FUNCTION SUPPORT AND LOCALIZATION

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QUANTUM UNCERTAINTY

$|\psi\rangle \, , \, \{ \, |i\rangle \} \quad : \quad$ How many ($\mathcal{N}$ out of N) $|i\rangle$ in state $|\psi\rangle$?

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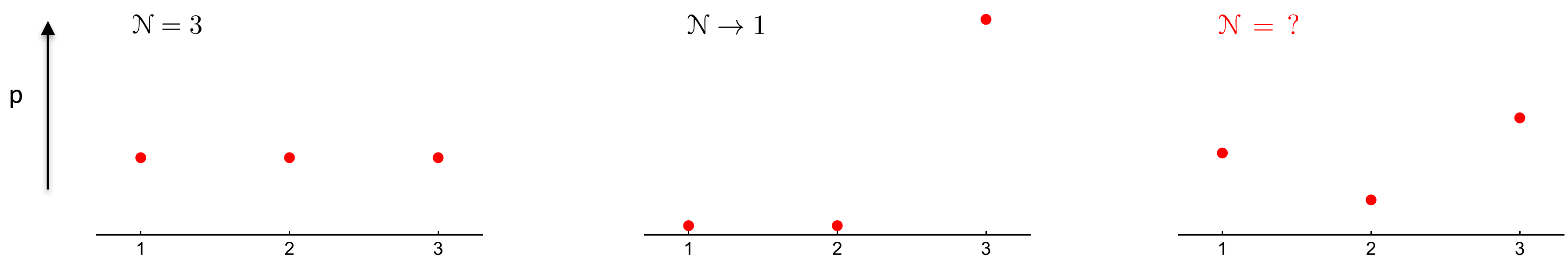
EFFECTIVE NUMBER OF OBJECTS WITH WEIGHTS

How many ($\mathcal{N}$ out of N) objects with weights $(p_1, p_2, \ldots p_N)$?

[structure of bonds, Anderson localization, many-body localization, quantum Hall effect, quantum chaos, ETH & thermalization, quantum information, dynamical localization, topological phases, Dirac eigenmodes, vacuum structure of gauge theories, holography…]

illustrative context : lattice Schrödinger particle wrt position basis

$|\psi\rangle \;\rightarrow\; \{\, \psi_i \,,\, i = 1, \ldots, N \,\}$ How many positions $(\mathcal{N})$ is particle simultaneously in?



Quantum mechanics: $|\psi\rangle \;\rightarrow\; P = (p_1, p_2, \ldots p_N)$ $\quad p_i = |\,\langle i \mid \psi\rangle\,|^2 = \psi_i^\star \psi_i$

Is quantum mechanics providing for $\mathcal{N} \;=\; \mathcal{N}[\,|\psi\rangle, \{|i\rangle\}\,] \;=\; \mathcal{N}[P]$?

Options:

(i) ill-posed question in QM

(ii) well-posed question in QM (how?)

?

Strategy: Define set $\mathfrak{N}$ of all functions $\mathcal{N} = \mathcal{N}[P]$ representing effective number of states

Convenient: $$P = (p_1, \ldots, p_N) \;\longrightarrow\; W \equiv NP = (w_1, \ldots, w_N)$$

$\mathcal{N} = \mathcal{N}[W]$: $$W \in \mathcal{R} \equiv \{\, (w_1, \ldots, w_N) \mid w_i \geq 0 \,,\, \sum_{i=1}^{N} w_i = N \,,\; N \in \mathbb{N} \,\}$$

Important "would be" examples:

