

Non perturbative determination of improvement b -coefficients in $N_f = 3$

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RG flow

- Renormalized couplings satisfy the RG flow equations in **mass-independent** scheme

$$\begin{cases} \mu \frac{\partial}{\partial \mu} g_R(\mu) &= \beta(g_R(\mu), m_R(\mu)) \\ \mu \frac{\partial}{\partial \mu} m_R(\mu) &= \tau(g_R(\mu), m_R(\mu)) m_R(\mu) \end{cases}$$

- Velocities β and τ have perturbative expansion

$$\begin{cases} \beta(g_R) & \stackrel{g_R \rightarrow 0}{\sim} -g_R^3 (b_0 + b_1 g_R^2 + b_2 g_R^4 + \mathcal{O}(g_R^6)) \\ \tau(g_R) & \stackrel{g_R \rightarrow 0}{\sim} -g_R^2 (d_0 + d_1 g_R^2 + d_2 g_R^4 + \mathcal{O}(g_R^6)) \end{cases}$$

with **universal** coefficients b_0 , b_1 , d_0 .

RGI quantities

- Integration of the equations yields to dimensionful finite constants, the renormalization group invariants (RGI)

$$\Lambda = \mu [b_0 g_R^2(\mu)]^{-\frac{b_1}{2b_0}} \exp \left\{ - \int_0^{g_R(\mu)} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] - \frac{1}{2b_0 g_R^2(\mu)} \right\}$$

$$M = m_R(\mu) [2b_0 g_R^2(\mu)]^{-\frac{d_0}{2b_0}} \exp \left\{ - \int_0^{g_R(\mu)} dg \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\}$$

- In the adimensional ratio, the function f depends only on $g_R^2(\mu)$

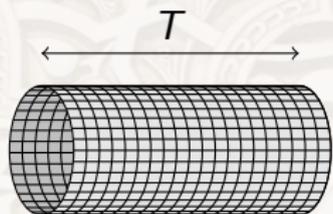
$$\frac{M}{\Lambda} = m_R(\mu) 1/\mu f(g_R^2(\mu))$$

Finite volume & mass independent renormalization scheme

- Finite size scaling approach to run between high energy and low energy scales:

$$\mu = 1/L \quad \frac{M}{\Lambda} = m_R(1/L) L f(g_R^2(L))$$

- Finite volume requires boundary conditions:



Schrödinger functional (SF) cylinder $L^3 \times T$
periodic bc in spatial directions,
Dirichlet bc in temporal direction

- Physical extent of the box and masses fixed by working at constant values of the renormalized couplings

$$\left(\mu, \frac{M}{\Lambda}\right) = \text{fixed} \iff \left(g_R^2(L), m_R(1/L) L\right) = \text{constant}$$

Improved & mass-independent

- In $O(a)$ improved & mass-independent schemes bare quantities must be renormalized with mass-dependent terms

$$\begin{aligned}g_R^2 &= Z_g(\tilde{g}_0^2, a\mu) \tilde{g}_0^2 & \tilde{g}_0^2 &= (1 + b_g(g_0^2)am_q) g_0^2 \\m_R &= Z_m(\tilde{g}_0^2, a\mu) \tilde{m} & \tilde{m} &= (1 + b_m(g_0^2)am_q) m_q \\X_R &= Z_X(\tilde{g}_0^2, a\mu) \tilde{X} & \tilde{X} &= (1 + b_X(g_0^2)am_q) X_l\end{aligned}$$

- Improvement parameters depend on the choice of the improvement conditions \equiv NP ambiguity of $O(a)$
-  ALPHA: improvement conditions at *constant physics*
 \implies improvement coefficients smooth functions of g_0^2

$$\left(g_R^2(L), m_R(1/L) L \right) = \text{constant}$$

History of non-perturbative b coefficients

- 1997 Quenched

[GMdD and R Petronzio, [hep-lat/9710071], Phys. Lett. B **419** (1998) 311]

- 2000 Quenched

[M Guagnelli *et al.* [ALPHA], [hep-lat/0009021], Nucl. Phys. B **595** (2001) 44]

- 2000 Quenched

[T Bhattacharya, *et al.*, [hep-lat/0009038], Phys. Rev. D **63** (2001) 074505]

- 2003 Quenched

[J Heitger *et al.* [ALPHA], [hep-lat/0312016], JHEP **0402** (2004) 064]

- 2010 Unquenched $N_f = 2$

[P Fritzscht, *et al.* [ALPHA], [arXiv:1004.3978 [hep-lat]], JHEP **1008** (2010) 074]

- 2016 Unquenched $N_f = 2 + 1$

[G S Bali *et al.* [RQCD Collaboration], [arXiv:1606.09039 [hep-lat]], Phys. Rev. D **94** (2016) no.7, 074501]

[P Korcyl and G S Bali, [arXiv:1607.07090 [hep-lat]], Phys. Rev. D **95** (2017) no.1, 014505]

- 2017(?) Unquenched $N_f = 3$

[GMdD, M Firrotta, J Heitger, C C Köster, A Vladikas [ALPHA], this work]

