Lattice QCD results for HVP contribution to anomalous magnetic moments of leptons

At Physical Point Mass with Full Systematics

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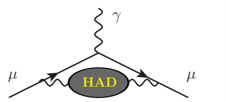
Budapest-Marseille-Wuppertal Collaboration
1612.02364 [hep-lat] and in preparation

Objective

Leading-Order (LO) Hadronic Vacuum Polarization (HVP) contribution to anomalous magnetic moments for all leptons:

$$a_{\ell=e,\mu,\tau}^{\text{LO-HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\ \omega(Q^2/m_\ell^2)\hat{\Pi}^f(Q^2)\ .$$

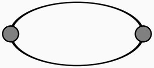


where suffix f stands for a flavor $f = l(u,d), s, c, disc$, and

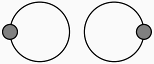
$$\hat{\Pi}^f(Q^2) = \Pi^f(Q^2) - \Pi^f(0) = \sum_t t^2 \left[1 - \left(\frac{\sin(z/2)}{z/2}\right)^2\right]_{z=Qt} \frac{1}{3}\sum_{i=1}^{3} C_{ii}^f(t)\ , \quad (1)$$

with

$$C_{\mu\nu}^{f=l,s,c}(t) = q_{f=l,s,c}^2 \sum_{\vec{x}} \langle\ j_\mu^f(x) j_\nu^f(0)\ \rangle|_{conn}\ ,$$



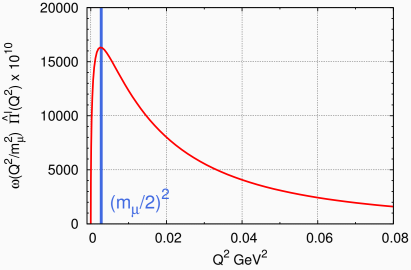
$$C_{\mu\nu}^{f=disc}(t) = q_{f=disc}^2 \sum_{\vec{x}} \langle(\bar{l}\gamma_\mu l - \bar{s}\gamma_\mu s)(\bar{l}\gamma_\nu l - \bar{s}\gamma_\nu s)\rangle|_{disc}\ .$$



Here, charge factors are given by $(q_l^2, q_s^2, q_c^2, q_{disc}^2) = (5/9,\ -1/9,\ 4/9,\ 1/9)$.

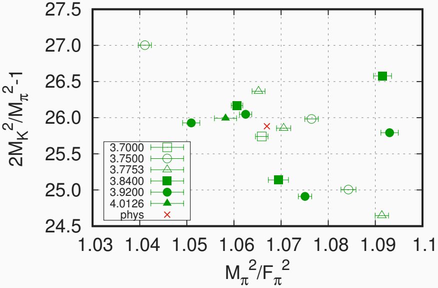
Our Challenges

- The integrand kurnel $\omega(Q^2/m_\mu^2)$ is known and makes a peak around $Q^2 \sim (m_\mu/2)^2 \sim (0.05 GeV)^2 \to 4 fm$: Our Lattice: $(L, T) \sim (6, 9-12) fm$.
- The Pion/Kaon dynamics precisely: Our simulations are performed with Physical Pion/Kaon Masses.
- Large distance signal: 10^4 Traj., 768 (9000) random sources for *ud*-conn. (*uds*-disc.) correlators.
- Need controled continuum limit: 15 lattice spacings ($a \sim 0.064 - 0.134$ fm).
- For a few % precision, we take account of: c quark w. matching onto perturb. theory.



Simulation Setup

- Tree-level improved Symanzik gauge action.
- Nf=(2+1+1) stout-smeared staggered quarks ($m_c/m_s = 11.85$).
- Scale setting $f_\pi = 130.41$ MeV via scale w_0 (L.Varnhorst, 21(Wed.)).
- Rational Hybrid Monte Carlo.



β	a[fm]	N_t	N_s	#traj.	M_π[MeV]	M_K[MeV]	#SRC (l,s,c,d)
3.7000	0.134	64	48	10000	~ 131	~ 479	$(768, 64, 64, 9000)$
3.7500	0.118	96	56	15000	~ 132	~ 483	$(768, 64, 64, 6000)$
3.7753	0.111	84	56	15000	~ 133	~ 483	$(768, 64, 64, 6144)$
3.8400	0.095	96	64	25000	~ 133	~ 488	$(768, 64, 64, 3600)$
3.9200	0.078	128	80	35000	~ 133	~ 488	$(768, 64, 64, 6144)$
4.0126	0.064	144	96	04500	~ 133	~ 490	$(768, 64, 64, -)$

