Comparative studies of the deformation techniques for the singular-drift problem in the complex Langevin method

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Introduction

The complex Langevin method is a promising approach to the sign problem.

However, in finite density QCD at low temperature and high density,
this method does not work due to the singular-drift problem.

◆ Singular-drift problem [Nishimura, Shimasaki '15]

- associated with appearance of near-zero eigenvalues of the Dirac operator

➢ We can avoid this problem by deforming the Dirac operator. [YI-Nishimura '16]

cf) the application to finite density QCD → Shimasaki's talk

◆ Note, however, that there are many ways to deform the Dirac operator.

It is important to look for a good deformation.

Plan of this talk

- Introduction
- Definition of the toy model
- Deformations of the Dirac operator
- Results
- Summary

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Definition of the toy model

- the Gaussian matrix model with a fermion determinant [Nishimura '02]

$$Z = \int \prod_{\mu=1}^{4} dX_\mu \, \underline{\det D(X)} \, e^{-S_{\rm b}[X_\mu]}$$

complex

$$S_{\rm b} = \frac{1}{2} N \sum_{\mu=1}^{4} \mathrm{tr}\,(X_\mu)^2$$

$$D_{i\alpha, j\beta}(X) = \sum_{\mu=1}^{4} (\Gamma_\mu)_{\alpha\beta} (X_\mu)_{ij}$$

X_μ : $N \times N$ Hermitian matrices

$$\Gamma_\mu = \begin{cases} i\sigma_i & \text{for } \mu = 1,2,3, \\ \mathbf{1}_2 & \text{for } \mu = 4 \end{cases}$$

- SO(4) rotational symmetry

→ broken down to SO(2) due to the fermion determinant

(predicted by the Gaussian expansion method)

[Nishimura-Okubo-Sugino '05]

- To see the SSB, introducing a symmetry breaking term,

$$\frac{N}{2} \mathrm{tr} X_\mu^2 \to \frac{N}{2} (1 + \epsilon_\mu) \, \mathrm{tr} X_\mu^2$$

We take the large N limit and then $\epsilon \to 0$ limit.

Application of the complex Langevin method

- complex Langevin equation

$$\frac{dX_\mu}{d\tau} = -\frac{dS}{dX_\mu} + \eta_\mu(\tau)$$

$\eta_\mu(\tau)$: white noise

$$S = \frac{N}{2}(1+\epsilon_\mu)\,\mathrm{tr}X_\mu^2 - \ln\det D(X)$$

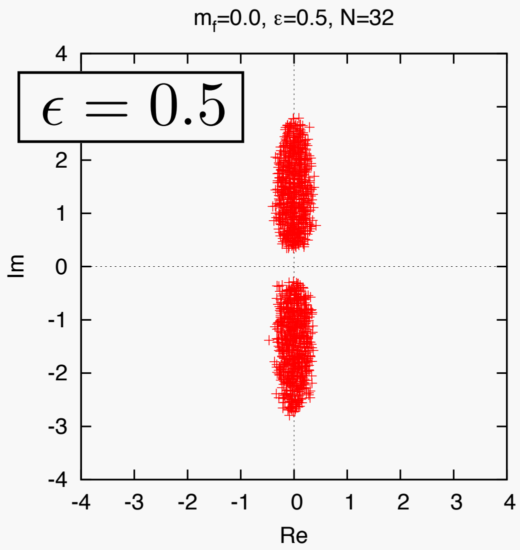
- drift term

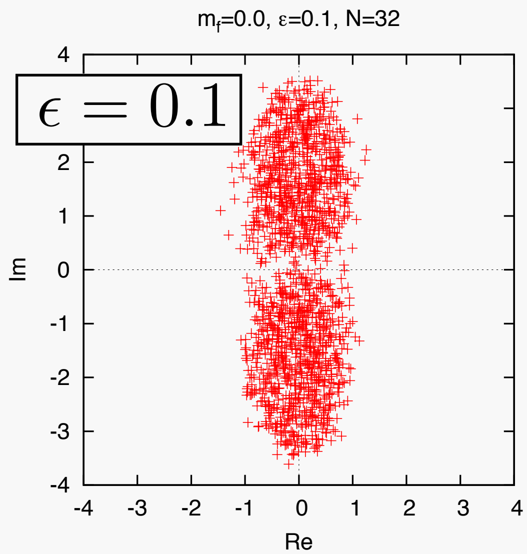
$$\frac{dS}{dX_\mu} = N(1+\epsilon_\mu)X_\mu - \underline{\mathrm{tr}_\alpha\left(D^{-1}\Gamma_\mu\right)}$$

Appearance of near-zero eigenvalues of the Dirac op.
causes the singular-drift problem.

