Heavy hadrons spectra on lattice using NRQCD

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- Lattice methods are powerful techniques in analyzing the spectrum of hadrons. However for hadrons containing heavy quarks particullarly bottom quark are difficult to analyze.
- For spectrum calculation it is necessary that $aM << 1$. For light quarks it is true but for charm quark $aM_c > 0.7$ and for bottom qurak $aM_b > 2$ with lattice spacing $a = 0.12fm$.
- However in hadrons containing heavy quarks the velocities of heavy quarks are non-relativistic. One can use effective theories like NRQCD.
  $M_\Upsilon = 9390$ MeV where as $2 \times M_b = 8360$ MeV ($\overline{MS}$ Scheme) and $M_{J/\psi} = 3096$ MeV where as $2 \times M_c = 2580$ MeV.

Foldy-Wouthuysen Transformation

- The Dirac equation $H\psi = i\frac{\partial\psi}{\partial t}$ where

$$H = \overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A}) + e\phi + m\beta$$

- Non-relativistic limit is reached by making the following transformation $\psi' = e^{iS}\psi$ where $S = -\frac{i}{2m}\beta\overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A})$.
- We get $i\frac{\partial\psi'}{\partial t} = H'\psi'$ where

$$\begin{aligned} H' &= e^{iS}He^{-iS} - ie^{iS}\frac{\partial e^{-iS}}{\partial t} \\ &= H + i[S,H] - \frac{1}{2}[S,[S,H]] - \frac{i}{6}[S,[S,[S,H]]] + ... \\ &\quad -\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + ... \end{aligned}$$

- Writting

$$\psi' = \begin{pmatrix} u \\ v \end{pmatrix}$$

- $i\frac{\partial \psi'}{\partial t} = H'\psi'$ gives

$$\begin{aligned} i\frac{\partial u}{\partial t} &= [m - \frac{1}{2m}\sum_j D_j^2 - \frac{e}{2m}\sigma.B - \frac{1}{8m^3}(\sum_j D_j^2)^2 \\ &+ e\phi - \frac{e}{8m^2}\nabla.E - \frac{ie}{8m^2}\sigma.(\nabla \times E - E \times \nabla)]u \end{aligned}$$

- Similarly like QED we write NRQCD Lagrangian upto $O[(v/c)^6]$
  $\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}_{v^4} + \delta\mathcal{L}_{v^6}$

$$\mathcal{L}_0 = \psi(x)^\dagger (iD_0 + \frac{\vec{D}^2}{2m})\psi(x)$$

$$\begin{aligned}\delta\mathcal{L}_{v^4} &= c_1 \frac{1}{8m^3}\psi^\dagger D^4 \psi + c_2 \frac{g}{8m^2}\psi^\dagger(\vec{D}.\vec{E} - \vec{E}.\vec{D})\psi \\ &+ c_3 \frac{ie}{8m^2}\psi^\dagger \vec{\sigma}.(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})\psi + c_4 \frac{g}{2m}\psi^\dagger\vec{\sigma}.\vec{B}\psi\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L}_{v^6} &= c_5 \frac{g}{8m^3}\psi^\dagger\{\vec{D}^2, \vec{\sigma}.\vec{B}\}\psi - c_6 \frac{ig^2}{8m^3}\psi^\dagger(\vec{\sigma}.\vec{E}\times\vec{E})\psi \\ &+ c_7 \frac{3}{64}\frac{ig}{m^4}\psi^\dagger\{\vec{D}^2, \vec{\sigma}.(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})\}\psi\end{aligned}$$

- $\mathcal{L}_0$ merely gives us Schrodinger equation.
- $c_1, c_2, c_3, c_4, c_5, c_6$ and $c_7 = 1$(tree level).

