The current matrix elements from HAL QCD method

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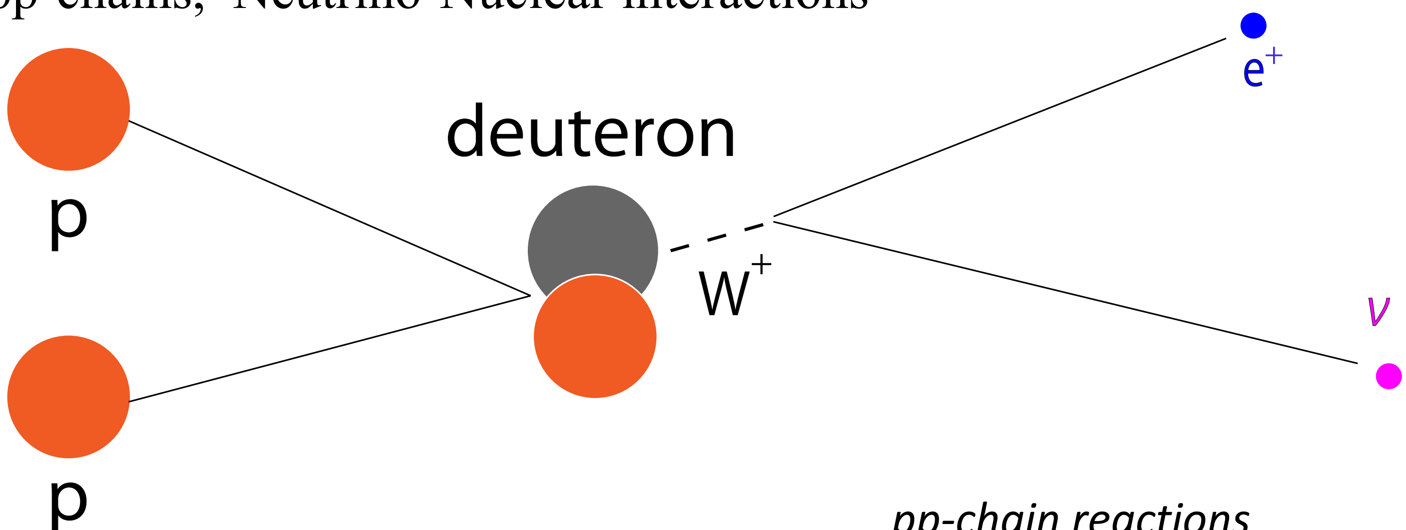
Introduction ;The back grounds

- Hadronic / Nuclear matrix elements (form factors) provide important information on
  - Hadron structures
  - Beyond Standard Model
    - WIMP and Nuclei interactions
  - Interactions
    - pp-chains, Neutrino-Nuclear interactions



pp-chain reactions

Introduction ; HAL QCD method

- *The method to construct the QM of multi-Hadrons **from QCD***

$$p_\alpha(\vec{r}) \equiv \varepsilon_{abc}\left(u_a^T(\vec{r})C\gamma_5 d_b(\vec{r})\right)u_{c,\alpha}(0)$$

$$n_\beta(\vec{r}) \equiv \varepsilon_{abc}\left(u_a^T(\vec{r})C\gamma_5 d_b(\vec{r})\right)d_{c,\beta}(0)$$

Operators for Baryons

$$\psi_{\alpha\beta}(\vec{r},t) = \langle 0 | n_\beta(\vec{r},t) p_\alpha(0,t) | p, B=2 \rangle$$

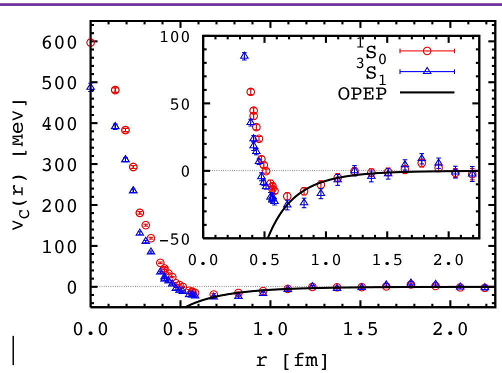
$$\underset{r\to\infty}{\simeq} e^{i\delta} \frac{\sin(pr+\delta)}{pr}$$

Equal time NBS wave function

$$\left(E + \frac{1}{m}\nabla_r^2\right)\psi_{\alpha\beta}(\vec{r},t) = \int d^3r' \, V(\vec{r},\vec{r}')\psi_{\alpha\beta}(\vec{r}',t)$$