Improvement condition

- (non-singlet) PCAC relation free of $O(a)$ violations

$$\tilde{\partial}_\mu \langle A_R^{ij}(x) \mathcal{O}^{ji} \rangle = (m_{R,i} + m_{R,j}) \langle P_R^{ij}(x) \mathcal{O}^{ji} \rangle + O(a^2)$$

- Standard renormalization pattern of improved lattice operators (for $\hat{m}^{(\text{sea})} \sim 0$)

$$A_R^{ij} = Z_A (1 + b_A a m_{q,ij} + \bar{b}_A a \text{tr} \hat{m}^{(\text{sea})}) \{ A_\mu^{ij} + c_A a \tilde{\partial}_\mu P^{ij} \}$$

$$P_R^{ij} = Z_P (1 + b_P a m_{q,ij} + \bar{b}_P a \text{tr} \hat{m}^{(\text{sea})}) P^{ij}$$

$$m_{R,i} = Z_m \left\{ m_{q,i} (1 + b_m a m_{q,i} + \bar{b}_m a \text{tr} \hat{m}^{(\text{sea})}) + x \text{tr} \hat{m}^{(\text{sea})} + y a \text{tr} \hat{m}^{2(\text{sea})} + z a (\text{tr} \hat{m}^{(\text{sea})})^2 \right\}$$

$x = (r_m - 1)/N_f, \quad y = (r_m d_m - b_m)/N_f, \quad z = (r_m \bar{d}_m - \bar{b}_m)/N_f$

$$A_\mu^{ij} = \bar{\psi}_i \gamma_\mu \gamma_5 \psi_j, \quad P^{ij} = \bar{\psi}_i \gamma_5 \psi_j$$

$$m_{q,ij} = \frac{1}{2}(m_{q,i} + m_{q,j}), \quad m_{q,i} = m_{0,i} - m_c = \frac{1}{2a} \left(\frac{1}{\kappa_i} - \frac{1}{\kappa_c} \right)$$

SF Correlation functions

$$m_{ij}(x_0) = \frac{\tilde{\partial}_0 f_A^{ij}(x_0) + a c_A \partial_0^* \partial_0 f_P^{ij}(x_0)}{2 f_P^{ij}(x_0)}$$

$$= \frac{Z_m Z_P}{Z_A} \left(x \text{tr} \hat{m}^{(\text{sea})} + (z+x(\bar{b}_P - \bar{b}_A)) a (\text{tr} \hat{m}^{(\text{sea})})^2 + y a \text{tr} \hat{m}^{2(\text{sea})} \right) \\ + m_{q,ij} (1 + (x(b_P - b_A) + \bar{b}_P - \bar{b}_A + \bar{b}_m) a \text{tr} \hat{m}^{(\text{sea})}) \\ + a m_{q,ij}^2 (b_P - b_A) + \frac{1}{2} a (m_{q,i}^2 + m_{q,j}^2) b_m)$$

$$f_A^{ij}(x_0) = -\frac{a^3}{3L^3} \sum_x \langle A_0^{ij}(x) \mathcal{O}^{ji} \rangle$$

$$f_P^{ij}(x_0) = -\frac{a^3}{3L^3} \sum_x \langle P^{ij}(x) \mathcal{O}^{ji} \rangle$$

$$\mathcal{O}^{ji} = a^6 \sum_{u,v} \bar{\zeta}_j(u) \gamma_5 \zeta_i(v)$$

$$g_A^{ij}(T-x_0) = -\frac{a^3}{3L^3} \sum_x \langle A_0^{ij}(x) \mathcal{O}'^{ji} \rangle$$

$$g_P^{ij}(T-x_0) = -\frac{a^3}{3L^3} \sum_x \langle P^{ij}(x) \mathcal{O}'^{ji} \rangle$$

$$\mathcal{O}'^{ji} = a^6 \sum_{u,v} \bar{\zeta}'_j(u) \gamma_5 \zeta'_i(v)$$



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Simulation details

- openQCD for the SF simulations

(SF-specific choices $C = C' = 0$ and $\theta = 0$)

[M Lüscher and S Schaefer, [arXiv:1206.2809 [hep-lat]], Comput. Phys. Commun. **184** (2013) 519]

<http://luscher.web.cern.ch/luscher/openQCD/>

- $N_f = 3$ mass-degenerate flavors Wilson-clover fermions
tree-level improved Lüscher–Weisz gauge action

- non-perturbative c_{SW}, c_A

[J Bulava and S Schaefer, [arXiv:1304.7093 [hep-lat]], Nucl. Phys. B **874** (2013) 188]

[J Bulava *et al.* [ALPHA], [arXiv:1604.05827 [hep-lat]], Phys. Rev. D **93** (2016) no.11, 114513]