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Light-Conn. (and Disc.) Correlator: An Example

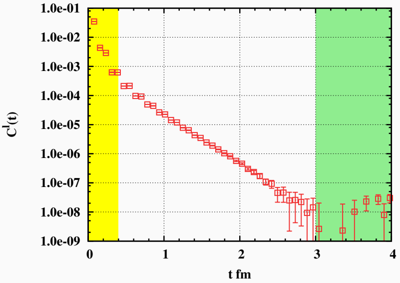
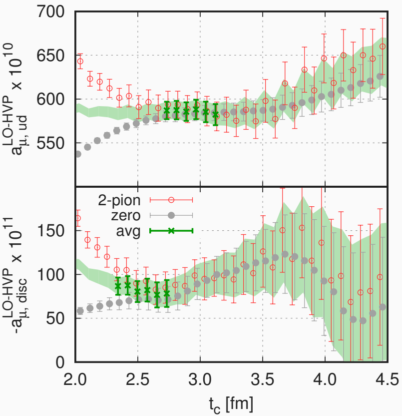


Figure: $C^{ud}(t) = \frac{5}{9}\sum_{\vec{x}}\frac{1}{3}\sum_{i=1}^{3}\langle j_i^{ud}(\vec{x},t)j_i^{ud}(0)\rangle$

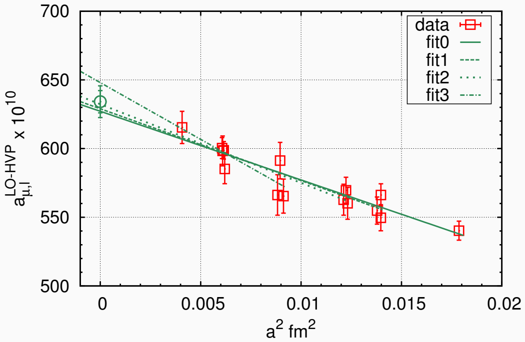
- The connected-light correlator $C^{ud}(t)$ loses signal for $t > 3fm$. To control statistical error, consider $C^{ud}(t > t_c) \to C^{ud}_{\mathrm{up/low}}(t, t_c)$, where $C^{ud}_{\mathrm{up}}(t, t_c) = C^{ud}(t_c)\ \varphi(t)/\varphi(t_c)$, $C^{ud}_{\mathrm{low}}(t, t_c) = 0.0$, with $\varphi(t) = \cosh[E_{2\pi}(T/2 - t)]$, and $E_{2\pi} = 2(M_\pi^2 + (2\pi/L)^2)^{1/2}$.
- Similarly, $C^{disc}(t) \to C^{disc}_{\mathrm{up/low}}(t, t_c)$, $-C^{disc}_{\mathrm{up}}(t > t_c) = 0.1C^{ud}(t_c)\ \varphi(t)/\varphi(t_c)$, $-C^{disc}_{\mathrm{low}}(t > t_c) = 0.0$.
- $C^{ud,disc}_{low}(t, t_c) \le C^{ud,disc}(t) \le C^{ud,disc}_{up}(t, t_c)$.

IR-CUT (t_c) Deps. of Light/Disc. Component: $a_{\ell,ud/disc}^{\text{LO-HVP}}$



- Corresponding to $C_{up/low}^{ud,disc}(t_c)$, we obtain upper/lower bounds for $g-2$: $a_{\ell,up/low}^{ud,disc}(t_c)$.
- Two bounds meet around $t_c = 3fm$. Consider the average of bounds: $\bar{a}_{\ell}^{ud,disc}(t_c) = 0.5(a_{\ell,up}^{ud,disc} + a_{\ell,low}^{ud,disc})(t_c)$, which is stable around $t_c = 3fm$.
- We pick up such averages $\bar{a}_{\ell}^{ud,disc}(t_c)$ with $4-6$ kinds of t_c around $3fm$. The average of average is adopted as $a_{\ell,\text{ud/disc}}^{\text{LO-HVP}}$ to be analysed, and a fluctuation over selected t_c is incorporated into the systematic error.