Baryon-Baryon interaction potentials

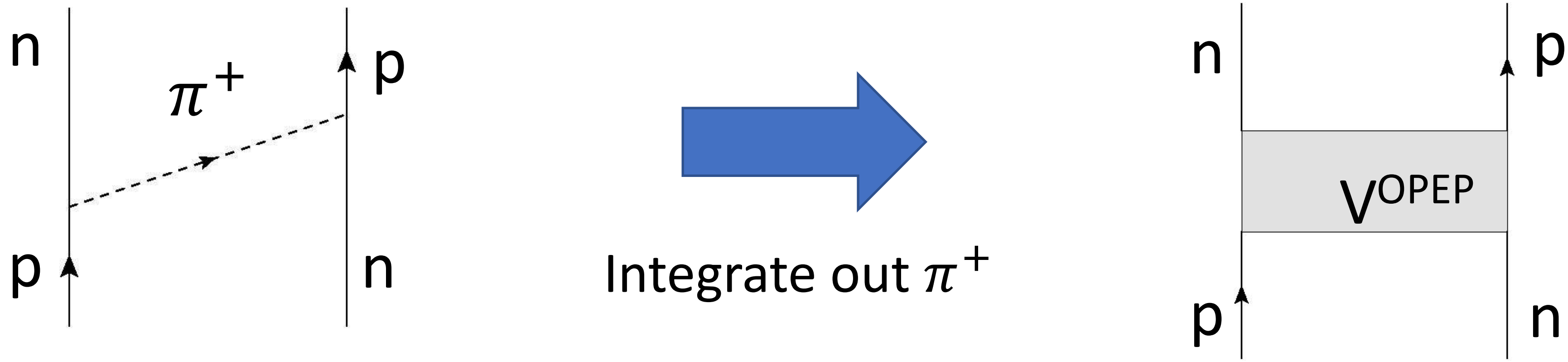
Correct scattering phase shift.



Introduction ; Conserved Current(1)

- Conserved currents should be derived carefully.
  - Example : one pion exchange potential (OPEP)

Original theory *Effective quantum mechanics*



Noether current = pionic part+ protonic part

The naïve current = protonic part

➢ *Since the pion current is missing*
the naïve current on the right does NOT conserve.

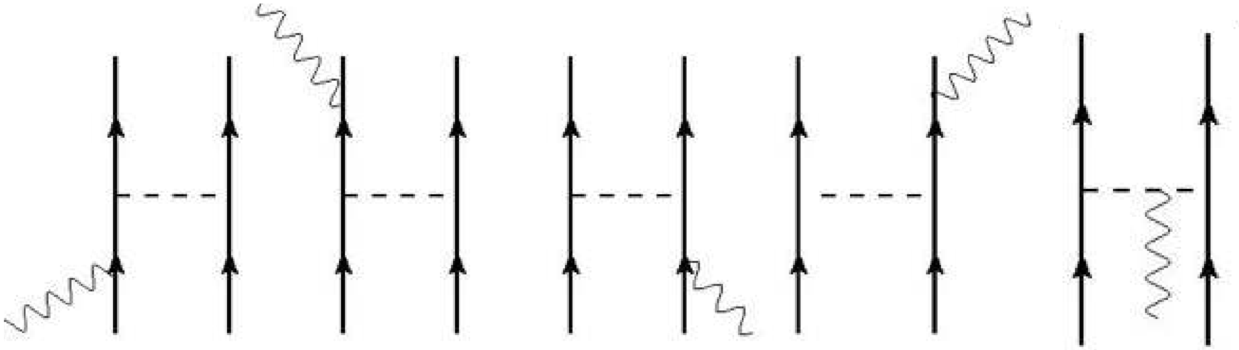
$$\partial_t\left[\psi^*(x)\psi(x)\right]-\vec{\nabla}\bullet\left[\frac{i}{2m}\psi^*(x)\overleftrightarrow{\nabla}\psi(x)\right]=0$$

Introduction ; Conserved current(2)

- Additional contribution is needed
  The exchange current / two body current.

Thompson and Heller.
P.R.C 7.6 (1973): 2355.

- Example ; Thompson and Heller's approach
  - Bremsstrahlung amplitudes of deuterons in the original theory.