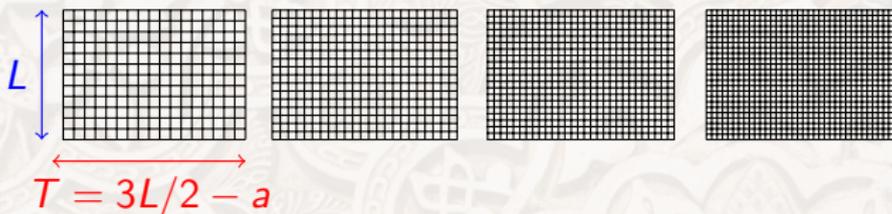
- RHMC reweighting factors

[A D Kennedy, I Horvath and S Sint, [hep-lat/9809092], Nucl. Phys. Proc. Suppl. **73** (1999) 834]

[M A Clark and A D Kennedy, [hep-lat/0608015], Phys. Rev. Lett. **98** (2007) 051601]

[M Lüscher and F Palombi, [arXiv:0810.0946 [hep-lat]], PoS LATTICE **2008** (2008) 049]

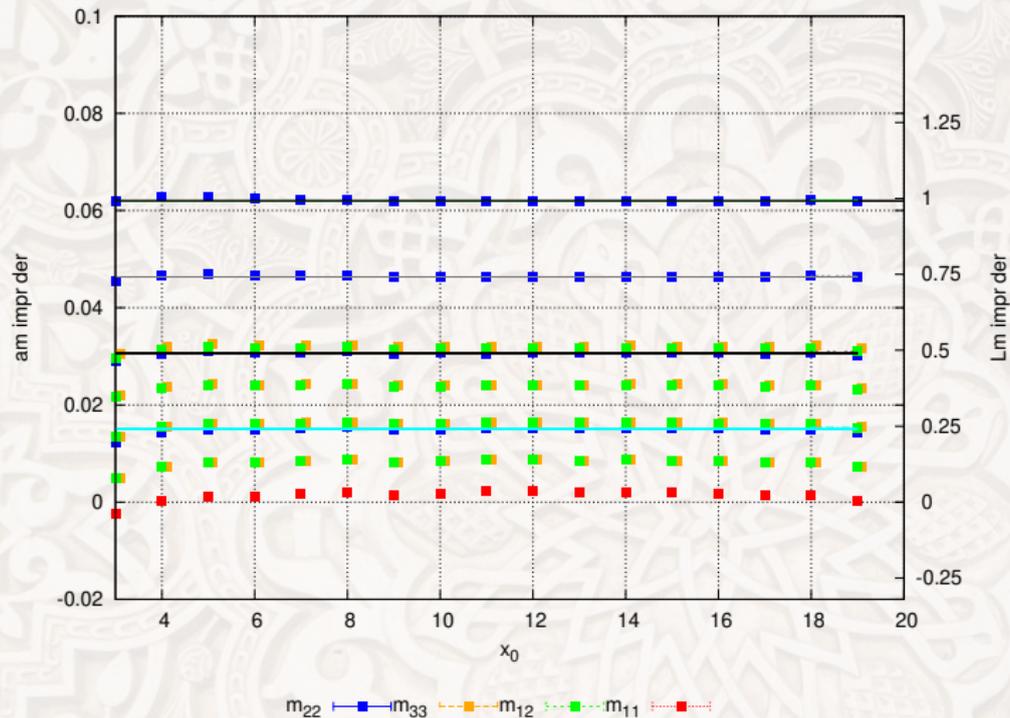
- $L \approx 1.2 \text{ fm}$, $Lm_{11} \approx 0.0$, $Lm_{22} \approx 0.25, 0.5, 0.75, 1.0$
- $a \approx 0.09 \text{ fm} \rightarrow 0.045 \text{ fm}$



$L^3 \times T/a^4$	β	κ	# REP	# MDU	ID
$12^3 \times 17$	3.3	0.13652	10	10240	A1k1
		0.13660	10	13048	A1k2
$12^3 \times 19$	3.3	0.13652	10	10468	A2k1
$16^3 \times 23$	3.512	0.13700	2	20480	B1k1
		0.13703	1	8192	B1k2
		0.13710	3	24560	B1k3
$16^3 \times 23$	3.47	0.13700	1	8176	B2k1
$20^3 \times 29$	3.676	0.13680	1	7848	C1k1
		0.13700	4	15232	C1k2
		0.13719	4	15472	C1k3
$24^3 \times 35$	3.810	0.13712	7	15448	D1k1

from: [J. Bulava et al. [ALPHA], [arXiv:1604.05827 [hep-lat]], Phys. Rev. D **93** (2016) no.11, 114513]

Masses (B1k3 ens)



