_{40}$
	$\frac{5}{2}$	2	1	$\frac{5}{2}$	3	1	$-i\frac{3\sqrt{3}}{35}\mathcal{W}_{10} - i\frac{2\sqrt{7}}{15}\mathcal{W}_{30} + i\frac{25}{21\sqrt{11}}\mathcal{W}_{50}$
	$\frac{5}{2}$	2	1	$\frac{5}{2}$	3	2	$5i\sqrt{\frac{2}{77}}\mathcal{W}_{54}$
	$\frac{5}{2}$	2	2	$\frac{5}{2}$	3	2	$i\frac{\sqrt{3}}{7}\mathcal{W}_{10} - i\frac{2}{3\sqrt{7}}\mathcal{W}_{30} + i\frac{5}{21\sqrt{11}}\mathcal{W}_{50}$

G1 sector:

$$\left| \begin{array}{cc} \mathcal{W}_{00} - \cot \delta_{\frac{1}{2}0} & -\frac{i}{\sqrt{3}}\mathcal{W}_{10} \\ \frac{i}{\sqrt{3}}\mathcal{W}_{10} & \mathcal{W}_{00} - \cot \delta_{\frac{1}{2}1} \end{array} \right| = 0$$

G2 sector:

$$\left| \begin{array}{cc} \mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20} - \cot \delta_{\frac{3}{2}1} & -i\frac{\sqrt{3}}{5}\mathcal{W}_{10} + i\frac{3}{5\sqrt{7}}\mathcal{W}_{30} \\ i\frac{\sqrt{3}}{5}\mathcal{W}_{10} - i\frac{3}{5\sqrt{7}}\mathcal{W}_{30} & \mathcal{W}_{00} - \frac{1}{\sqrt{5}}\mathcal{W}_{20} - \cot \delta_{\frac{3}{2}2} \end{array} \right| = 0$$

Boosting in elongated box

- Boost in z-direction: from ${}^2D_{4h}$ to ${}^2C_{4v}$.
- Same formulas as in the cubic case; the only difference is in the modified zeta functions labeled by γ and \mathbf{d} .

$$\mathcal{W}_{lm}(1, q^2, \gamma, \eta) = \frac{\mathcal{Z}_{lm}^{\mathbf{d}}(1, q^2, \gamma, \eta)}{\pi^{\frac{3}{2}} \eta \gamma q^{l+1}}$$

$$\mathcal{Z}_{lm}^{\mathbf{d}}(s, q^2, \gamma, \eta) = \sum_{\tilde{\mathbf{n}} \in P_{\mathbf{d}}(\gamma, \eta)} \frac{\mathcal{Y}_{lm}(\tilde{\mathbf{n}})}{(\tilde{\mathbf{n}}^2 - q^2)^s}$$

$$P_{\mathbf{d}}(\gamma, \eta) = \left\{ \tilde{\mathbf{n}} \in R^3 \mid \tilde{\mathbf{n}} = \hat{\gamma}^{-1} \hat{\eta}^{-1} \left(\mathbf{m} + \frac{1}{2} A \mathbf{d} \right), \mathbf{m} \in Z^3 \right\}$$