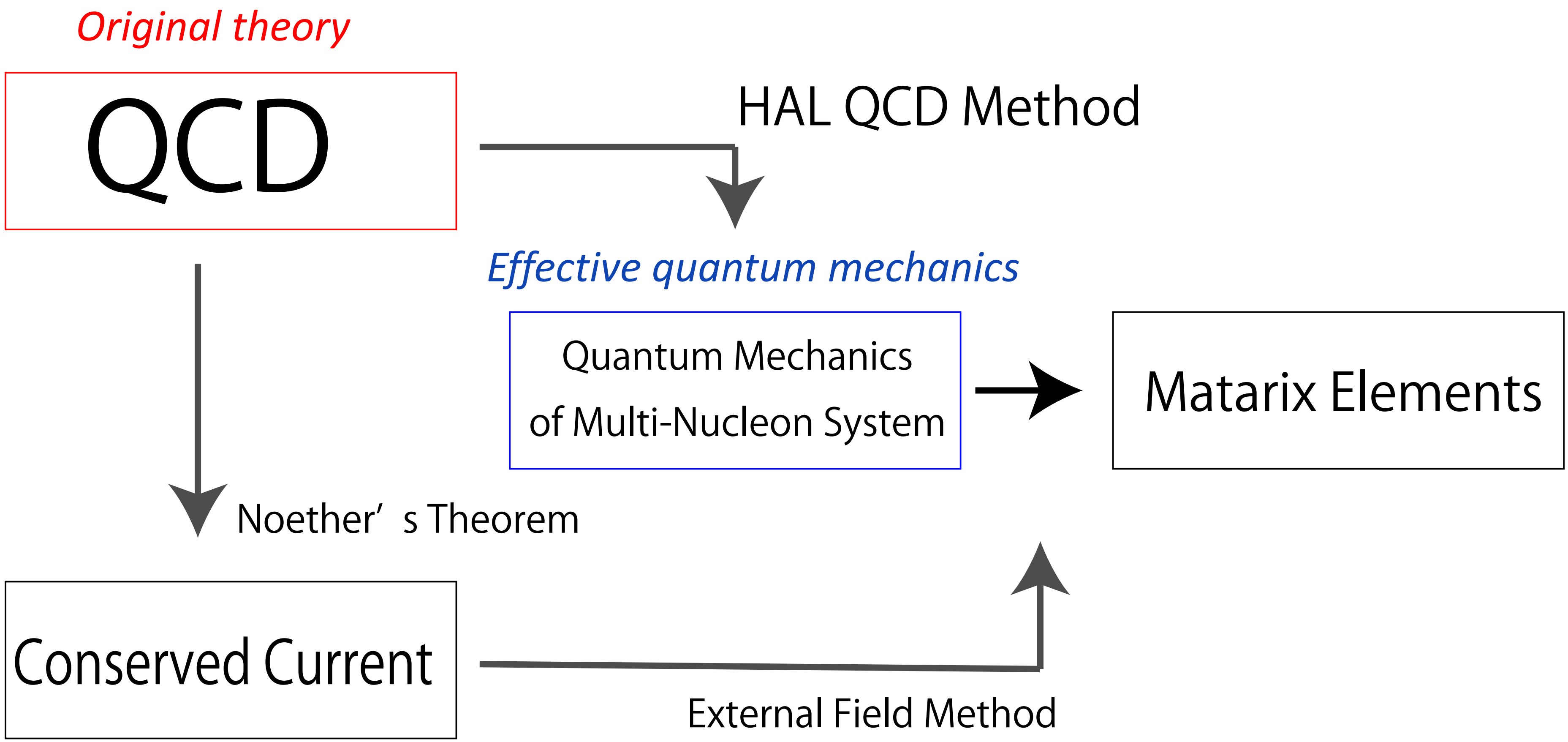
  - Current in the effective QM that reproduce the Amp.

$$i\partial_t \psi_{NN} = \left(-\frac{\partial^2}{2M}\right)\psi_{NN} + V^{OPEP}\psi_{NN}$$

ψ_{NN} ; 2 nucleon wave func.

V^{OPEP} ; OPEP.

Strategy



- As a first step we consider
  - A non-relativistic solvable field theory
  - Theory of elemental bosons, instead of Nucleons
  - Simplest non-trivial model

The model(1)

- Non-relativistic two channel coupling model.

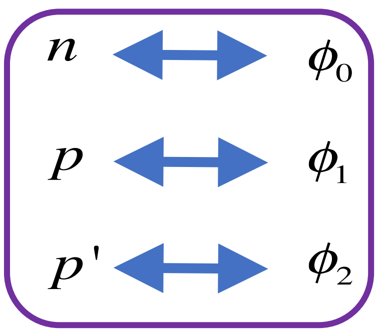
$$H = \sum_{\alpha=0,1,2} \hat{T}_\alpha + \sum_{i,j=1,2} \hat{V}_{ij}$$

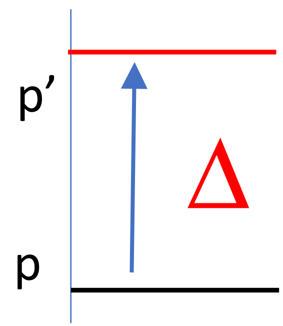
$$\hat{T}_\alpha \equiv \int d^3x\ \phi^\dagger_\alpha(\vec{x})\left[-\frac{\nabla^2}{2m}\right]\phi_\alpha(\vec{x}) \qquad (\alpha = 0,1)$$