☐ $$\frac{1}{\mathcal{N}_p[W]} = \frac{1}{N^2} \sum_{i=1}^{N} w_i^2$$

participation number

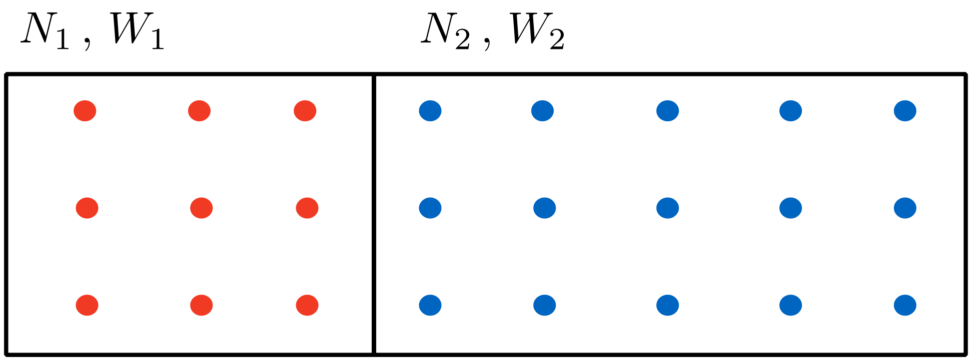
Bell & Dean, 1970

☐ $$\mathcal{N}_+[W] = \sum_{i=1}^{N} \mathfrak{n}(w_i) \qquad \mathfrak{n}(w) \equiv \begin{cases} 0 \,, & w = 0 \\ 1 \,, & w > 0 \end{cases}$$

"mathematical support" number

$\mathcal{N}_p \notin \mathfrak{N}$ $\quad$ $\mathcal{N}_+ \notin \mathfrak{N}$

Key new ingredient: ADDITIVITY for effective number of states



$$N \;\rightarrow\; \mathcal{N}[W]$$

Effective number of states has to be measure-like!

$$\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2) \qquad S_1 \cap S_2 = \emptyset$$

$$N_{12} \;=\; N_1 + N_2$$

$$\mathcal{N}[W_{12}] \;=\; \mathcal{N}[W_1] + \mathcal{N}[W_2]$$

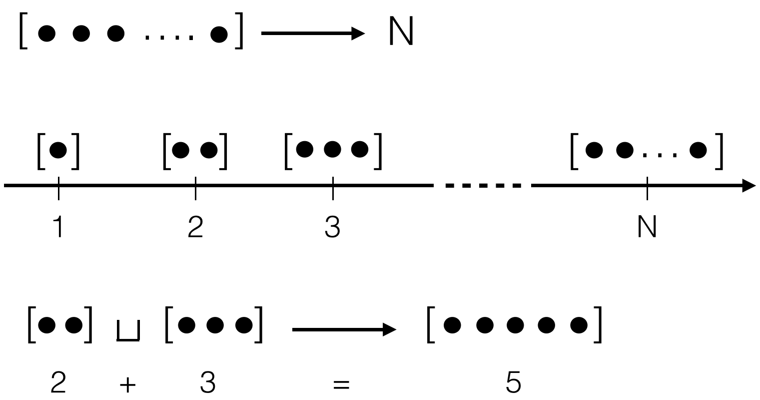
Note: $W_1 \in \mathcal{R}_{N_1} \;\;,\;\; W_2 \in \mathcal{R}_{N_2} \;\Rightarrow\; W_1 \boxplus W_2 \in \mathcal{R}_{N_1+N_2}$

$$(a_1, \ldots, a_N) \;\boxplus\; (b_1, \ldots, b_M) \equiv (a_1, \ldots, a_N, b_1, \ldots, b_M)$$

$$\mathcal{N}[W_1 \boxplus W_2, N_1 + N_2] \;=\; \mathcal{N}[W_1, N_1] + \mathcal{N}[W_2, N_2] \quad , \quad \forall\, W_1, W_2, N_1, N_2$$