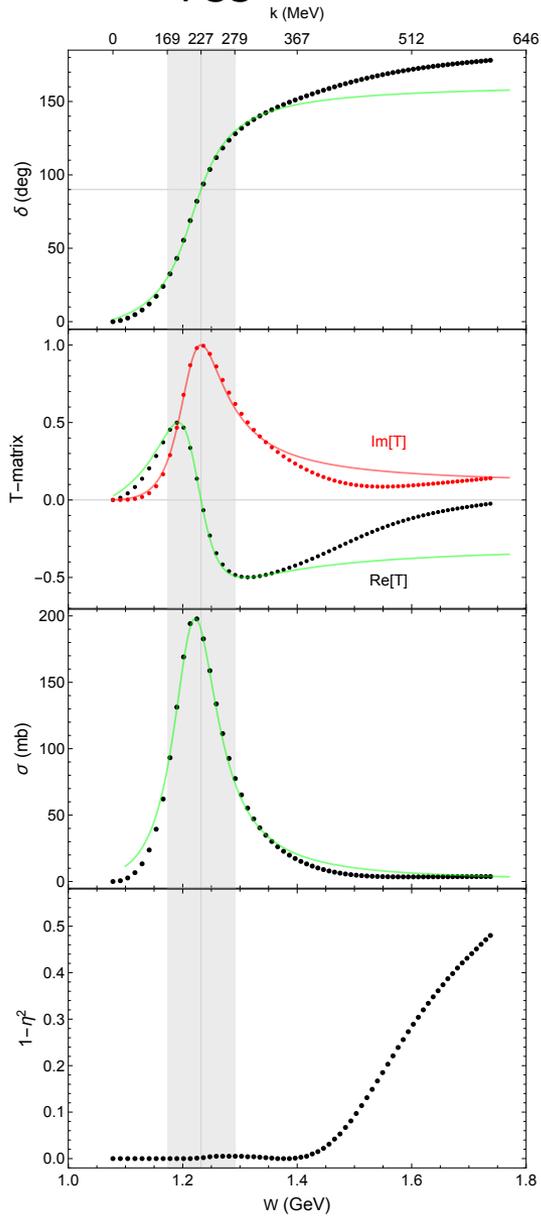
Conclusion

- We derived Lüscher phaseshift formulas in elongated boxes for elastic scattering of
 - two spinless mesons ($\pi\pi\rightarrow\rho$ or $\pi K\rightarrow K^*$)
 - one spinless and one spinful in s-wave ($\pi\rho\rightarrow a_1$)
 - spinless meson and spin-1/2 baryon ($\pi N\rightarrow\Delta$, S_{11} , or P_{11})
- Results checked against those in the cubic box
 - Relationship between the two follows subduction rules
- Boosting is considered in both elongated and cubic boxes
 - allows wider coverage of energy
 - but more complicated formulas
- Details can be found in [arXiv:1706.00262](https://arxiv.org/abs/1706.00262)