The singular-drift problem

◆ Eigenvalue distribution of the Dirac operator





→ It is difficult to make ϵ_μ sufficiently small.

We consider the deformation of the Dirac op. to avoid the singular-drift problem and extrapolate the deformation parameter to zero using only reliable data points.

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Deformation of the Dirac operator

$$D(X) = \sum_{\mu=1}^{4} \Gamma_\mu \otimes X_\mu \qquad \Gamma_\mu = \begin{cases} i\sigma_i & \text{for } \mu = 1,2,3, \\ \mathbf{1}_2 & \text{for } \mu = 4 \end{cases}$$

To avoid singular-drift problem, we consider two types of deformation.

◆ Deformation 1 [YI-Nishimura '16]

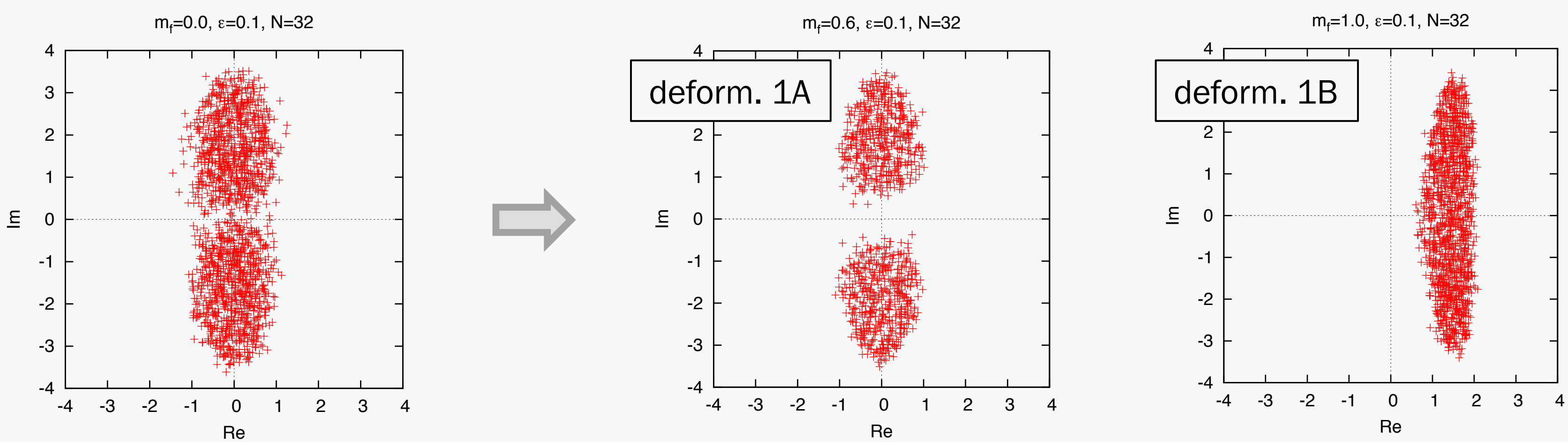
$$D(X; m_f) = \sum_{\mu=1}^{4} \Gamma_\mu \otimes X_\mu + \begin{cases} m_f \Gamma_3 \otimes \mathbf{1}_N & \text{: deformation 1A} \\ m_f \Gamma_4 \otimes \mathbf{1}_N & \text{: deformation 1B} \end{cases}$$

$m_f \in R$: deformation parameter

This term splits or shifts the eigenvalue distribution.

$m_f \to 0$

◆ Eigenvalue distribution



➢ The singularity at the origin is avoided.