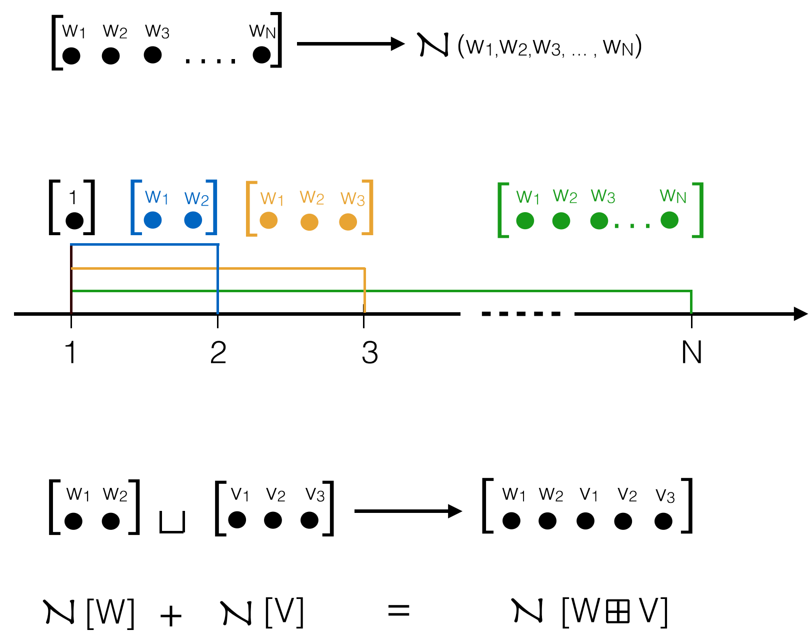
number of objects

natural numbers



number of objects with weights

effective numbers



Effective Numbers

(A) $\mathcal{N}[W_1 \boxplus W_2] \;=\; \mathcal{N}[W_1] + \mathcal{N}[W_2]$

(S) $\mathcal{N}(\ldots w_i \ldots w_j \ldots) \;=\; \mathcal{N}(\ldots w_j \ldots w_i \ldots)$

(B1) $\mathcal{N}(1, 1, \ldots, 1) \;=\; N$

(B2) $\mathcal{N}(N, 0, \ldots, 0) \;=\; 1$

(B) $1 \;\leq\; \mathcal{N}[W] \;\leq\; N$

(C) $\mathcal{N}[W]$ is continuous on $\mathcal{R}$

(M-) $\mathcal{N}(\ldots w_i - \epsilon \ldots w_j + \epsilon \ldots) \leq \mathcal{N}(\ldots w_i \ldots w_j \ldots) \quad , \quad w_i \leq w_j$

monotonicity wrt cumulation

$\mathfrak{N}$: set of functions satisfying (A) , (S) , (B2) , (C) , (M-)

[(B1) and (B) follow]

$\mathcal{N}_p \notin \mathfrak{N}$ (not additive) $\qquad$ $\mathcal{N}_+ \notin \mathfrak{N}$ (not continuous)

Theorem

Let $\mathcal{N}_\star \in \mathfrak{N}$ and $\mathcal{N}_+ \notin \mathfrak{N}$ be functions on $\mathcal{R}$ defined previously. Then

$$(a) \qquad \mathcal{N}_\star[W] \leq \mathcal{N}[W] \leq \mathcal{N}_+[W] \qquad \forall\, \mathcal{N} \in \mathfrak{N} \;,\; \forall\, W \in \mathcal{R}$$

$$(b) \qquad \mathcal{N}_\star[W] < \mathcal{N}_+[W] \quad \Longrightarrow \quad \{\, \mathcal{N}[W] \mid \mathcal{N} \in \mathfrak{N} \,\} \supseteq [\alpha, \beta\,)$$

where W in (b) is arbitrary but fixed and $\alpha = \mathcal{N}_\star[W]$, $\beta = \mathcal{N}_+[W]$.

$$\mathcal{N}_\star[W] = \sum_{i=1}^{N} \mathfrak{n}_\star(w_i) \qquad \mathfrak{n}_\star(w) \equiv \begin{cases} w \,, & 0 \leq w \leq 1 \\ 1 \,, & w > 1 \end{cases}$$