Consistency check

$$\left\{ \begin{array}{l} \frac{1}{2}(m_{22}-m_{11}) = Z \delta (1+aA^{(\text{sea})}+2a\bar{m} b_{\text{mPA}})+\dots \\ (m_{22}-m_{33}) = Z \delta (1+aA^{(\text{sea})}+(2a\bar{m}+a\delta) b_{\text{mPA}})+\dots \\ (m_{33}-m_{11}) = Z \delta (1+aA^{(\text{sea})}+(2a\bar{m}-a\delta) b_{\text{mPA}})+\dots \\ (m_{22}-m_{12}) = Z \delta (1+aA^{(\text{sea})}+2a\bar{m} b_{\text{mPA}}+a\delta b_{\text{PA}})+\dots \\ (m_{12}-m_{11}) = Z \delta (1+aA^{(\text{sea})}+2a\bar{m} b_{\text{mPA}}-a\delta b_{\text{PA}})+\dots \end{array} \right.$$

$$a\bar{m}=(am_{q,2}+am_{q,1})/2, \quad a\delta=(am_{q,2}-am_{q,1})/2$$

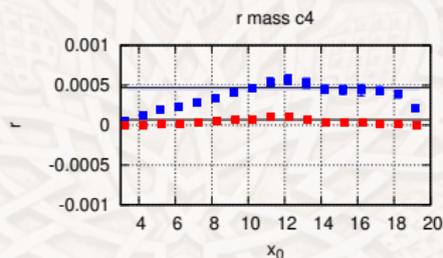
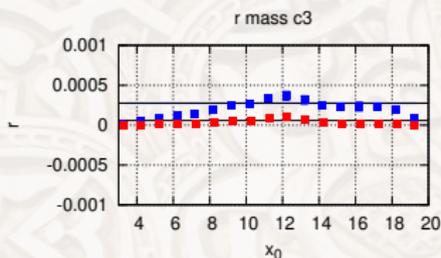
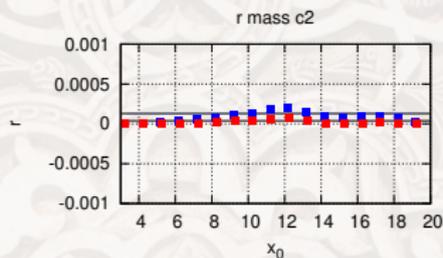
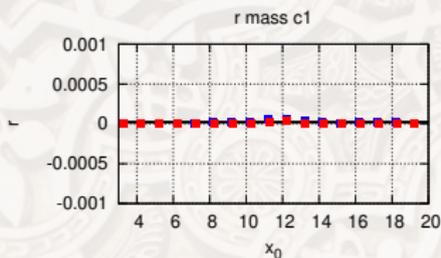
$$Z=Z(g_0)=\frac{Z_{\text{m}}(g_0,L)Z_{\text{P}}(g_0,L)}{Z_{\text{A}}(g_0)}, \quad aA^{(\text{sea})}=(x b_{\text{PA}}+\bar{b}_{\text{mPA}}) a \text{tr } \hat{m}^{(\text{sea})}$$

$$b_{\text{mPA}}=b_{\text{m}}+b_{\text{P}}-b_{\text{A}}, \quad b_{\text{PA}}=b_{\text{P}}-b_{\text{A}}$$

$$\frac{1}{4} \frac{(m_{22}-m_{11})(m_{22}-m_{11})}{(m_{22}-m_{33})(m_{33}-m_{11})} = 1 + O(a^2)$$

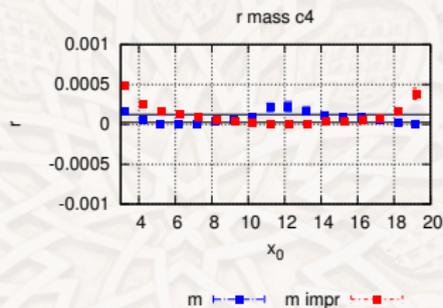
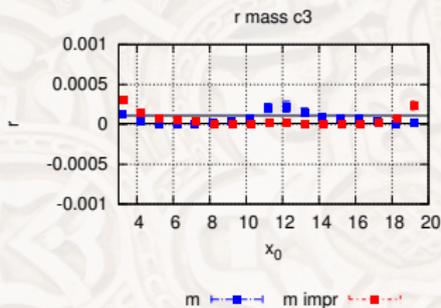
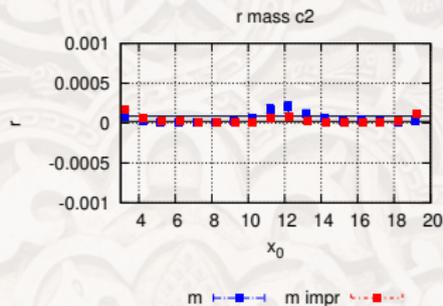
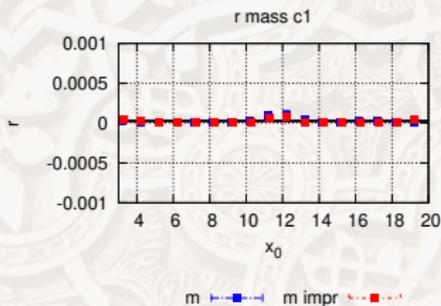
$$\frac{1}{4} \frac{(m_{22}-m_{11})(m_{22}-m_{11})}{(m_{22}-m_{12})(m_{12}-m_{11})} = 1 + O(a^2)$$

Consistency check (B1k3 ens)



$$r_1 = \frac{1}{4} \frac{(m_{22} - m_{11})(m_{22} - m_{11})}{(m_{22} - m_{33})(m_{33} - m_{11})} - 1 = O(a^2)$$