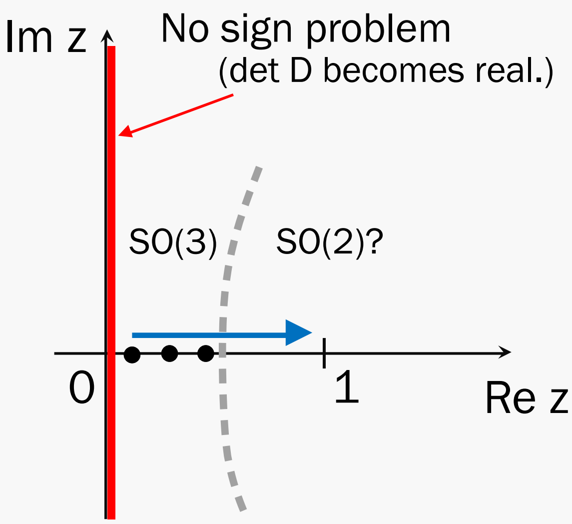
Deformation of the Dirac operator (cont'd)

◆ Deformation 2

$$D\left(X;z\right)=\sum_{i=1}^{3}\Gamma_i\otimes X_i+z\Gamma_4\otimes X_4 \qquad z\in C$$

z=1 : undeformed model

This deformation ($z \neq 1$) breaks SO(4) symmetry to SO(3).



⇨ z can be regarded as an analogue of the chemical potential in finite density QCD.

◆ We restrict z to be real and approach z = 1.

The phase transition is expected to occur at some z within $0 < z < 1$.

Plan of this talk

Observables

- order parameters for the SSB of SO(4)

$$\rho_\mu(\epsilon, m_\mathrm{f}) = \lim_{N\to\infty} \frac{\left\langle \frac{1}{N}\mathrm{tr}X_\mu^2 \right\rangle_{\epsilon, m_\mathrm{f}}}{\sum_{\nu=1}^{4} \left\langle \frac{1}{N}\mathrm{tr}X_\nu^2 \right\rangle_{\epsilon, m_\mathrm{f}}}$$

- We take $\epsilon \to 0$ limit and $m_\mathrm{f} \to 0$ limit.

- prediction from the Gaussian expansion method

$$\rho_\mu(\epsilon = 0, m_\mathrm{f} = 0) \simeq \underline{0.35, 0.35,} \, 0.167, 0.133$$

→ This indicates the SSB from SO(4) to SO(2)

- Note that these results may contain uncontrollable systematic errors.