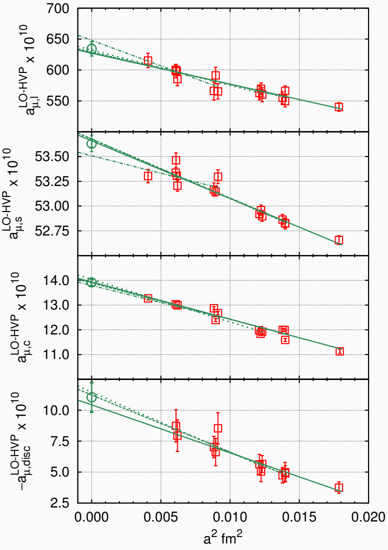
Continuum Extrap. of Light Component: $a_{\mu,ud}^{\text{LO-HVP}}$



$$F(a_{\mu,ud}^{\text{LO-HVP}}, A, C_{M_\pi}, C_{M_K}) = a_{\mu,ud}^{\text{LO-HVP}}(1 + Aa^2)\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24)\ , \quad \chi^2/\text{d.o.f.} = 7.8/12 \text{ (fit1 case)}.$$

Continuum Extrapolations, Summary



- With 6 $\beta's = 15$ $a^2[fm]$ simulations, allowing full control over continuum limit.
- Get systematic uncertainty from various cuttings: no-cut, or cutting $a \geq 0.134,\ 0.111,$ or 0.095.
- Get good χ^2/dof with extrapolation linear in a^2, and interpolation linear in M_π^2 and M_K^2 (strange/charm).
- Strong a^2 dependences for $a_{\mu,ud/disc}^{\text{LO-HVP}}$ due to taste violations, and for $a_{\mu,c}^{\text{LO-HVP}}$ due to large m_c.

Perturbative Corrections

Consider separation of momentum integral range as

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{2}$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ , \tag{3}$$

where

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max})$: lattice simulations, investigated so far ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: $\gamma_l(Q_{max})\hat{\Pi}^f(Q_{max}) + \Delta^{pert} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$.

For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{max})$ is small. For example,

$$a_{\mu}^{\text{LO-HVP}}(Q > Q_{max}) \xrightarrow{Q_{max}=2GeV} 0.678(1)(1) \ , (0.1\%). \tag{4}$$

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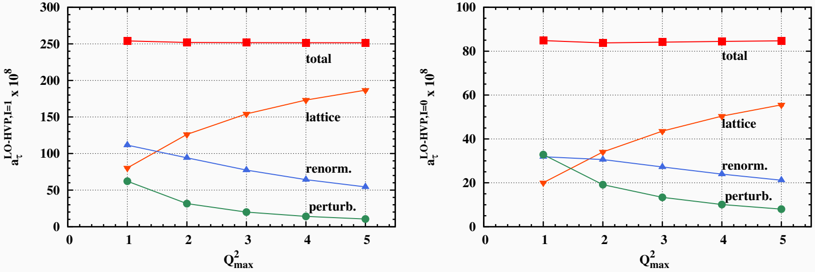
Perturbative Corrections for tau

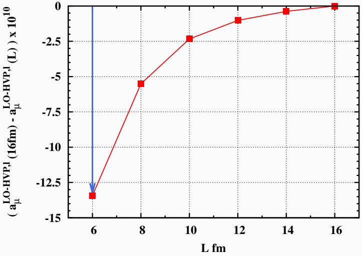
$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max}) + \gamma_l(Q_{max})\hat{\Pi}^f(Q_{max}) + \Delta^{pert} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ .$$

- For tau, $Q > Q_{max}$ effects are large, while the total is stable.
- The fluctuation over $Q^2_{\text{max}} = 2.0 - 5.0 GeV^2$ are incorporated into systematic errors.



FV for $a_{\mu,ud}^{\text{LO-HVP}}$ by XPT

- The $a_\mu^{\text{LO-HVP}}$ comes from Euclidean mommenta; Exponentially suppressed FV with $LM_\pi \sim 4$ for $L \sim 6fm$.
- We work with $L \sim$ *fixed*, and FV effects cannot be estimated from simulations and need model.
- Long-distance $I = 1$ $(I = 0)$ contribution dominated by 2-pions (3-pions). The dominant FV in $I = 1$ channel could be estimated by XPT for $\pi^+\pi^-$ loop (Aubin et al '16).



$$(a_{\mu,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,I=1}^{\text{LO-HVP}}(6fm))|_{\text{XPT}}$$
$$= 13.42(13.42) \times 10^{-10} \,, (1.9\%).$$

Isospin breaking effects (Preliminary)

Get missing effects from phenomenology

Effect	corr. to $a_\mu^{\text{LO-HVP}} \times 10^{10}$
$\rho{-}\omega$ mix.	2.71
$\rho{-}\gamma$ mix.	−2.74
FSR	4.22
EM in M_π, M_ρ, Γ_ρ	−11.17
$\pi^0\gamma$	4.64(4)
$\eta\gamma$	0.65(1)
Total	−1.69(20)