Improvement upto $O(a^4)$

- For $a = 0.12fm$ it is desirable to correct operators upto order $O(a^4)$.
- Symmetric derivative

$$\begin{aligned}
\Delta_i^{\pm} f(x) &= \frac{1}{2a}[f(x+a\hat{i}) - f(x-a\hat{i})] \\
&= \partial_i f + \frac{a^2}{6}\partial_i^3 f \\
&= \partial_i f + \frac{a^2}{6}\Delta_i^{\pm}\Delta_i^{+}\Delta_i^{-} f \\
\partial_i f &= \Delta_i^{\pm} f - \frac{a^2}{6}\Delta_i^{+}\Delta_i^{\pm}\Delta_i^{-} f \\
\tilde{\Delta}_i^{\pm} f &= \Delta_i^{\pm} f - \frac{a^2}{6}\Delta_i^{+}\Delta_i^{\pm}\Delta_i^{-} f
\end{aligned}$$

- Laplacian

$$\tilde{\Delta}^2 = \Delta^2 - \frac{a^2}{12}\sum_i [\Delta_i^{+}\Delta_i^{-}]^2$$

- Gauge fields corrected upto $O(a^4)$ {using cloverleaf}

$$g\tilde{F}_{\mu\nu}(x) = gF_{\mu\nu}(x) - \frac{a^4}{6}[\Delta_\mu^{+}\Delta_\mu^{-} + \Delta_\nu^{+}\Delta_\nu^{-}]gF_{\mu\nu}(x)$$

Green's Function

- The Lagrangian has the following form

$$\mathcal{L} = \psi^\dagger(x,t) D_4 \psi(x,t) + \psi^\dagger(x,t) H \psi(x,t)$$

- H contains spatial derivaties only.
- Green's function obeys.

$$\begin{aligned} G(x,t+1;0,0) &= (1-\frac{aH_0}{2})(1-\frac{a\delta H}{2})U_t(x,t)^\dagger \\ &\quad (1-\frac{a\delta H}{2})(1-\frac{aH_0}{2})G(x,t;0,0) \end{aligned}$$

$H = H_0 + \delta H$. For stability purpose we modify

$$\begin{aligned} G(x,t+1;0,0) &= (1-\frac{aH_0}{2n})^n(1-\frac{a\delta H}{2})U_t(x,t)^\dagger \\ &\quad (1-\frac{a\delta H}{2})(1-\frac{aH_0}{2n})^n G(x,t;0,0) \end{aligned}$$

with $G(x,t;0,0) = 0$ for $t < 0$ and $G(x,t;0,0) = \delta_{x,0}$ for $t = 0$. From the above equation it is evident that $n > \frac{3}{2m}$.

Tuning and mass calculation

- In NRQCD Lagrangian the rest mass term is not included.
- In order to tune b-quark we calculated 'kinetic mass' of η_b meson

$$
\begin{aligned}
E(p) - E(0) &= \sqrt{p^2 + M^2} - M \\
\Rightarrow \Delta E + M &= \sqrt{p^2 + M^2} \text{ where } \Delta E = E(p) - E(0) \\
\Rightarrow (\Delta E)^2 + 2M\Delta E &= p^2 \\
\Rightarrow M &= \frac{p^2 - (\Delta E)^2}{2\Delta E}
\end{aligned}
$$

Heavy-heavy correlator

- For mesons containing both heavy quarks let the heavy quark and anti-quark are created by two component spinor $\psi^\dagger$ and χ and their destruction operators are ψ and $\chi^\dagger$. As anti-quaks transform by $\bar{3}$ under color rotation so it is convinient to rename the anti-quark spinor.

$$
\begin{aligned}
C(\vec{p},t) &= \sum_x \langle 0|e^{i\vec{p}.\vec{x}} O(\vec{x},t) O^\dagger(\vec{0},0)|0\rangle \\
&= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|\chi^\dagger(x)\Gamma_{sk}(x)\psi(x)\psi^\dagger(0)\Gamma^\dagger_{sc}(0)\chi(0)|0\rangle \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|\chi(0)\chi^\dagger(x)\Gamma_{sk}(x)\psi(x)\psi^\dagger(0)\Gamma^\dagger_{sc}(0)0\rangle \\
&= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[G^\dagger(x,0)\Gamma_{sk}(x)G(x,0)\Gamma^\dagger_{sc}(0)]
\end{aligned}
$$