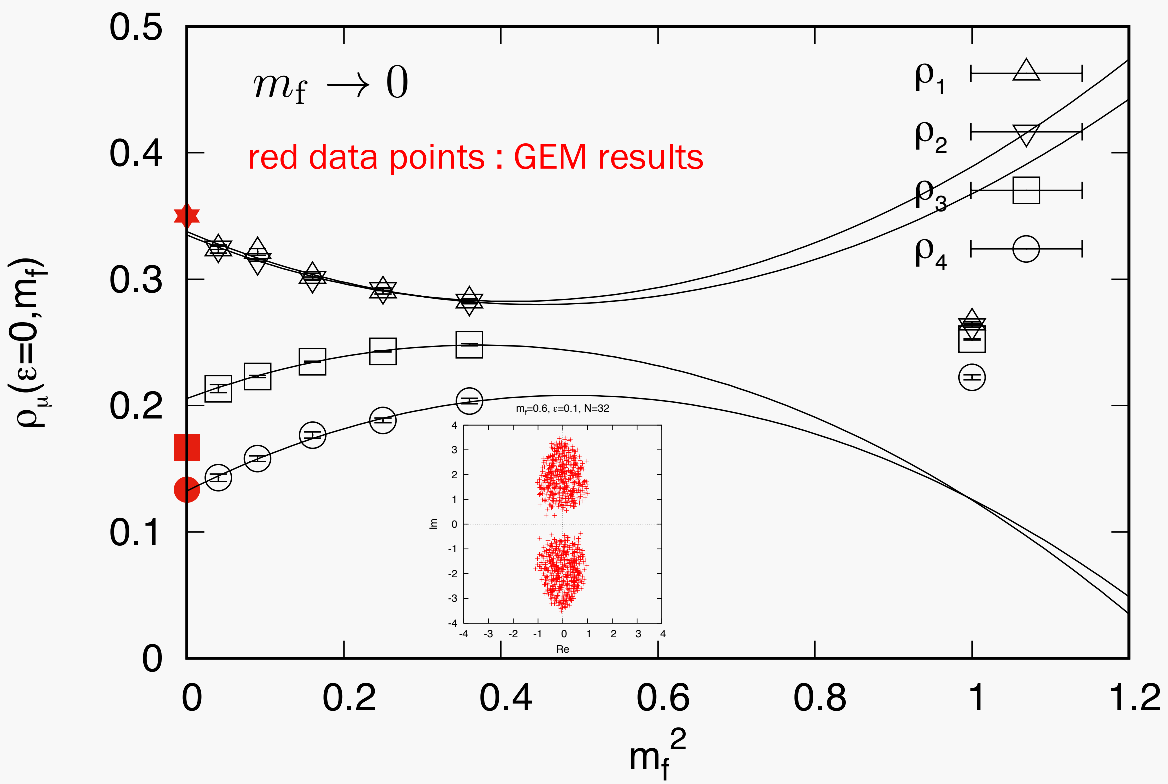
Results

- Deformation 1A

$$D(X;m_{\mathrm{f}})=\sum_{\mu=1}^{4}\Gamma_\mu\otimes X_\mu+\underline{m_{\mathrm{f}}\Gamma_3\otimes\mathbf{1}_N}$$

invariant under $m_{\mathrm{f}}\to -m_{\mathrm{f}}$

- The lines are fits to $f(m_{\mathrm{f}})=c_1+c_2m_{\mathrm{f}}^2+c_3m_{\mathrm{f}}^4$



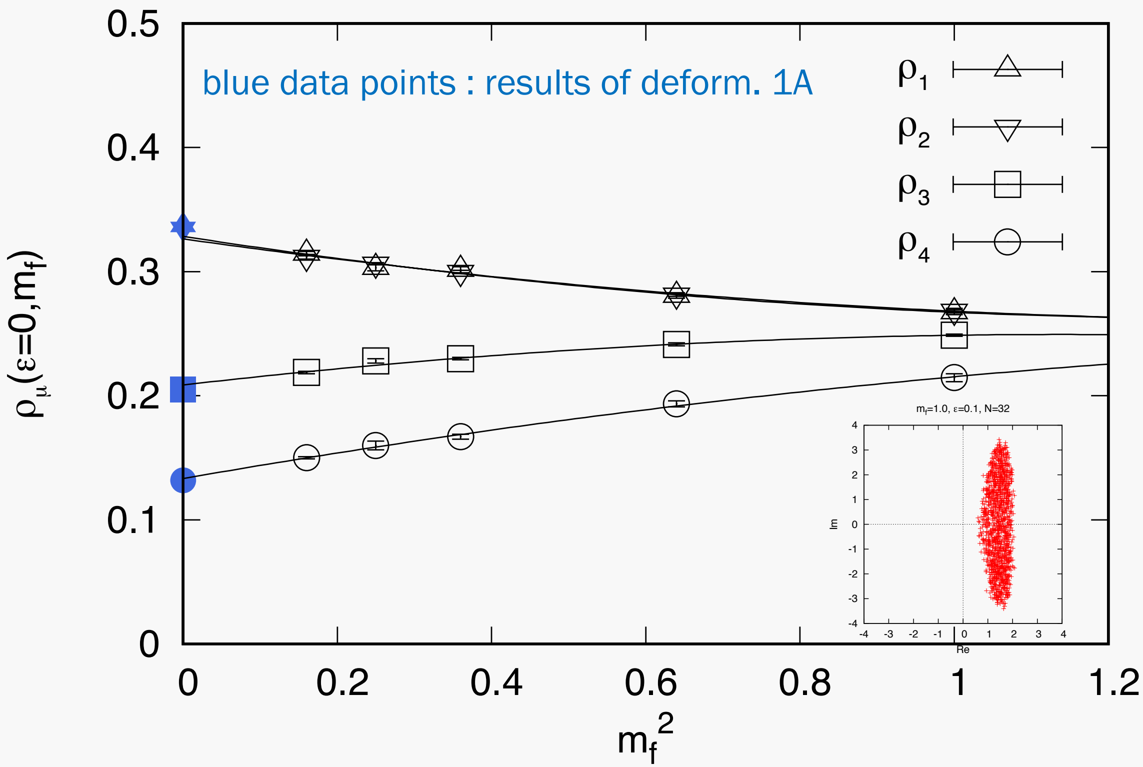
[YI-Nishimura '16]

Results

◆ Deformation 1B

$$D\left(X;m_{\mathrm{f}}\right)=\sum_{\mu=1}^{4}\Gamma_{\mu}\otimes X_{\mu}+m_{\mathrm{f}}\Gamma_{4}\otimes\mathbf{1}_{N}$$

invariant under $m_{\mathrm{f}}\to -m_{\mathrm{f}}$



[YI-Nishimura '16]

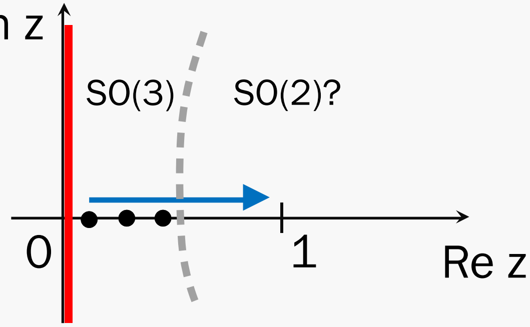
These results are consistent with that of deformation 1A.
This suggests that the GEM results have certain systematic errors.