Consistency check (B1k3 ens)



$$r_2 = \frac{1}{4} \frac{(m_{22} - m_{11})(m_{22} - m_{11})}{(m_{22} - m_{12})(m_{12} - m_{11})} - 1 = O(a^2)$$

Numerical definitions of $b_A - b_P$, b_m , Z

$$R_{AP} = \frac{2(2m_{12} - m_{11} - m_{22})}{(m_{11} - m_{22})(am_{q,1} - am_{q,2})} = b_A - b_P + O(am_{q,1} + am_{q,2})$$

$$R_m = \frac{4(m_{12} - m_{33})}{(m_{11} - m_{22})(am_{q,1} - am_{q,2})} = b_m + O(am_{q,1} + am_{q,2})$$

$$R_Z = \frac{m_{11} - m_{22}}{m_{q,1} - m_{q,2}} + (R_{AP} - R_m)(am_{11} + am_{22}) = Z + O(a \operatorname{tr} \hat{m}^{(\text{sea})})$$

$$Z = Z(g_0) = \frac{Z_m(g_0, L) Z_P(g_0, L)}{Z_A(g_0)}$$

$$L \approx 1.2 \text{ fm} \quad Lm_{11} \approx 0.0$$

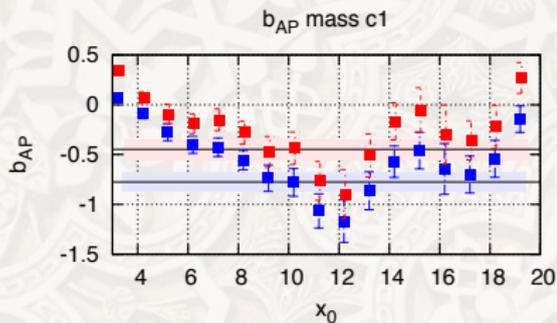
$$Lm_{22} \approx 0.25, 0.5, 0.75, 1.0$$

$$m_{0,3} = \frac{1}{2}(m_{0,1} + m_{0,2})$$

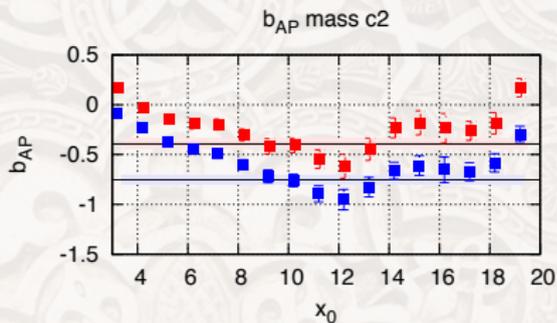
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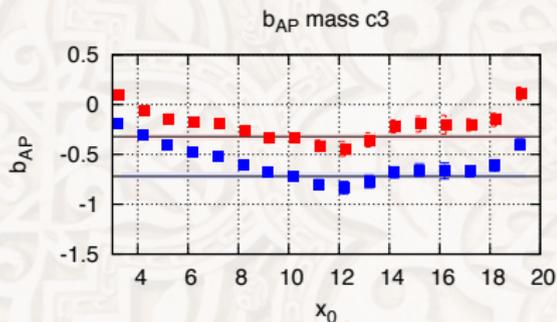
$(b_A - b_P)$ vs x_0 (B1k3 ens)



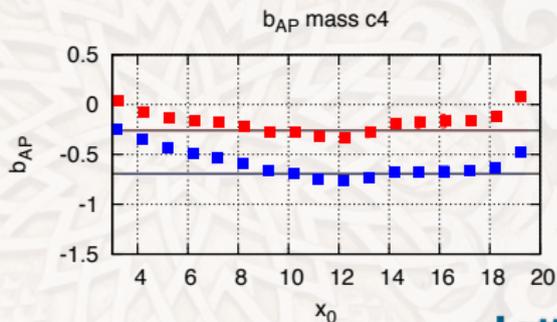
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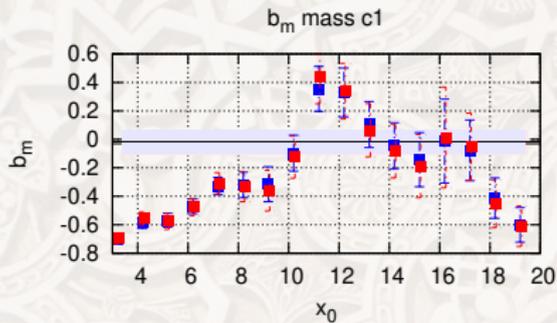


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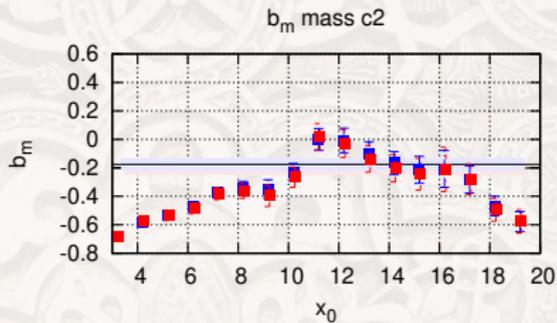