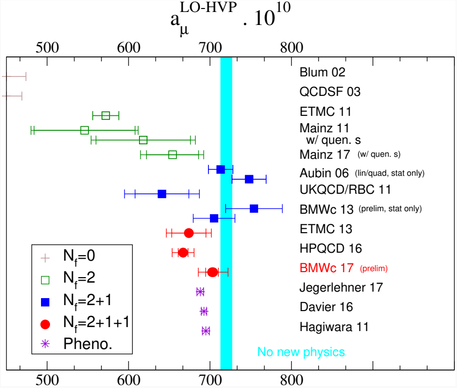
- Thanks to F.Jegerlehner (& M. Benayoun) for correspondance and numbers
- Results based on Gounaris-Sakurai fit to e^+e^-, from $2M_\pi$ to $1\,\mathrm{GeV}$
- EM modes from M. Benayoun et al '12
- F.J. estimates error to $\sim 10\%$ of total (i.e. 0.2×10^{-10}), we take 50% of largest contribution (i.e. $5.6 \times 10^{-10} = 300\%$ of total)
- Thus: $\Delta_{\text{IB}} a_\mu^{\text{LO-HVP}} = (-1.7 \pm 5.6) \times 10^{-10} (\textit{Preliminary})$

Summary on $a_\mu^{\text{LO-HVP}}$ (Preliminary)

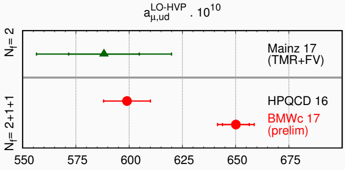
$a_\mu^{\text{LO-HVP}}$ BMWc, Preliminary

$I = 1$	$583(7)_{st}(7)_{acut}(0)_{tcut}(0)_{qcut}(13)_{fv}$
$I = 0$	$118(4)_{st}(3)_{acut}(0)_{tcut}(0)_{qcut}$
total	$703(8)_{st}(8)_{acut}(0)_{tcut}(0)_{qcut}(13)_{fv}(6)_{iso}$

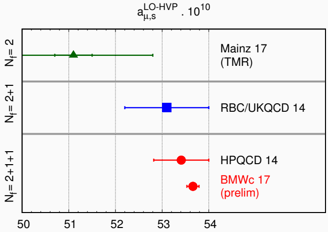
- Total error is 2.6%, dominated FV effects.
- Our results are consistent with both "No New Physics" and Dispersive Method. There is some tension with HPQCD 16.

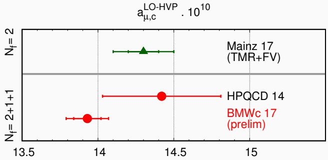


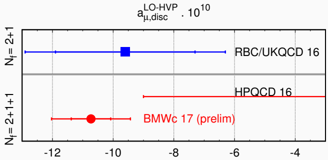
More detailed comparison



- BMWc '17 *ud* contribution is signficantly larger than other Nf=2+1+1 results
  → difference with HPQCD '14/Mainz '17 is $\sim 2.4/1.5\sigma$.
- BMWc '17 *c* contribution is slightly smaller than other Nf=2+1+1 results
- BMWc '17 is only calculation performed directly at physical quark masses with 6 β's to fully control continuum extrapolation.
- BMWc '17 $\delta a_{\mu,\,\text{disc}}^{\text{LO-HVP}} = 1.5 \times 10^{-10}$
  → contributes only 0.2% to error on $a_\mu^{\text{LO-HVP}}$.







Summary on $a_{e,\tau}^{\text{LO-HVP}}$ (Preliminary)

$a_e^{\text{LO-HVP}}$ BMWc, Preliminary

$I=1$	$157.0(2.4)_{st}(2.2)_{acut}(0.0)_{tcut}(0.0)_{qcut}(4.6)_{fv}$
$I=0$	$30.1(1.2)_{st}(1.1)_{acut}(0.1)_{tcut}(0.0)_{qcut}$
total	$187.6(2.7)_{st}(2.4)_{acut}(0.1)_{tcut}(0.0)_{qcut}(4.6)_{fv}(1.6)_{iso}$

$a_\tau^{\text{LO-HVP}}$ BMWc, Preliminary

$I=1$	$252(1)_{st}(1)_{acut}(0)_{tcut}(0)_{qcut}(2)_{fv}$
$I=0$	$83(0)_{st}(1)_{acut}(0)_{tcut}(1)_{qcut}$
total	$336(1)_{st}(1)_{acut}(0)_{tcut}(1)_{qcut}(2)_{fv}(2)_{iso}$