- THERE IS A BOTTOM TO EFFECTIVE NUMBER OF OBJECTS WITH WEIGHTS
- CONSISTENT VALUES UP TO NUMBER OF NON-ZERO WEIGHTS
- PERFECTLY NATURAL : $\mathcal{N}_\star$ is the actual content in effective number of objects

recall: $$\mathcal{N}_+[W] = \sum_{i=1}^{N} \mathfrak{n}_+(w_i) \qquad \mathfrak{n}_+(w) \equiv \begin{cases} 0 \,, & w = 0 \\ 1 \,, & w > 0 \end{cases}$$

$|\psi\rangle\ ,\ \{\,|i\rangle\}\quad :\quad \text{How many } (\mathcal{N} \text{ out of } N)\ |i\rangle \text{ in state } |\psi\rangle\,?$

(i) ill-posed question in QM

(ii) well-posed question in QM (how?) ✓

Answer:

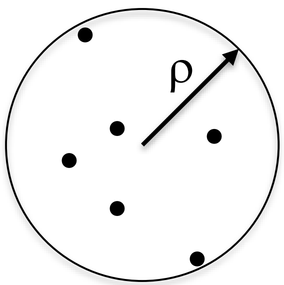
$$\mathcal{N}[\,|\psi\rangle, \{|i\rangle\}\,] = \mathcal{N}_\star[W] \qquad W = (w_1, \ldots, w_N) \qquad w_i = N\,|\langle i\,|\,\psi\rangle\,|^2$$

Localization

- ❑ Additivity at the root of all applications discussed here

(1) $|\psi\rangle = |\psi\rangle_{(N)}$: $\lim_{N\to\infty} \frac{\mathcal{N}_\star[W,N]}{N} = 0$ collapse of $|\psi\rangle$ wrt $\{\, |i\rangle \}$

(2) $N \propto L^D$ $\mathcal{N}[W,L] \propto L^d$, $d \leq D$ effective dimension of domain



$\mathcal{N}_\star[W,N]$ most probable states inside

defines $\rho[W,N]$

(3) $\lim_{N\to\infty} \rho[W,N] < \infty$ definition of localization

NOTE: This definition is universal : does not invoke exponential decays

NOTE: Applies equally to Euclidean Dirac eigenmodes, composite fields and such

Quantum Uncertainty

- Minimality Theorem gives basis for different angle on uncertainty principle
- QM uncertainty due to states ``sharing parts of themselves" in very real sense
- The degree of this sharing is described by the uncertainty statement

Let the description of a quantum system be regularized so that the dimension of Hilbert space is N*. The uncertainty of a system in state* $|\psi\rangle$ *with respect to basis* $\{\,|i\rangle\}$ is $\mathcal{N}_\star(w_1,\ldots,w_N)$ states, where $w_i = N\,|\langle i \mid \psi\rangle|^2$.

Entropy of a Quantum State

- Specifying quantum state requires (frequently infinitely) many numbers
- Statistical concepts natural even though no temperature
- Minimality Theorem specifies how many states $|i\rangle$ are accessible to system in $|\Psi\rangle$

Entropy of a quantum state $|\psi\rangle$ wrt basis $\{|i\rangle\}$:

$$S_q[\,|\psi\rangle, \{|i\rangle\}\,] = \log \mathcal{N}_\star[W] \qquad W = (w_1, \ldots, w_N)\ ,\ w_i = N\,|\langle i\,|\,\psi\rangle\,|^2$$

- Vanishes after measurement of associated observable (describe the measurement process?)
- Proportional to volume for states of many-body systems
- Tool to characterize many-body states (QCD vacuum) : producing infinitely-many characteristics
- Rationale very different in nature from entanglement entropy (study entangled states?)
- Underlying idea: quantum uncertainty produces quantum entropy
- Similarly ``quantum temperature" with respect to a given basis

Conclusion

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