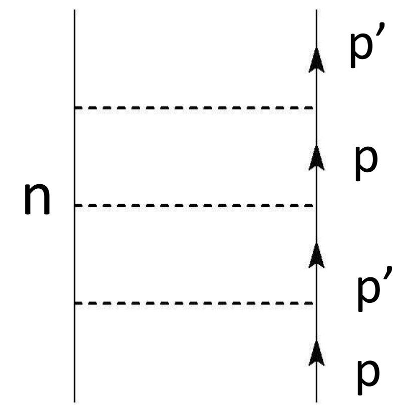
$$\hat{T}_0 \equiv \int d^3x\ \phi^\dagger_\alpha(\vec{x})\left[-\frac{\nabla^2}{2m} + \Delta\right]\phi_\alpha(\vec{x})$$

$$\hat{V}_{ij} \equiv \int d^3x\ d^3y\ \phi^\dagger_0(\vec{x})\phi^\dagger_i(\vec{y})V_{ij}(\vec{x}-\vec{y})\phi_j(\vec{y})\phi_0(\vec{x})$$

- The model mimics the (np)-(np') coupling.







- The conserved current accompanied by U(1) symmetry.

$$j_\mu(x) = \sum_{j=1,2}\left[\delta^0_\mu \phi^\dagger_j(x)\phi_j(x) - \delta^i_\mu \frac{i}{2m}\phi^\dagger_j(x)\overleftrightarrow{\nabla}_i\phi_j(x)\right]$$

The model(2)

- The equal time ($x^0 = y^0 = t$)NBS wave functions (i=1,2)

$$\psi_{i,nP}(x,y) \equiv \langle 0 | \phi_0(y)\phi_i(x) | n,P \rangle$$

$$\begin{cases} \vec{R} = \frac{1}{2}(\vec{x}+\vec{y}) \\ \vec{r} = \vec{x}-\vec{y} \end{cases}$$

$$\equiv \tilde{\psi}_{i,n}(\vec{r}) e^{i\vec{P}\cdot\vec{R}} e^{-iE_n(P)t}$$

$$E_n(P) = E_n^{rel} + \frac{1}{4m}\vec{P}^2$$

- Coupled Channel Equation.

$$\left(E_n^{rel} + \tfrac{1}{m}\nabla_r^2\right)\tilde{\psi}_{1,n}(\vec{r}) = V_{11}(\vec{r})\tilde{\psi}_{1,n}(\vec{r}) + V_{12}(\vec{r})\tilde{\psi}_{2,n}(\vec{r})$$

$$\left(E_n^{rel} + \tfrac{1}{m}\nabla_r^2 + \Delta\right)\tilde{\psi}_{1,n}(\vec{r}) = V_{11}(\vec{r})\tilde{\psi}_{1,n}(\vec{r}) + V_{12}(\vec{r})\tilde{\psi}_{2,n}(\vec{r})$$

The HAL QCD method

- Construct the potential for ch.1 in the elastic region
  - The HAL QCD potential.

$$V_\Delta(\vec{r},\vec{r}') \equiv \sum_m^{E^{rel}_m<\Delta} \left[V_{11}(\vec{r})\widetilde{\psi}_{1,m}(\vec{r}) + V_{12}(\vec{r})\widetilde{\psi}_{2,m}(\vec{r}) \right] \widetilde{\psi}^{v}_{1,m}(\vec{r}')$$

    - The dual vector

$\widetilde{\psi}^{v}_{1,m}(\vec{r})$ Where $\int d^3r \widetilde{\psi}^{v}_{1,m}(\vec{r})\widetilde{\psi}_{1,n}(\vec{r}) = \delta_{mn}$

- The Schrödinger equation is satisfied by the potential

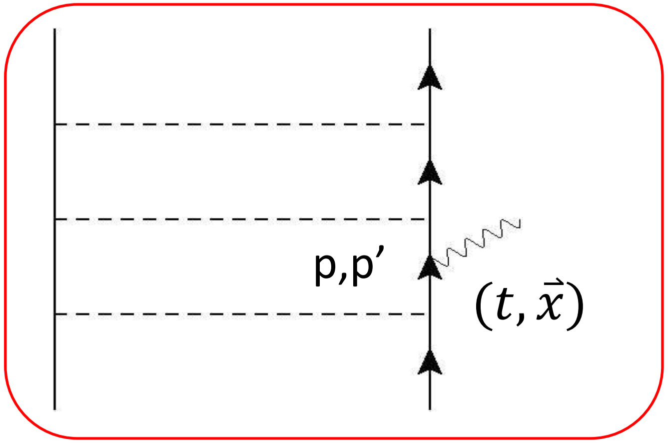
$$\left(E^{rel}_n + \frac{1}{m}\nabla^2_r\right)\widetilde{\psi}_{1,n}(\vec{r},t) = \int d^3r' \, V_\Delta(\vec{r},\vec{r}')\widetilde{\psi}_{1,n}(\vec{r}',t)$$

integrate over r' ⟶ $\left(E^{rel}_n + \frac{1}{m}\nabla^2_r\right)\widetilde{\psi}_{1,n}(\vec{r}) = V_{11}(\vec{r})\widetilde{\psi}_{1,n}(\vec{r}) + V_{12}(\vec{r})\widetilde{\psi}_{2,n}(\vec{r})$