Burger et.al.('15): $a_e^{\text{LO-HVP}} = 178.2(6.4)(8.6)$, $a_\tau^{\text{LO-HVP}} = 341(8)(6)$
HPQCD ('16): $a_e^{\text{LO-HVP}} = 177.9(3.9)$
Jeherlehner('16): $a_e^{\text{LO-HVP}} = 185.11(1.24)$.
Eidelman et.al.('07): $a_\tau^{\text{LO-HVP}} = 338(4)$.

Summary of Summary and Perspective

- We have obtained $a_\mu^{\text{LO-HVP}}$ (as well as slope/curvature of $\hat{\Pi}(Q^2 = 0)$, 1612.02364 [hep-lat]) directly at physical point masses.
- Full controlled continuum extrapolation and matching to perturbation theory.
- Model/pheno. assumptions are put on only for small corrections from FV, QED and isospin breaking.
- Our results are consistent with both "No New Physics" and Dispersive Methods with a conservative systematic errors. There is some tension with HPQCD 16 on $a_{\mu,ud}^{\text{LO-HVP}}$.
- Total error is 2.6%, dominated by FV effects.
- Need $\sim 0.2\%$ precision to match forthcoming experiments!!
  1. increase statistics by $50 - 100$ times.
  2. control FV effects directly with simulations.
  3. simulations with QED and isospin breaking corrections taken account (T.Balint, 21(Wed)).

Backup

Continuum Extrap. of Light Component: $a_{\mu,ud}^{\text{LO-HVP}}$

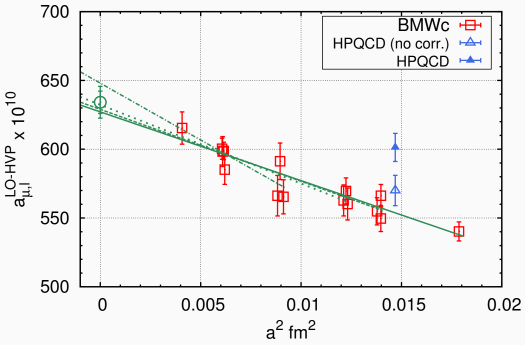
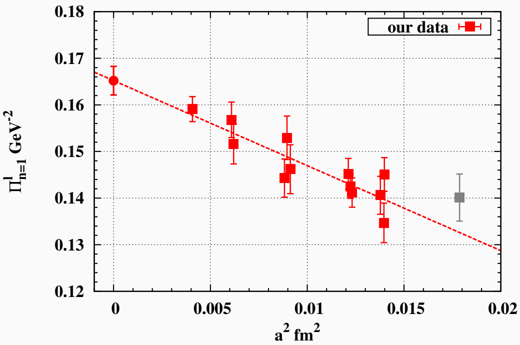


Figure: Red-squares = Our data. Blue-triangles = HPQCD, 1403.1778.

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24)\ , \quad \chi^2/\text{d.o.f.} = 7.8/12\ \text{(fit1 case)}.$$

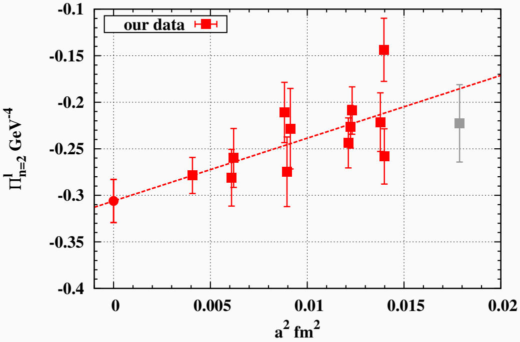
Continuum Extrapolation of $\Pi^l_{n=1}$ I



$$F(C_\Pi^{(2)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi^{(2)}}{a^2}\,\frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^l_{n=1}|_{a^2\to 0} = 0.1652(31) \ , \quad \chi^2/\mathrm{d.o.f.} = 24.3/20$$