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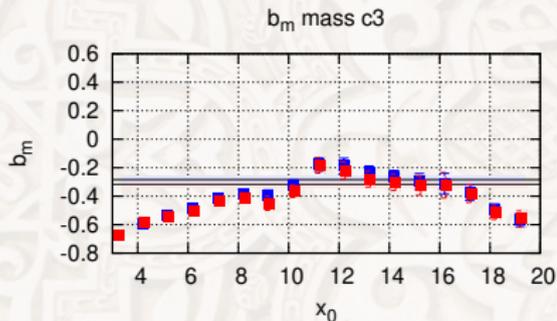
b_m VS x_0 (B1k3 ens)



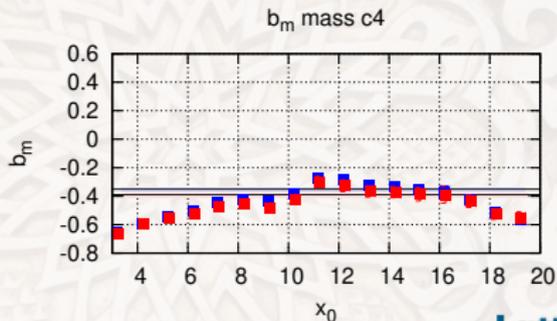
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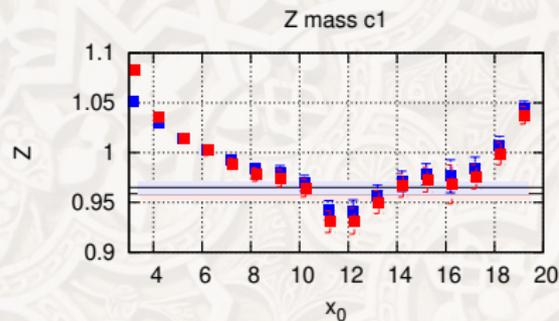


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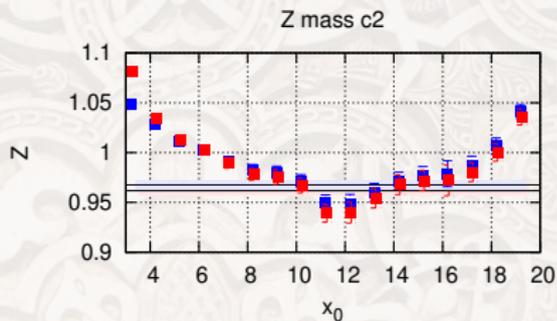


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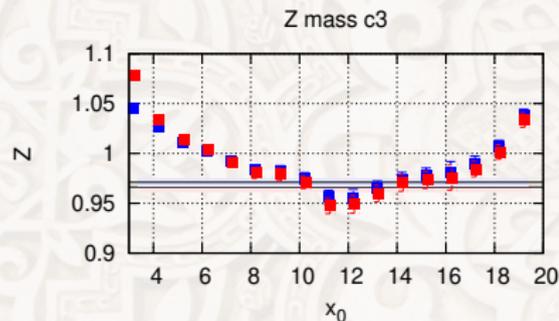
Z vs x_0 (B1k3 ens)



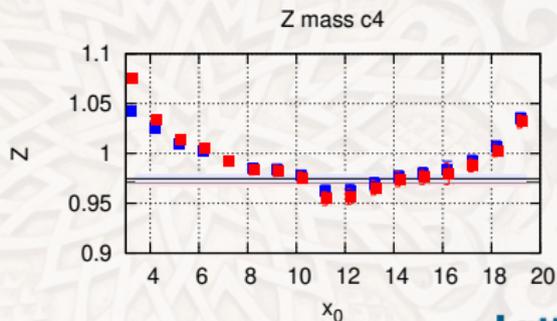
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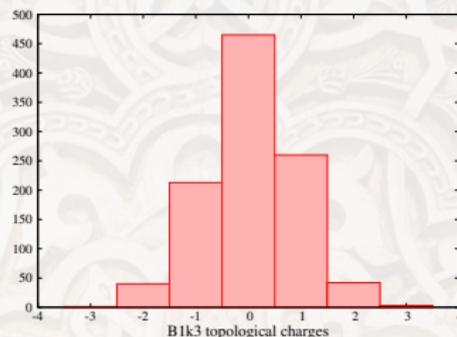
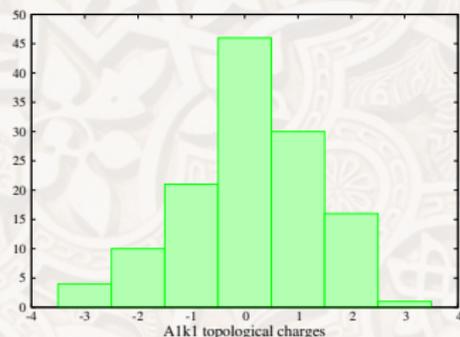