Current Matrix Elements(1)

- The Hamiltonian in the external field

$$H[A_t] = \hat{T}_0 + \sum_{i=1,2} \hat{T}_i[A_t] + \sum_{i,j=1,2} \hat{V}_{ij}$$



$$\hat{T}_1[A_t] \equiv \int d^3x\ \phi_1^\dagger(\vec{x})\left[-\frac{\{\nabla - i\vec{A}(\vec{x},t)\}^2}{2m} - A_0(\vec{x},t)\right]\phi_1(\vec{x})$$

$$\hat{T}_2[A_t] \equiv \int d^3x\ \phi_2^\dagger(\vec{x})\left[-\frac{\{\nabla - i\vec{A}(\vec{x},t)\}^2}{2m} - A_0(\vec{x},t) + \Delta\right]\phi_2(\vec{x})$$

- The external field couples to the kinetic parts of p and p'

Current Matrix Elements(2)

- Hamiltonian in the external fields with projection.

$$H_\Delta[A_t] = P_\Delta H[A_t] P_\Delta$$

  - Projection op. on to the elastic region

$$P_\Delta \equiv \int \frac{d^3P}{(2\pi)^3} \sum_k^{E^{rel}_n<\Delta} |P,n\rangle\langle P,n|$$

    - Unwanted transition to ch.2 is induced by the ext. field
    - The projection is introduced to eliminate this contribution

- The Heisenberg op.

$$\phi_\alpha(x;A) = U_\Delta(0,t;A)\phi_\alpha(\vec{x})U_\Delta(t,0;A) \qquad U_\Delta(t,s;A) \equiv T\exp\left\{i\int_s^t dt\, H_\Delta[A_t]\right\}$$

Current Matrix Elements(3)

- The NBS wave functions in external field.

$$\psi_{i,nP}(x,y;A) \equiv \langle 0 | \phi_0(y;A)\phi_i(x;A) | P,n \rangle \qquad i = 1,2$$

- Satisfy the modified coupled channel equation

$$\left(iD_x^0 + \frac{1}{2m}\vec{D}_x^2 + \frac{1}{2m}\nabla_y^2\right)\psi_{1,nP}(x,y;A) = \sum_{\beta=1,2} V_{1\beta}(\vec{x}-\vec{y})\psi_{\beta,nP}(x,y;A)$$
$$+\int d^3x'\; d^3y'\; \delta V_{1\beta}(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t)\psi_{\beta,nP}(x',y';A)$$

  - The Additional term from the cut off

$$\delta V_{1\beta}(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t) \equiv \langle 0 | \phi_0(\vec{y})\phi_1(\vec{x})(1-P_\Delta)H[A_t]P_\Delta\phi_0(\vec{y})\phi_\beta(\vec{x}) | 0 \rangle$$

    - We write

$$V_{1\beta;\Delta}(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t) = V_{1\beta}(\vec{x}'-\vec{y}')\delta(\vec{x}'-\vec{x})\delta(\vec{y}'-\vec{y}) + \delta V_{1\beta}(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t)$$

Current Matrix Elements(4)

- The HAL QCD potential in the external field.

$$V_\Delta(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t) \equiv \sum_m^{E^{rel}_m<\Delta} \sum_{\beta=1,2} V_{1\beta;\Delta}(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t)\psi_{\beta,mP}(\vec{x}',\vec{y}')\underbrace{\psi^{v}{}_{1,mP}(\vec{x}',\vec{y}')}_{\text{Dual vector}}$$

  - The dual vector satisfies orthogonality relation

$$\int d^3x\ d^3y\ \psi^{v}{}_{1,nQ}(\vec{x},\vec{y})\psi_{1,mP}(\vec{x},\vec{y}) = \delta_{nm}\delta^3(\vec{P}-\vec{Q})$$

- The Schrödinger equation is satisfied by the potential.
  - NBS wave func. is reproduced.

$$\left(iD_x^0 + \frac{1}{2m}\vec{D}_x^2 + \frac{1}{2m}\nabla_y^2\right)\psi_{1,nP}(x,y;A) = \int d^3x'\ d^3y'\ V_\Delta(\vec{x},\vec{y};\vec{x}',\vec{y}';A_t)\psi_{1,nP}(x',y';A)$$

Current Matrix Elements(5)

$$H_0 \equiv -\frac{1}{2m}\nabla_x^2 - \frac{1}{2m}\nabla_y^2$$

$$(i\partial_0 - H_0[A_t] - V[A_t])\psi(x;A) = 0$$

Multiply $\frac{\delta}{\delta A_\mu(z)}$ on both hand sides

$$\left(i\partial_0 + H_0 - V\right)\frac{\delta\psi(x;A)}{\delta A_\mu(z)} = \frac{\delta\left\{H_0[A_t] + V[A_t]\right\}}{\delta A_\mu(z)}\psi(x)$$