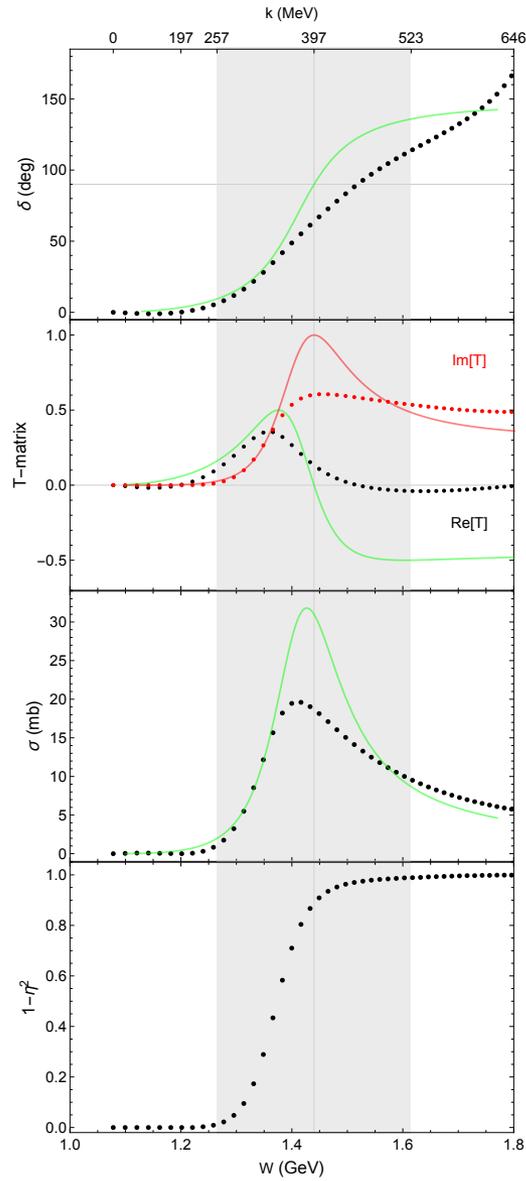
Reserve slides

πN scattering experiment

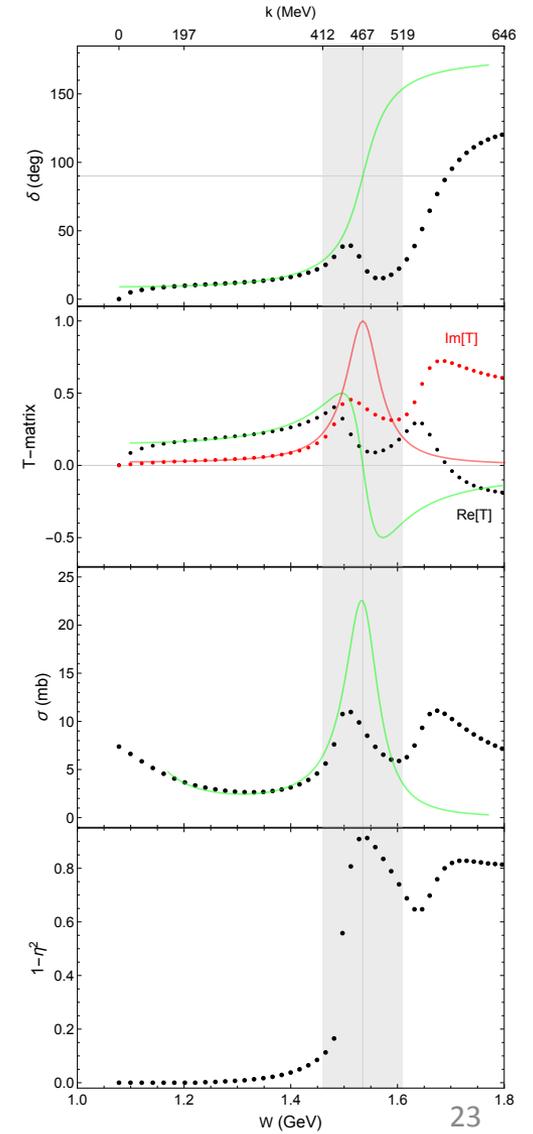
P33