Results (Preliminary)

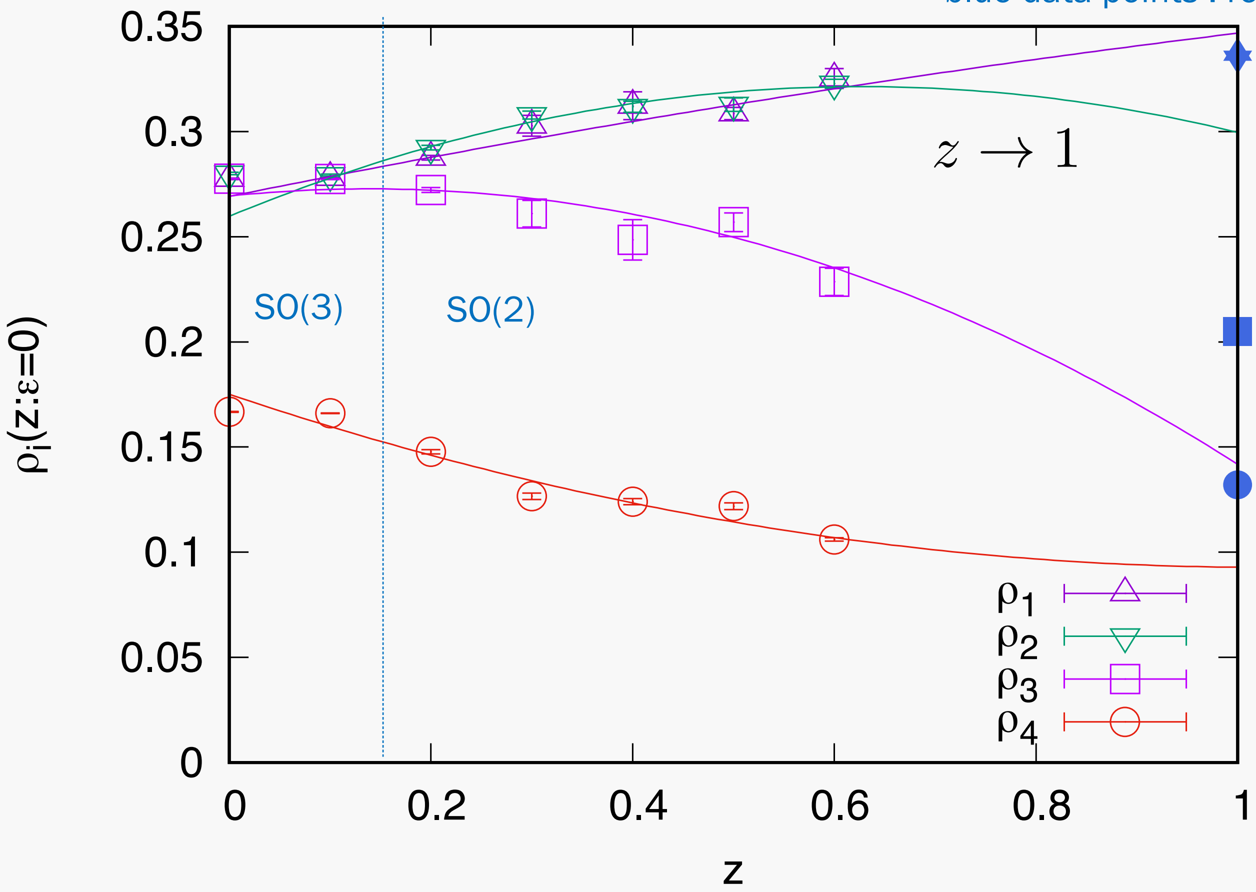
◆ Deformation 2

$$D\left(X;z\right)=\sum_{i=1}^{3}\Gamma_i\otimes X_i+z\Gamma_4\otimes X_4$$

➢ The lines are fits to $g\left(z\right)=c_1+c_2z+c_3z^2$