Use the Green's function
to solve the Eq.

$$\frac{\delta\psi(x;A)}{\delta A_\mu(z)} = \left(i\partial_0 + H_0 - V\right)^{-1}\frac{\delta\left\{H_0[A_t] + V[A_t]\right\}}{\delta A_\mu(z)}\psi(x)$$

Current Matrix Elements(6) ; Result(1)

- The Current matrix element formula

$$\langle l,P | j^\mu(z) | m,Q \rangle = \int d^3x \int d^3y \, \psi^L{}_{1,lP}(\vec{x},\vec{y}) K^\mu(x,y;z) \psi^R{}_{1,mQ}(\vec{x},\vec{y})$$

  - The functional derivative

$$K^\mu(x,y;z) = \frac{\delta}{\delta A_\mu(z)} \left[A_0(x) + \frac{1}{2m} \vec{D}_x^2 + V_\Delta[A_t] \right] \Bigg|_{A\equiv 0}$$

  - Left and right eigenvectors
    - coming from the Green's func.

$$\begin{cases} E_m(P)\psi^R_{1,mP} = (H_0 + V)\psi^R_{1,mP} \\ E_m(P)\psi^L_{1,mP} = \psi^L_{1,m}(H_0 + V) \end{cases}$$

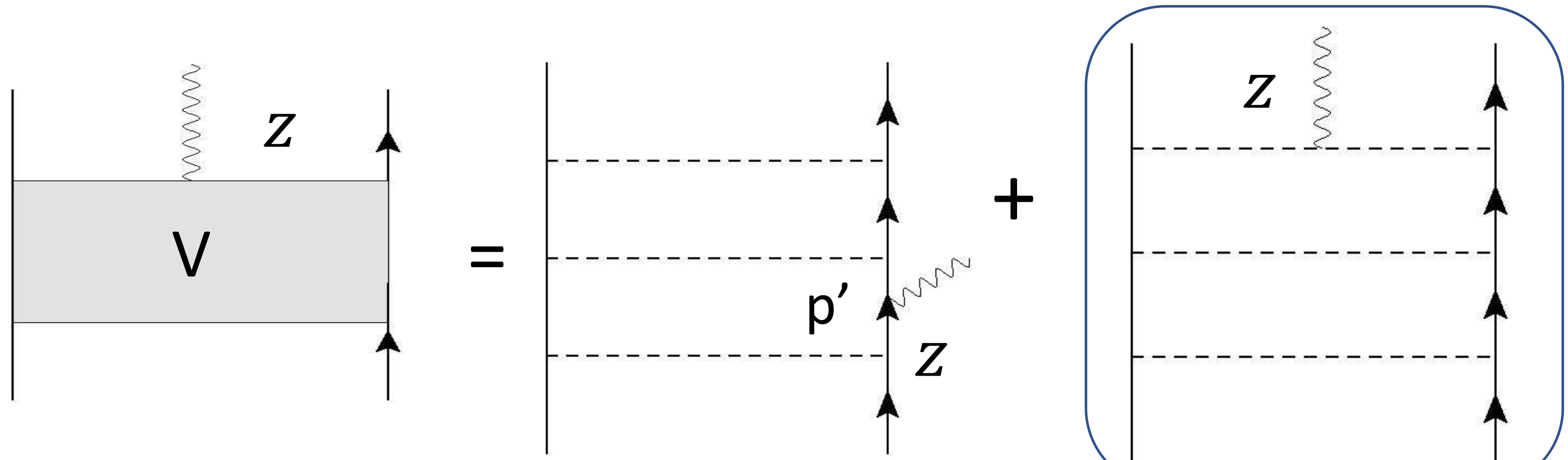
Current Matrix Elements(6) ; Result(2)

- Physical meanings
  - Time component / Charge density

$$\langle l,P|\,j^0(z)\,|m,Q\rangle = \int d^3y\,\widetilde{\psi}^{L}{}_{1,l}(\vec{z},\vec{y})\,\widetilde{\psi}^{R}{}_{1,m}(\vec{z},\vec{y})\,e^{-iE_n z^0}$$

$$+\int d^3x\ d^3y\ \psi^{L}{}_{1,lP}(\vec{x},\vec{y})\left.\frac{\delta V_\Delta[A_t]}{\delta A_0(z)}\right|_{A=0}\psi^{R}{}_{1,mQ}(\vec{x},\vec{y})$$

  - Corresponding diagrams



Summary

- We have given an example how to consider the matrix element in an effective Q.M. which is defined by the HAL QCD's phase shift equiv. potentials.
  - We have constructed the HAL QCD potential in a 2nd quantized non-relativistic 2 channel coupling model.
  - In an effective Q.M. defined by the HAL QCD potential, we have formulated how to calculate the current matrix element so that the result agrees with those obtained from the original theory.
- Remaining problems
  - Extension to
    - the relativistic theories
    - models with composite particles
  - To use it on the lattice, we have somehow to deal with
    - the time-evolution in an external field with a cutoff
    - the numerical evaluation of the functional derivative

Back up ; B.C for the Green's func.