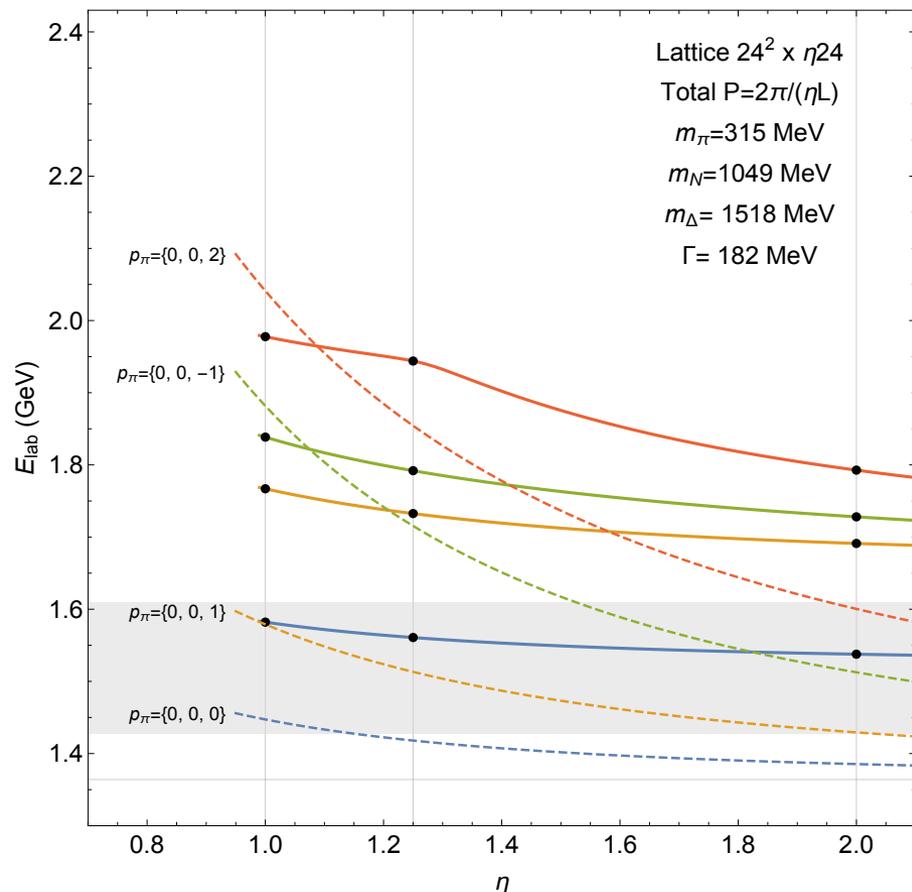
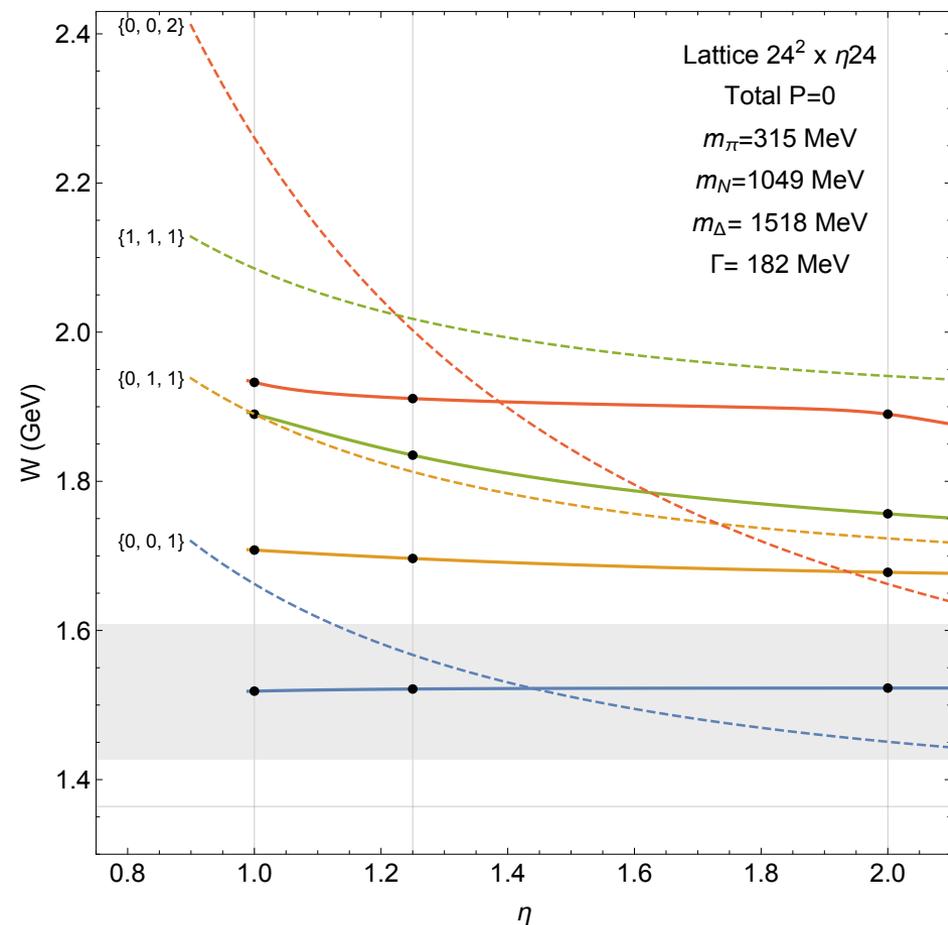
P11



S11



Expected energy spectrum for P33



Expected energy spectrum for P33