Continuum Extrapolation of $\Pi^{l}_{n=2}$



$$F(C_\Pi^{(4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi^{(4)}}{a^4}\,\frac{1+Aa^2+\cdots}{1+Ba^2+\cdots}\left(1+C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right)\ .$$

$$\Pi^{l}_{n=2}\big|_{a^2\to 0} = -0.306(23)\ .$$

Continuum Extrapolation of $\Pi^s_{n=1,2}$

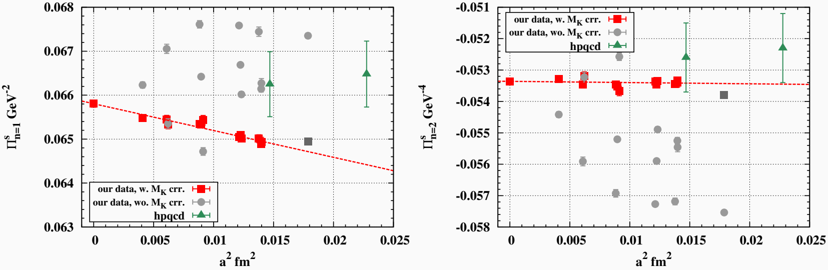


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} (1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K) \ .$$

$$\Pi^s_{n=1}|_{a^2\to 0} = 0.0658(1) \ , \quad \Pi^s_{n=2}|_{a^2\to 0} = -0.0534(2) \ , \quad \chi^2/\text{d.o.f.} = 20.9/18$$

Continuum Extrapolation of $\Pi^c_{n=1,2}$

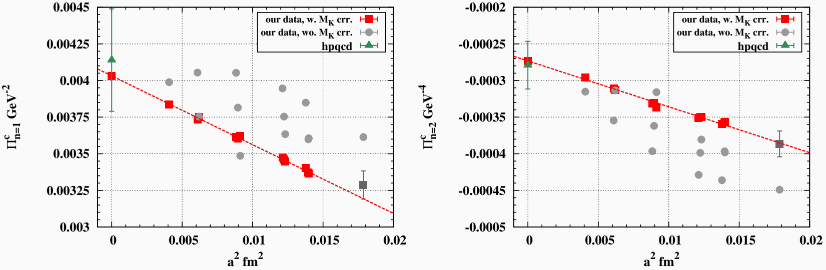


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1208.2855.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \, \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K\right) .$$

$$\Pi^c_{n=1}|_{a^2 \to 0} = 0.00403(2) \ , \quad \Pi^c_{n=2}|_{a^2 \to 0} = -2.73(2) \times 10^{-4} \ .$$

Disconnected Contributions

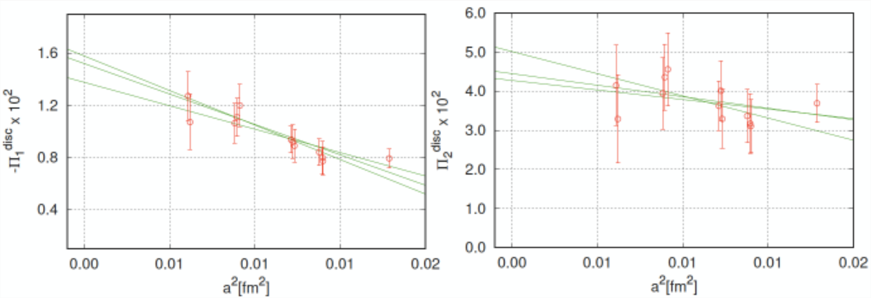


Figure: From Presentaion of T.Kawanai in Lattice 2016.

$$\Pi_1^{\rm disc} = -1.5(2)(1) \times 10^{-3}\ {\rm GeV}^{-2}\ , \quad (5)$$

$$\Pi_2^{\rm disc} = -4.6(1.0)(0.4) \times 10^{-3}\ {\rm GeV}^{-4}\ . \quad (6)$$

Summary Table of Moments (Preliminary)

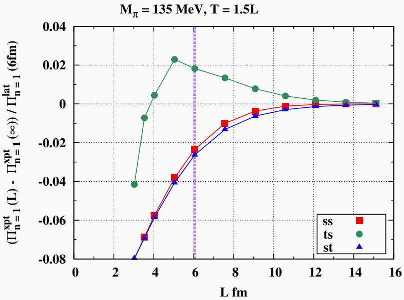
	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)

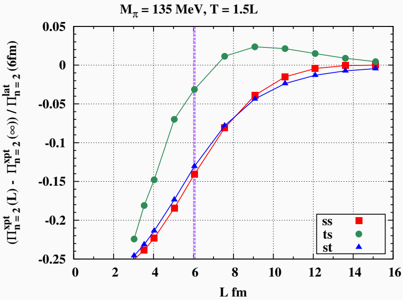
Table: Preliminary results on the first two moments of the HVP function.