- In the last line we have used $G^\dagger(x,0) = -[\chi(x)\chi^\dagger(0)]^\dagger$. Here $\Gamma(x) = \Omega\phi(x)$. ϕ is the smearing operator and Ω is a 2×2 matrix in spin space. $\Omega = I$ for pseudoscalar particles and $\Omega = \sigma_i$ for vector particles.

- We ran our code on 40, $24^3 64$ milc lattices. Nr-loop $n = 3$ and mass is tuned to $m = 0.759$.
- Here we shown the correlators for η_b obtained at momenta $\vec{p} = \frac{2\pi}{L}(2, 0, 0)$ and $\vec{p} = \frac{2\pi}{L}(0, 0, 0)$ with $L = 24$. We find kinetic mass of $\eta_b = 9.42$ GeV.

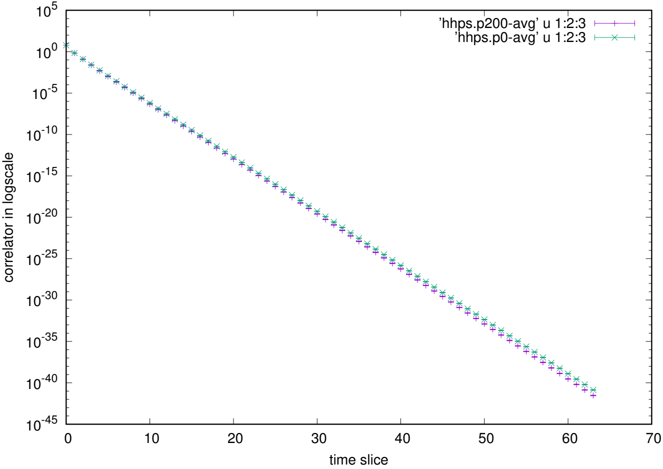

Figure: Heavy-heavy p200 vs p0

Mass difference of Υ and η_b

- The following plot shows the mass differencebetween Υ and η_b. For fit range 7-17 we find the splitting to be $= 131$ MeV.

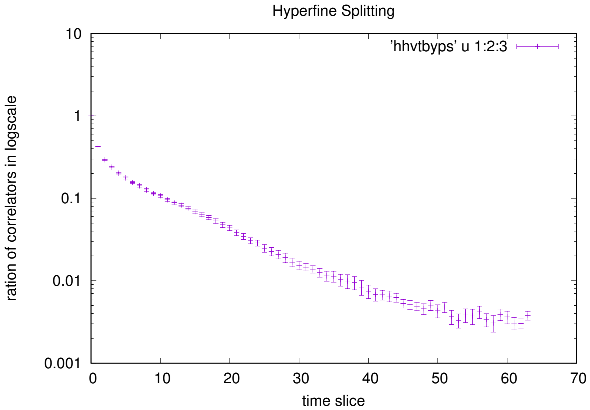

Figure: Hyperfine splitting

Heavy-light correlator(light = overlap)

- For mesons containing one heavy quark and one light anti-quark the interpolating operator is $Q^\dagger(x)\Gamma(x)q(x)$.

$$
\begin{aligned}
C(\vec{p},t) &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|q^\dagger(x)\Gamma^\dagger_{sk}(x)Q(x)Q^\dagger(0)\Gamma_{sc}(0)q(0)|0\rangle \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|q(0)q^\dagger(x)\Gamma^\dagger_{sk}(x)Q(x)Q^\dagger(0)\Gamma_{sc}(0)|0\rangle \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[M(0,x)\gamma_4\Gamma^\dagger_{sk}(x)G(x,0)\Gamma_{sc}(0)] \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[\gamma_5 M(x,0)^\dagger\gamma_5\gamma_4\Gamma^\dagger_{sk}(x)G(x,0)\Gamma_{sc}(0)]
\end{aligned}
$$

η_c meson

- Charm is tuned using the kinetic mass.