- The Green's function

$$\left(i\partial_0 + H_0 - V\right)^{-1} = G(t,t')$$

  - The solution (without writing the coordinates)

$$\frac{\delta\psi(\vec{x},t;A)}{\delta A_\mu(z)} = G(t,z^0)K_\mu(z^0)\psi(z^0)$$

  - The functional derivative

$$\frac{\delta\psi(\vec{x},t;A)}{\delta A_\mu(z)} = 0 \qquad (t < z^0)$$

- The B.C for the Green's func.

$$G(t,t') = 0 \qquad (t < t')$$

Back up ; The Green's func.

- The explicit form

$$G(\vec{x},\vec{y};\vec{x}'\vec{y}',t,t') = -i\theta(t-t')\sum_n^{\infty}\int \frac{d^3P}{(2\pi)^3}\psi^L_{1;nP}(\vec{x},\vec{y})\psi^R_{1;nP}(\vec{x'},\vec{y'})e^{-iE_n(\vec{P})(t-t')}$$

- Satisfies

$$\left(i\partial_t + \frac{1}{2m}\nabla_x^2 + \frac{1}{2m}\nabla_y^2\right)G(\vec{x},\vec{y};\vec{x'},\vec{y'},t,t') - \int d^3x'' \int d^3y'' V(\vec{x},\vec{y};\vec{x''},\vec{y''})G(\vec{x''},\vec{y''};\vec{x'},\vec{y'},t,t')$$

$$= \delta(\vec{x}-\vec{x'})\delta(\vec{y}-\vec{y'})\delta(t-t').$$

Back up ; Ward-Takahashi id.

- Ward-Takahashi identity gives the conservation rule.
  - Ex; A p-n interaction model with non-local interaction.

$$S = S_p + S_n + S_I \qquad \begin{cases} S_p = \int d^4x \frac{1}{2}\left[i\phi_p^*(x)\overleftrightarrow{\partial_t}\phi_p(x) + \frac{1}{2m}(\nabla\phi_p^*(x))(\nabla\phi_p(x))\right] \\ S_n = \int d^4x \frac{1}{2}\left[i\phi_n^*(x)\overleftrightarrow{\partial_t}\phi_n(x) + \frac{1}{2m}(\nabla\phi_n^*(x))(\nabla\phi_n(x))\right] \\ S_I = -\int dt \int d^3x \int d^3y \phi_n^*(\vec{x},t)\phi_p^*(\vec{y},t)V(\vec{x}-\vec{y})\phi_n(\vec{y},t)\phi_p(\vec{x},t) \end{cases}$$

- The Word-Takahashi id
  - Obtained from local U(1) invariance of the observables.

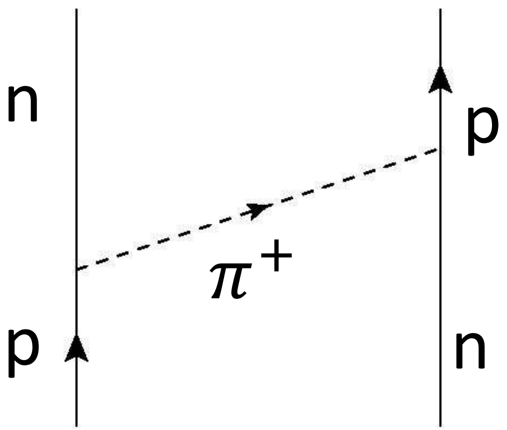
$$-(i\partial_\mu \hat{J}_\mu(z)) + \hat{V}(z) = 0 \qquad \hat{V}(z) = \int d^3x\ d^3y\ \{\delta(\vec{x}-\vec{z}) - \delta(\vec{y}-\vec{z})\}\phi_n^*(\vec{x},z^0)\phi_p^*(\vec{y},z^0)V(\vec{x}-\vec{y})\phi_p(\vec{y},z^0)\phi_n(\vec{x},z^0)$$

  - If $\hat{V}(z)$ is a divergence of an current op. , we call it an exchange current.