TOTAL ERROR: 1.4% for Π_1, and 4.0% for Π_2 .

FV via Box Asymmetry, XPT Estimate for Various L I

c.f. Aubin et.al., PRD (2016).



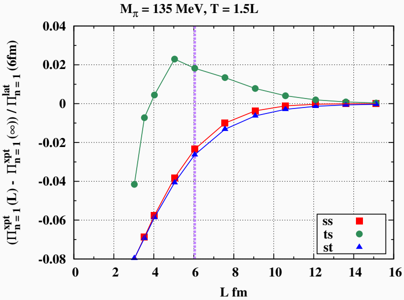


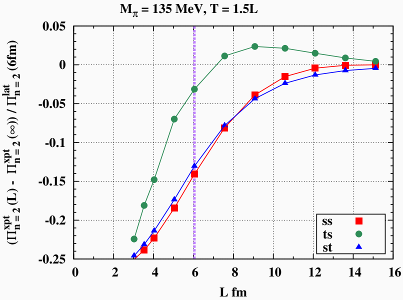
$$\Delta^{i}_{n=1,2}(L) = \left[\Pi^{\mathrm{xpt},i}_{n=1,2}(L) - \Pi^{\mathrm{xpt}}_{n=1,2}(\infty)\right] , \quad i = ss, ts, st ,$$

$$\frac{\Delta^{i}_{n}(L = 6\mathrm{fm})}{\Pi^{\mathrm{lat},i}_{n}} \sim \begin{cases} 2\% & \text{(for the 1st moment, } n = 1) , \\ 10\% & \text{(for the 2nd moment, } n = 2) . \end{cases} \tag{7}$$

FV via Box Asymmetry, XPT Estimate for Various L II

c.f. Aubin et.al., PRD (2016).





$$\Delta^i_{n=1,2}(L) = \left[\Pi^{\mathrm{xpt},i}_{n=1,2}(L) - \Pi^{\mathrm{xpt}}_{n=1,2}(\infty)\right] \ , \quad i = ss, ts, st \ ,$$

$$FV.(L) \pm dFV.(L) = \left[\max\{\Delta_i\} + \min\{\Delta_i\}\right]/2 \pm \left[\max\{\Delta_i\} - \min\{\Delta_i\}\right]/2$$

$$\xrightarrow[L\to 6fm]{} \begin{cases} 0.0006(22) & \text{(for the 1st moment, } n=1) \ , \\ -0.015(19) & \text{(for the 2nd moment, } n=2) \ . \end{cases} \qquad (8)$$

Summary Table of Moments with FV I (Preliminary)

	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)
$I = 1$ FV corr.	0.0006(23)	−0.015(10)
$I = 1$ + FV corr.	0.0834(8)(9)(23)	−0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	−0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. HPQCD(arXiv:1601.03071 and PRD2014):

$$\Pi_1^l = 0.1606(25)\ \mathrm{GeV}^{-2}, \quad \Pi_2^l = -0.368(16)\ \mathrm{GeV}^{-4},$$
$$\Pi_1^s = 0.06625(74)\ \mathrm{GeV}^{-2}, \quad \Pi_2^s = -0.0526(11)\ \mathrm{GeV}^{-4}.$$

Summary Table of Moments with FV II (Preliminary)

	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)
$I = 1$ FV corr.	0.0006(23)	−0.015(10)
$I = 1$ + FV corr.	0.0834(8)(9)(23)	−0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	−0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. Phenomenology(Benayoun et.al.1605.04474):
$\Pi_1 = 0.990(7)\ \mathrm{GeV}^{-2}, \quad \Pi_2 = -0.206(2)\ \mathrm{GeV}^{-4}.$

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{9}$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ , \tag{10}$$

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: computed by lattice $\hat{\Pi}^f(Q_{max})$ and perturbations .

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \qquad (9)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ , \qquad (10)$$

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: computed by lattice $\hat{\Pi}^f(Q_{max})$ and perturbations .

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{11}$$

$$\begin{aligned}
&= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) + \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \right] \ , \\
&= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) \quad \left(\to \Delta a_{\ell,f}^{\text{LO-HVP}}\right) \\
&\quad + \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \quad \left(\to \gamma_\ell(Q_{max}^2) \hat{\Pi}^f(Q_{max}^2)\right) \ .
\end{aligned} \tag{12}$$

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \,, \tag{11}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) + \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \right] \,,$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) \quad \left(\to \Delta a_{\ell,f}^{\text{LO-HVP}}\right)$$

$$+ \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \quad \left(\to \gamma_\ell(Q_{max}^2) \hat{\Pi}^f(Q_{max}^2)\right) \,. \tag{12}$$

Why $\hat{\Pi}^f$?