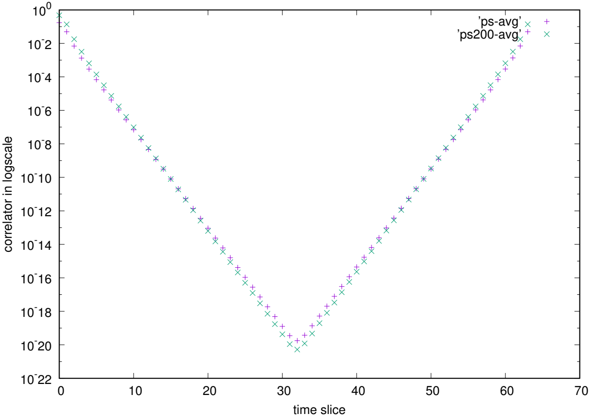

Figure: Light-light p200 vs p0

- We find the kinetic mass of $\eta_c = 3.05$ GeV.

B_c meson

- Plot for B_c meson correlators.

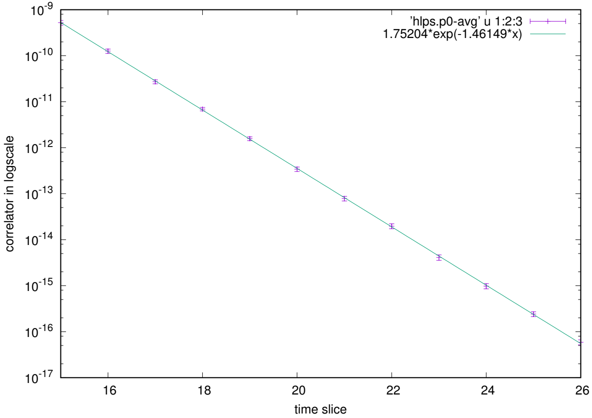

Figure: Heavy-light meson correlators obtained at zero momentum

- From fit $E_{B_c} = 1.46149 = 1.46149 \times 197.3/0.12 = 2402$ Mev

- As we used kinetic mass in tuning the bottom and charm masses we had to use the following formula to calculate the mass of B_c.

$$M_{B_c} = E_{B_c} + \frac{1}{2}(M_{\eta_b} - E_{\eta_b}) + \frac{1}{2}(M_{\eta_c} - E_{\eta_c})$$

  Here $E_{B_c}, E_{\eta_b}, E_{\eta_c}$ are the simulated masses and M_{η_b}, M_{η_c} are their pdg values.
- $M_{B_c} = 2402 + [(2980 - 2190)/2] + [(9391 - 2472)/2] = 6256.5$ Mev
  with error $= \pm$ 20 MeV

B_s meson

- s-quark has been directly tuned to produce $s\bar{s}$ pseudoscalar mass to be 686 MeV. Here we used the following formula to calculate the mass of pseudoscalar s-quarkonium state

$$M_{s\bar{s}} = \sqrt{2M_K^2 - M_\pi^2}$$

where M_K and M_π are kaon and pion masses.