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Topological sectors



- Project to trivial topological sector ($Q_{top}(t) = 0$)

$$Q_{top}(t) = -\frac{a^4}{32\pi^2} \sum_x \epsilon_{\mu\nu\alpha\beta} \text{tr}\{G_{\mu\nu}(x, t) G_{\alpha\beta}(x, t)\}$$

t = “flow time” of the gradient flow (covariant) diffusion

$$\begin{cases} \partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) \\ B_\mu(0, x) = A_\mu(x) \end{cases}$$

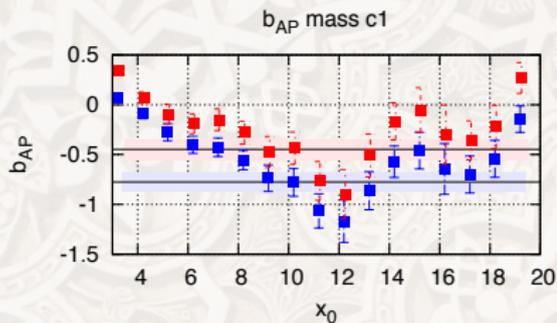
$$D_\mu = \partial_\mu + [B_\mu, \]$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

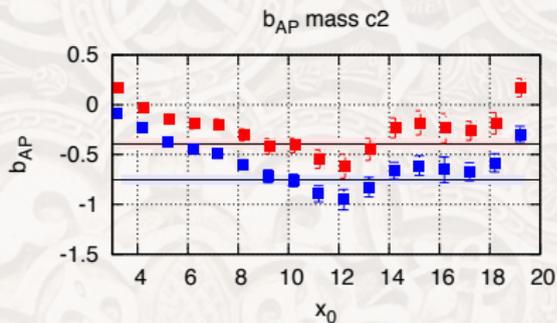
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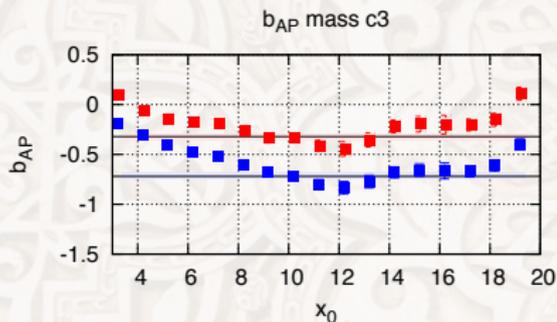
$(b_A - b_P)$ vs x_0 , $Q_{top}(t) = 0$ (B1k3 ens)



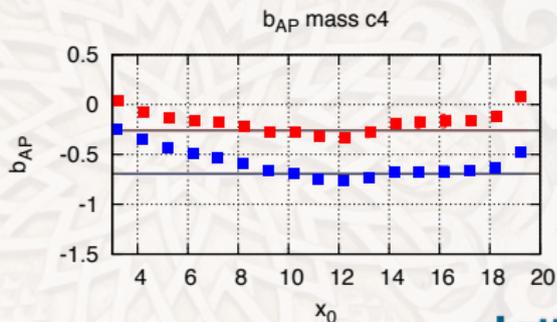
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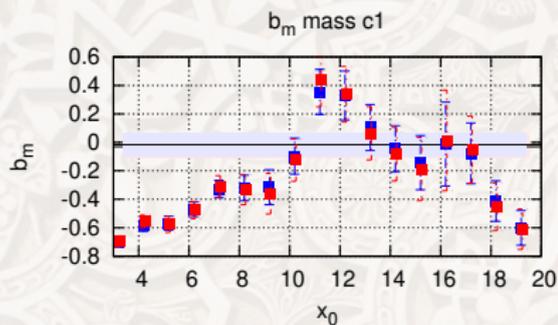


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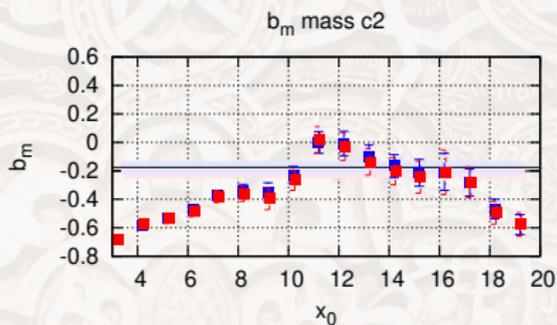


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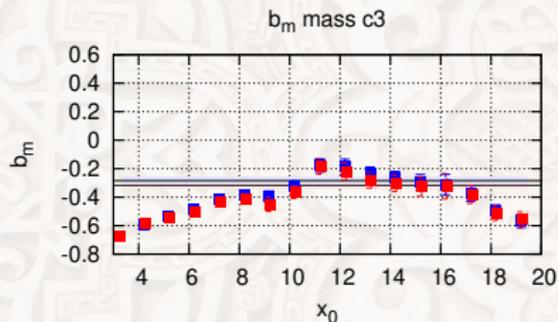
b_m vs x_0 , $Q_{top}(t) = 0$ (B1k3 ens)