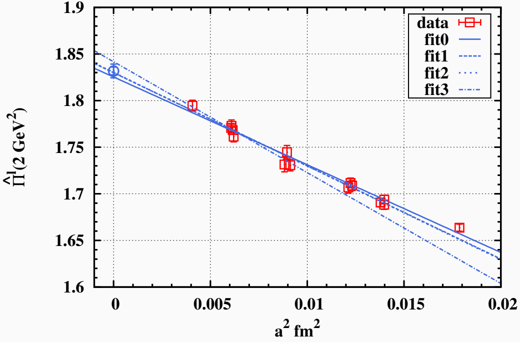
$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)\hat{\Pi}^f(Q^2) \;, \tag{11}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \boldsymbol{\Pi}^f(Q_{max}^2)\right) + \left(\boldsymbol{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right)\right] \;,$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \boldsymbol{\Pi}^f(Q_{max}^2)\right) \quad \left(\to \Delta a_{\ell,f}^{\text{LO-HVP}}\right)$$

$$+\left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)\right]\left(\boldsymbol{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \quad \left(\to \gamma_\ell(Q_{max}^2)\hat{\Pi}^f(Q_{max}^2)\right) . \tag{12}$$

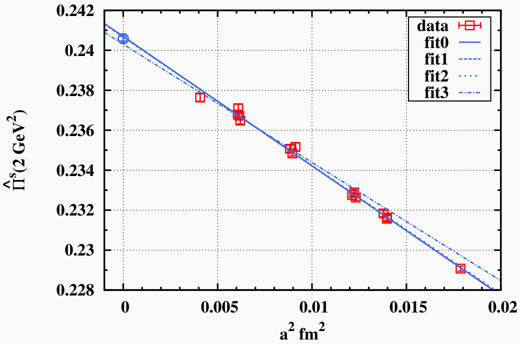
Continuum Extrap. of Light Component: $\hat{\Pi}^l$



$$F(\hat{\Pi}^l, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^l \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\hat{\Pi}^l = 1.8318(42)(60) , \quad \chi^2/\mathrm{d.o.f.} = 8.2/12 \text{ (fit1 case).}$$

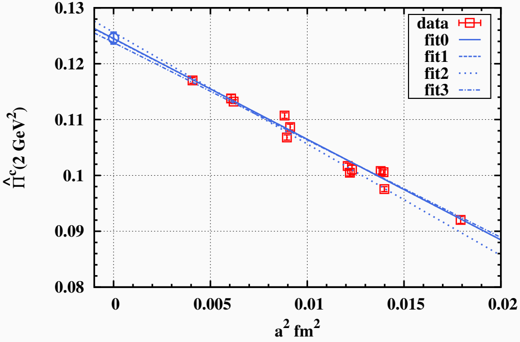
Continuum Extrap. of Strange Component: $\hat{\Pi}^s$



$$F(\hat{\Pi}^s, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^s \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K\right) .$$

$\hat{\Pi}^s = 0.2406(1)(2)$, $\chi^2/\mathrm{d.o.f.} = 13.6/11$ (fit1 case).

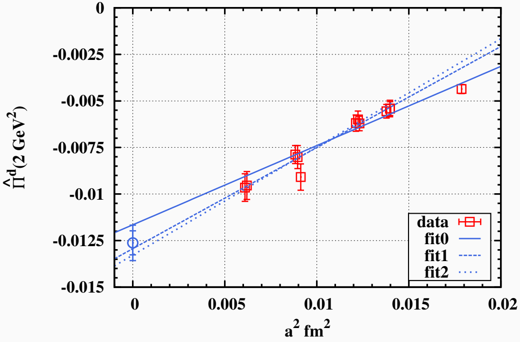
Continuum Extrap. of Charm Component: $\hat{\Pi}^c$



$$F(\hat{\Pi}^c, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^c \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) \ .$$

$\hat{\Pi}^c = 0.1246(9)(7)$, $\chi^2/\mathrm{d.o.f.} = 26.1/9$ (fit1 case).

Continuum Extrap. of Disc. Component: $\hat{\Pi}^d$



$$F(\hat{\Pi}^d, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^d \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K) .$$

$$\hat{\Pi}^d = -0.0126(6)(7) , \quad \chi^2/\text{d.o.f.} = 3.5/9 \text{ (fit1 case).}$$