- B_s meson mass is calculated as

$$M_{B_s} = E_{B_s} + \frac{1}{2}(M_{\eta_b} - E_{\eta_b})$$

$M_{B_s} = 1640 + (9389 - 2476)/2 = 5096$ MeV

Heavy-light correlator(light = hisq)

- $Q = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$, $\Gamma = \gamma_5$ or $\Gamma = \gamma_k$

$$
\begin{aligned}
C(\vec{p}, t) &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|q^\dagger(x)\Gamma^\dagger_{sk}(x)Q(x)Q^\dagger(0)\Gamma_{sc}(0)q(0)|0\rangle \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|q(0)q^\dagger(x)\Gamma Q(x)Q^\dagger(0)\Gamma|0\rangle \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[M(0,x)\gamma_4\Gamma G(x,0)\Gamma] \\
&= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[\gamma_5 M(x,0)^\dagger\gamma_5\Gamma G(x,0)\Gamma]
\end{aligned}
$$

- $G(x,0)$ is now a 4×4 matrix in spinor space having vanishing lower components but it is in Dirac representation of gamma matrices. We can convert it to milc gamma representation by an unitary transformation $S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_y & \sigma_y \\ -\sigma_y & \sigma_y \end{pmatrix}$

- Plot for B_s meson correlators.

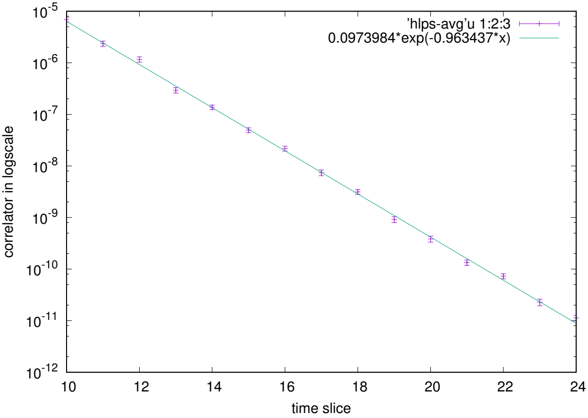

Figure: Heavy-light meson correlators obtained at zero momentum

- $M_{B_s} = 1584 + (9389 - 2476)/2 = 5040$ MeV

Ω_{bbb} baryon

- Interpolator $(\mathcal{O}_k)_\alpha = \epsilon_{abc}({Q^a}^T C\gamma_k Q^b)Q^c_\alpha$ with $C = \gamma_4\gamma_4$

$$
\begin{aligned}
C_{ij\alpha\beta}(t) &= \sum_{\vec{x}} \langle 0|[\mathcal{O}_i(\vec{x},t)]_\alpha [\mathcal{O}_j^\dagger(\vec{0},0)]_\beta|0\rangle \\
&= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh} G^{ch}_{\alpha\beta}(x,0)\, Tr[C\gamma_i G^{bg}(x,0)\overline{C\gamma_j}{G^{af}}^T(x,0)]
\end{aligned}
$$

- The correlator has overlap with both spin 3/2 and spin 1/2 states

$$C_{ij}(t) = Z_{3/2}e^{-E_{3/2}t}\Pi P^{3/2}_{ij} + Z_{1/2}e^{-E_{1/2}t}\Pi P^{1/2}_{ij}$$

$\Pi = \frac{1}{2}(1+\gamma_4)$, $P^{3/2}_{ij} = \delta_{ij} - \frac{1}{3}\gamma_i\gamma_j$, $P^{1/2}_{ij} = \frac{1}{3}\gamma_i\gamma_j$ and $P^{3/2}_{ij}.P^{1/2}_{jk} = 0$.

- $P^{3/2}_{xx}.C_{xx} + P^{3/2}_{xy}.C_{yx} + P^{3/2}_{xz}.C_{zx} = \frac{2}{3}Z_{3/2}\Pi e^{-E_{3/2}t}$