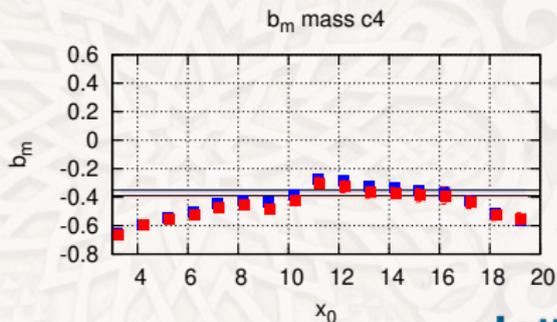
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m —■— m impr —■—

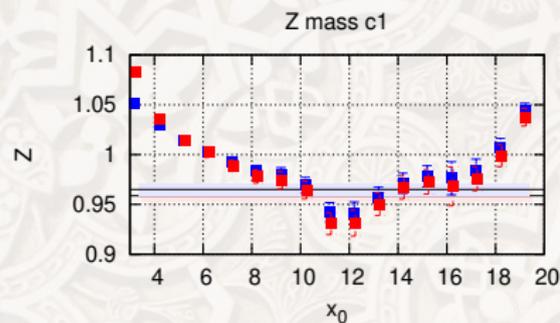


m —■— m impr —■—

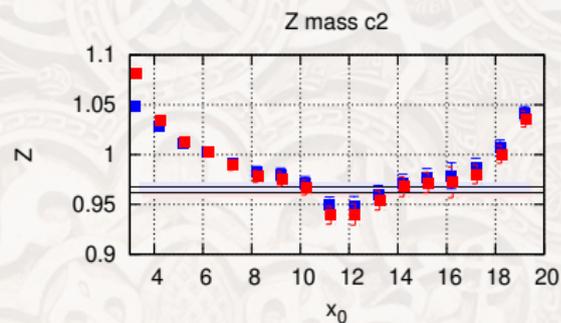


m —■— m impr —■—

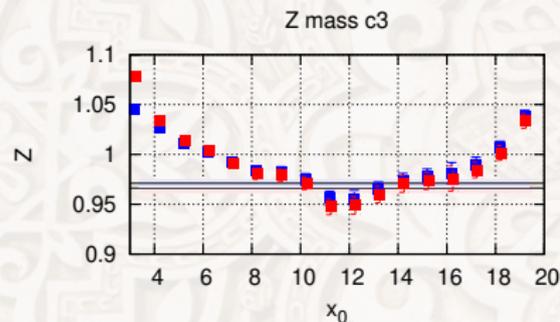
Z vs x_0 , $Q_{top}(t) = 0$ (B1k3 ens)



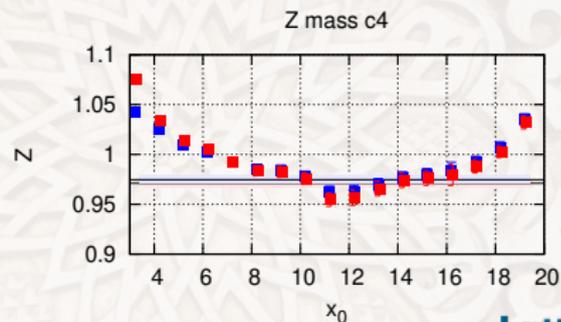
m —■— m impr —■—



m —■— m impr —■—



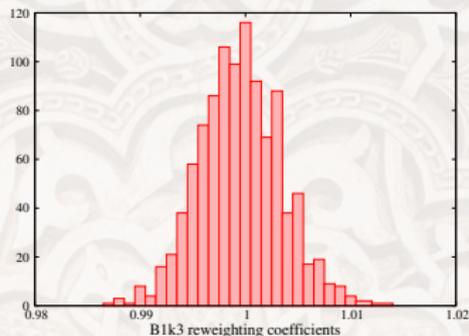
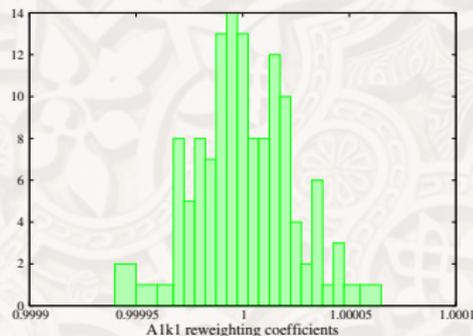
m —■— m impr —■—



m —■— m impr —■—

Conclusion & Outlooks

- Results for b_{AP} sensitive to the choice of the lattice derivative (standard or improved)
- No significant dependence on the topological charge
- \implies Compute the correlators for the finer lattices
- \implies Reconstruct the Padè fits



■ RHMC reweighting factors

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[M A Clark and A D Kennedy, [hep-lat/0608015], Phys. Rev. Lett. **98** (2007) 051601]

[M Lüscher and F Palombi, [arXiv:0810.0946 [hep-lat]], PoS LATTICE **2008** (2008) 049]

$$\langle A \rangle = \frac{\langle AW \rangle_W}{\langle W \rangle_W}$$