blue data points : results of deform. 1A



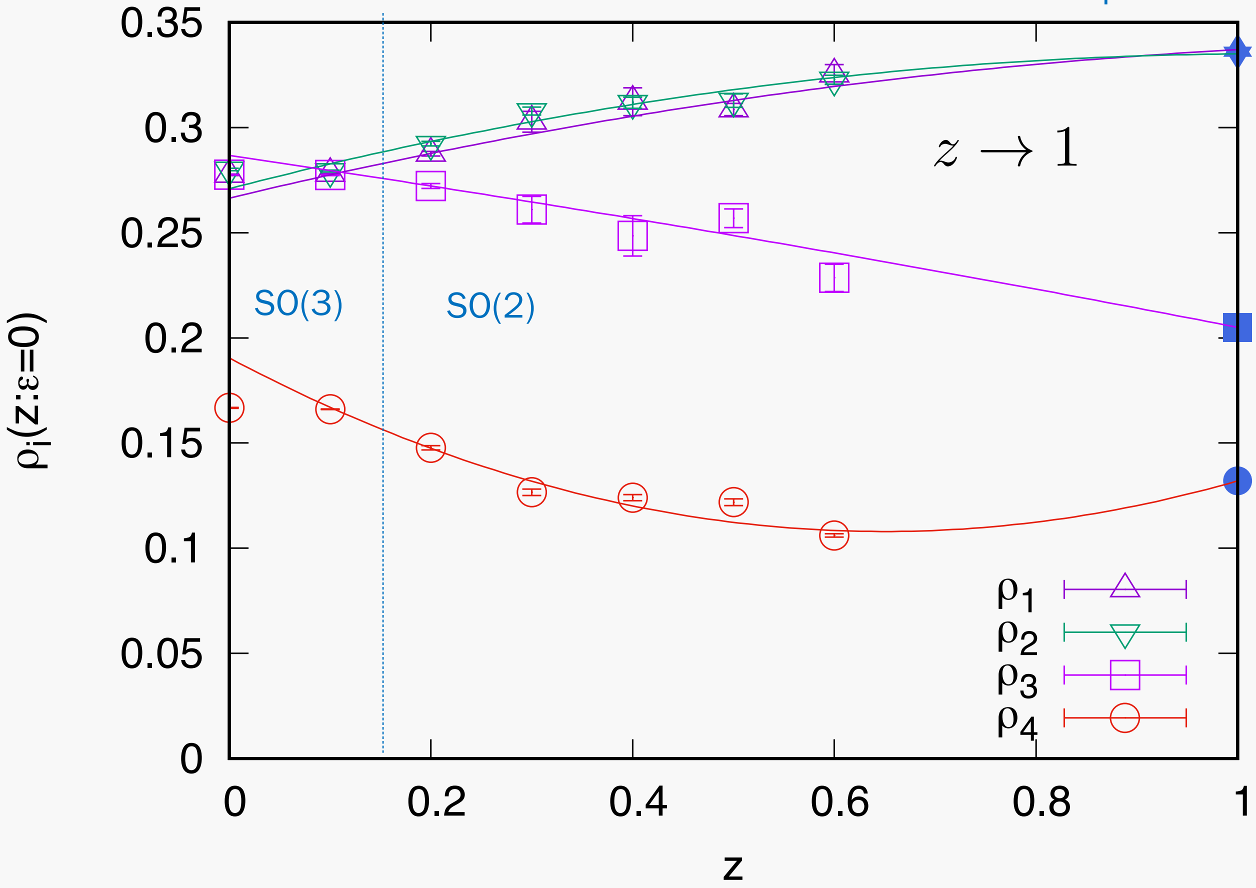
Results (Preliminary)

◆ Deformation 2

$$D(X; z) = \sum_{i=1}^{3} \Gamma_i \otimes X_i + z\Gamma_4 \otimes X_4$$

➢ Fit using the results obtained by the "deformation 1A".