- Plot for Ω_{bbb} correlator

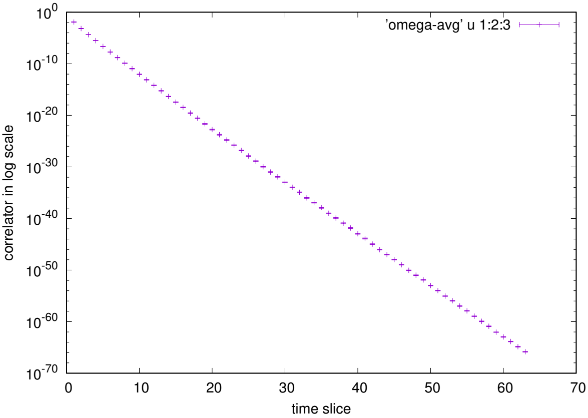

Figure: Omega 3/2

- $M_{\Omega_{bbb}} = E_{\Omega_{bbb}} + \frac{3}{2}(M_{\eta_b} - E_{\eta_b})$=14.38 GeV with error = $\pm$ 20 MeV

- Interpolator $(\mathcal{O}_k)_\alpha = \epsilon_{abc}({Q^a}^T C\gamma_k Q^b)s^c_\alpha$

$$
\begin{aligned}
C_{ij\alpha\beta}(t) &= \sum_{\vec{x}} \langle 0|[\mathcal{O}_i(\vec{x},t)]_\alpha [\mathcal{O}_j^\dagger(\vec{0},0)]_\beta|0\rangle \\
&= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh}[M^{ch}(x,0).\gamma_4]_{\alpha\beta}\, Tr[C\gamma_i G^{bg}(x,0)\overline{C\gamma_j}\, {G^{af}}^T(x,0)]
\end{aligned}
$$

- Change $G(x,0)$ into milc gamma representation.
- $P^{3/2}_{xx}.C_{xx} + P^{3/2}_{xy}.C_{yx} + P^{3/2}_{xz}.C_{zx} = \frac{2}{3} Z_{3/2}\Pi e^{-E_{3/2}t}$

- Plot for Ω_{bbs} correlator

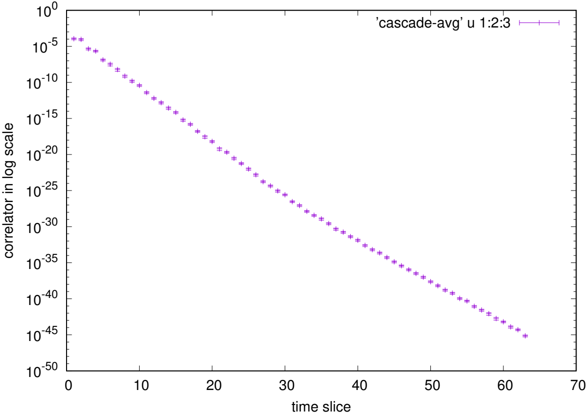

Figure: $Omega_{bbs}$ 3/2

- $M_{\Omega_{bbs}} = E_{\Omega_{bbs}} + (M_{\eta_b} - E_{\eta_b})$=9.81 GeV with error = $\pm$ 40 MeV

THANK YOU

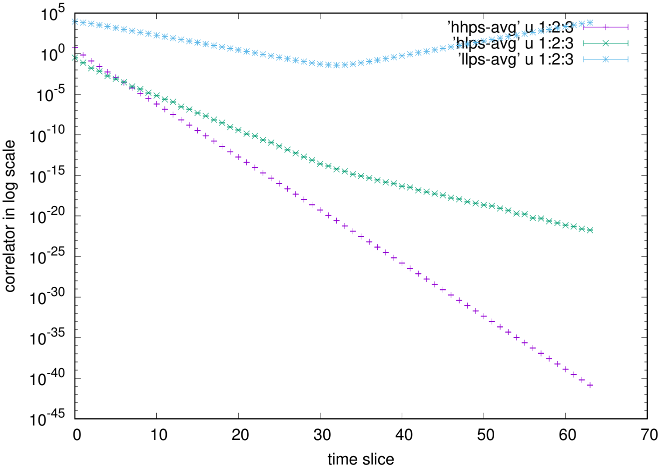

Figure: hh hl ll mesons