Introduction ; “Noether’s theorem”

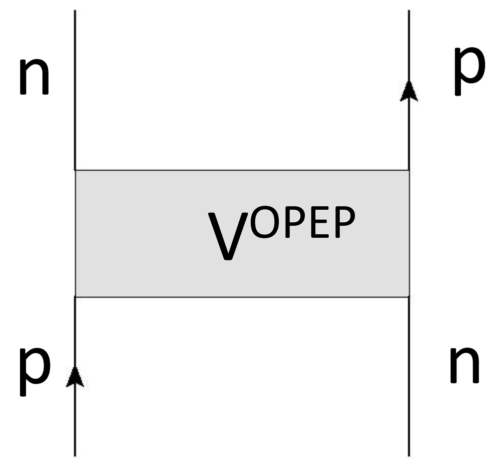
- “Noether’s currents” are not always the conserved currents.
  - Example : p-n pion exchange interaction

Original theory



Integrate out π^+

Effective quantum mechanics



Noether current = pionic part+ protonic part

“Noether current” = protonic part

- Since the pion current is missing
  the “Noether current” in the right do not conserve.

Asymptotic form of NBS wave func.

- Example ; S-wave

$$\langle 0|\phi(\vec{x})\phi(0)|\phi(+\vec{p})\phi(-\vec{p}),\mathrm{in}\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2k_0}\langle 0|\phi(\vec{x})|\phi(k)\rangle\langle\phi(k)|\phi(0)|\phi(+\vec{p})\phi(-\vec{p}),\mathrm{in}\rangle + I(\vec{x})$$

$$= e^{i\vec{p}\cdot\vec{x}} + \int \frac{d^3k}{(2\pi)^3 2k_0}\frac{\mathcal{T}}{4E(\vec{p})\left(E(\vec{k})-E(\vec{p})-i\epsilon\right)}e^{i\vec{k}\cdot\vec{x}} + I(x)$$

$$\vdots$$

$$= e^{i\vec{p}\cdot\vec{x}} + \frac{1}{2i}\left(e^{2i\delta(p)}-1\right)\frac{e^{ipr}}{pr} + \cdots$$

$$= e^{i\delta(p)}\frac{\sin\left(pr+\delta(p)\right)}{pr} + \cdots$$

$$\phi_{\mathrm{as}}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0}\left(a_{\mathrm{as}}(k)e^{-ikx} + a^\dagger_{\mathrm{as}}(k)e^{+ikx}\right)$$

$$\left[a_{\mathrm{as}}(k), a^\dagger_{\mathrm{as}}(k')\right] = (2\pi)^3 2k_0\delta^3(k-k')$$

$$\langle 0|\phi(x)|\phi(k)\rangle = Z^{1/2}e^{-ikx}$$

$$\phi(x) \to Z^{1/2}\phi_{\mathrm{in}}(x) \quad \text{as} \quad x_0 \to -\infty$$

  - From LSZ reduction formula

$$\langle\phi(k_1)\phi(k_2),\mathrm{out}|\phi(p_1)\phi(p_2),\mathrm{in}\rangle$$

$$= \langle\phi(k_1)\phi(k_2),\mathrm{in}|\phi(p_1)\phi(p_2),\mathrm{in}\rangle$$

$$+(i)^2\int d^4x_1\int d^4x_2\, e^{ik_1x_1}\left(\Box_1+m^2\right)e^{ik_2x_2}\left(\Box+m^2\right)\langle 0|T\phi(x_1)\phi(x_2)|\phi(p_1)\phi(p_2),\mathrm{in}\rangle$$

$$= \langle\phi(k_1)\phi(k_2),\mathrm{in}|\phi(p_1)\phi(p_2),\mathrm{in}\rangle$$

$$+i(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\underbrace{\left(m^2-(k_1-p_1-p_2)^2\right)\times i\int d^4x\, e^{ik_1x}\left(\Box+m^2\right)\langle 0|T\phi(x)\phi(0)|\phi(p_1)\phi(p_2),\mathrm{in}\rangle}_{=\mathcal{T}}$$

Asymptotic form of NBS wave func.

$$\langle \phi(k_1)\phi(k_2),\mathrm{out} | \phi(p_1)\phi(p_2),\mathrm{in}\rangle$$
$$= \langle \phi(k_1)\phi(k_2),\mathrm{in} | \phi(p_1)\phi(p_2),\mathrm{in}\rangle$$
$$+(i)^2 \int d^4x_1 \int d^4x_2\, e^{ik_1x_1}\left(\Box_1 + m^2\right) e^{ik_2x_2}\left(\Box + m^2\right)\langle 0 | T\phi(x_1)\phi(x_2) | \phi(p_1)\phi(p_2),\mathrm{in}\rangle$$
$$= \langle \phi(k_1)\phi(k_2),\mathrm{in} | \phi(p_1)\phi(p_2),\mathrm{in}\rangle$$
$$+i(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\underbrace{\left(m^2-(k_1-p_1-p_2)^2\right)\times i\int d^4x\, e^{ik_1x}\left(\Box + m^2\right)\langle 0 | T\phi(x)\phi(0) | \phi(p_1)\phi(p_2),\mathrm{in}\rangle}_{=\mathcal{T}}$$