blue data points : results of deform. 1A

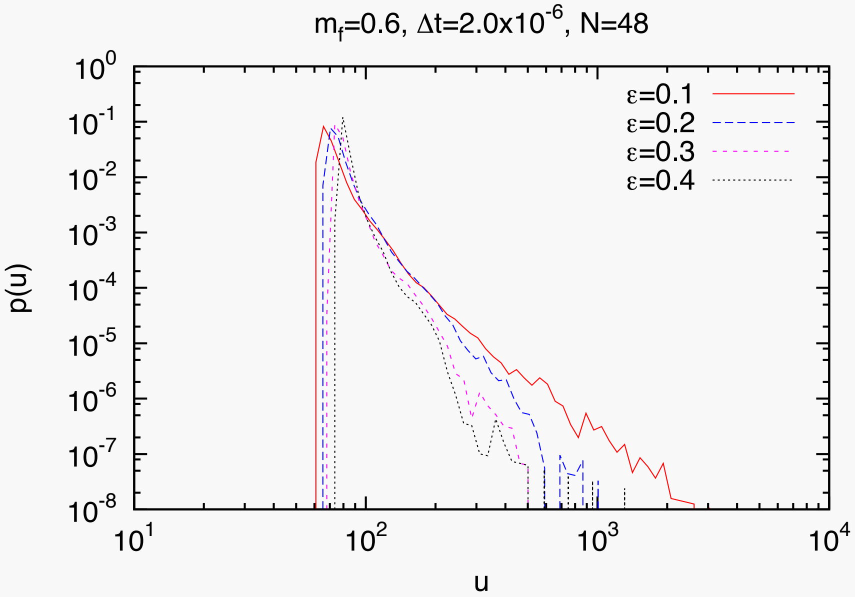


Summary

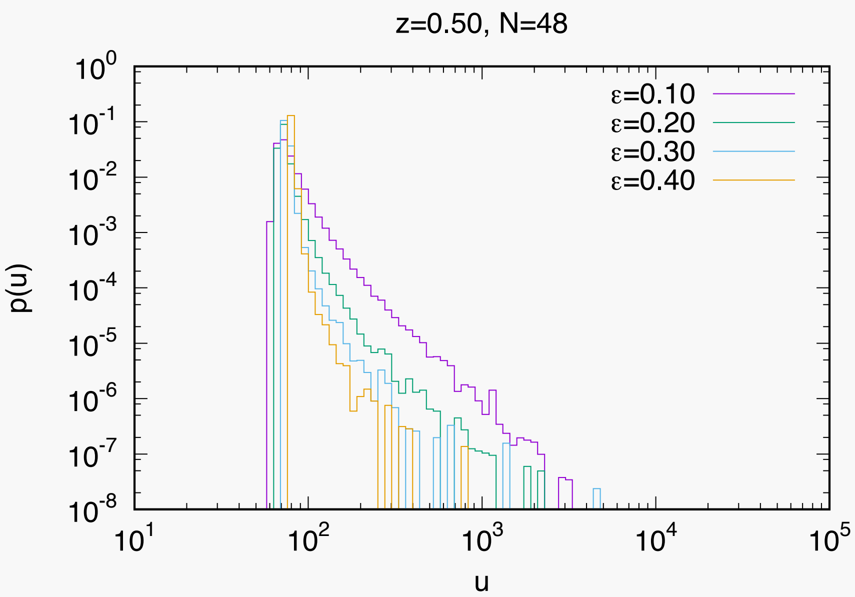
- In order to avoid the singular-drift problem in the CLM, we have deformed the Dirac operator.
- We considered two types of deformation and compared the results.
- After taking $m_{\mathrm{f}} \to 0$ limit, the results obtained for deformation 1A are consistent with that for deformation 1B, and both the results show the SSB from SO(4) to SO(2).
- However, there is a slight deviation from the prediction by the GEM, which indicates that the result of GEM contains certain systematic errors.
- More calculations are needed for deformation 2.
- Our work implies the importance of trying various deformations and seeing the consistency.

The Validity of the simulations

- histogram of the norm of drift
  - deformation1



  - deformation2



$$\rho_\mu\,(\epsilon = 0, m_{\mathrm{f}} = 0) \simeq 0.35, 0.35, 0.167, 0.133$$

$$\lim_{m_{\mathrm{f}}\to 0}\lim_{\epsilon\to 0}\rho_\mu(\epsilon, m_{\mathrm{f}}) = 0.328(4), 0.326(2), 0.208(2), 0.133(2)$$

$$\lim_{m_{\mathrm{f}}\to 0}\lim_{\epsilon\to 0}\rho_\mu(\epsilon, m_{\mathrm{f}}) = 0.337(6), 0.335(2), 0.205(2), 0.132(4)$$