An Alternative Lattice Field Theory Formulation Inspired by Lattice Supersymmetry

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Motivations

- Original motivation: quest for exact Lattice supersymmetry. This has two fundamental difficulties:
  - *Violation of the Leibniz rule* for the derivative (difference) operator on the lattice.
  - *Species doubler problem* for *chiral fermions*
- These two problems have a common origin: in the continuum the momentum is both the derivative operator and a additive and conserved quantity. In the standard lattice formulation the derivative operator does not coincide with the momentum and is not additive and conserved quantity $\Rightarrow$ violation of the Leibniz rule.
- We shall then investigate a lattice formulation in which it is the derivative operator to be additive and conserved.
- Locality and translational invariance are lost on the lattice, but in the end recovered as we shall see.

The "new lattice" - Assumptions.

We work in momentum representation.

- All fields are periodic with period $\frac{4\pi}{a}$, lattice spacing $\frac{a}{2}$.
- The continuum derivative operator $i\partial_\mu$ (denoted by $\hat{p}_\mu$ in momentum representation) is replaced on the lattice by a function $\Delta(p_\mu)$ that satisfies the following properties:
  - $\Delta(p_\mu)$ is periodic with period $\frac{4\pi}{a}$
  - $\Delta(p_\mu)$ is an odd function of p_μ: $\Delta(-p_\mu) = -\Delta(p_\mu)$.
  - For $\frac{ap_\mu}{2} \ll 1$ the lattice derivative coincides with p_μ:

    $\Delta(p_\mu) = p_\mu + O\left(\left(\frac{ap_\mu}{2}\right)^3\right)$
  - $\Delta(p_\mu)$ has the symmetry (more about this later)

    $\Delta(\frac{2\pi}{a} - p_\mu) = \Delta(p_\mu)$
  - Simplest (ultralocal) example: $\Delta(p_\mu) = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$.
    Non local derivatives on the lattice will need to be introduced later on.

The "new lattice" - Assumptions (cont.)

- $\Delta(p_\mu)$ is an additive and conserved quantity, just like the momentum $\hat{p}_\mu$ in the continuum.

  This is a crucial assumption and it implies:

  - additive: the local product of two fields of the continuum theory is replaced on the lattice by a non-local product which we call star product (see following slides).

    Important: with respect of the star product $\Delta(p_\mu)$ satisfies the Leibniz rule.

  - conserved: the delta function of continuum momentum conservation

    $$\delta\left(\hat{p}_{1\mu} + \hat{p}_{2\mu} + \cdots + \hat{p}_{n\mu}\right)$$

    is replaced on the lattice by:

    $$\delta\left(\Delta(p_{1\mu}) + \Delta(p_{2\mu}) + \cdots + \Delta(p_{n\mu})\right)$$

    Dondi-Nicolai, 1977

The doublers problem - 1

Since $\Delta(p_\mu)$ is additive and conserved it must be identified with the momentum $\hat{p}_\mu$ of the continuum theory. This defines a map from the lattice momentum to the continuum momentum:

$$\hat{p}_\mu = \Delta(p_\mu)$$

This is not a one-to-one correspondence.

Consider for instance $\Delta(p_\mu) = \frac{2}{a} \sin \frac{ap_\mu}{2}$. For each choice of μ the above correspondence has

- for $|\hat{p}_\mu| > \frac{2}{a}$ no solution for p_μ
- for $|\hat{p}_\mu| < \frac{2}{a}$ two solutions for p_μ

In particular:

$$\hat{p}_\mu = 0 \qquad \Longrightarrow \qquad p_\mu = (0, \frac{2\pi}{a})$$

- In d dimensions a constant field configuration in the continuum ($\hat{p}_\mu = 0$) has 2^d images on the lattice.
- One d.o.f. on the lattice (bosonic or fermionic alike) corresponds to 2^d d.o.f. of the continuum theory $\Rightarrow$ doublers.

The doublers problem - 2

Species doublers are present in both bosonic and fermionic sector (which is good for supersymmetry) but the real novelty is that

- All fermionic doublers have the same chirality
- Doublers (bosonic and fermionic) can be identified by imposing that in momentum representation all fields are invariant, for each component labeled by μ, under $p_\mu \rightarrow \frac{2\pi}{a} - p_\mu$, namely in one dimension:

$$\tilde{\varphi}_A \left(\frac{2\pi}{a} - p \right) = \tilde{\varphi}_A (p)$$

In coordinate representation:

$$\varphi \left(-\frac{na}{2} \right) = (-1)^n \varphi \left(\frac{na}{2} \right)$$

The doublers problem - 3

- In some theories with extended supersymmetry the doublers can be treated as independent degrees of freedom and associated to different members of a supermultiplet.

  AD,A.Feo,N.Kawamoto,I.Kanamori,J.Saito, 2010 and 2012

The doublers change of chirality with respect to the conventional lattice is due to

$$\left.\frac{d\hat{p}_\mu}{dp_\mu}\right|_{p_\mu=0} = 1 \qquad \left.\frac{d\hat{p}_\mu}{dp_\mu}\right|_{p_\mu=\frac{2\pi}{a}} = -1$$

This is different from the conventional lattice where the lattice momentum itself p_μ is additive and conserved *modulo* $\frac{2\pi}{a}$ and hence the map from p_μ to $\hat{p}_\mu$ is of the form:

$$\hat{p}_\mu - p_\mu = \frac{2\pi}{a}k_\mu \qquad -\frac{2\pi}{a} < p_\mu < \frac{2\pi}{a}$$

$$\Rightarrow \qquad \frac{d\hat{p}_\mu}{dp_\mu} = 1 \qquad \text{everywhere}$$

The star product - 1

- In the continuum the local product

  $\Phi_{12}(x) \equiv \Phi_1 \cdot \Phi_2(x) = \Phi_1(x)\Phi_2(x)$

  becomes in momentum space a convolution (locality $\Rightarrow$ additive $\hat{p}$)

  $\tilde{\Phi}_{12}(\hat{p}) = \int_{-\infty}^{+\infty} d\hat{p}_1 \, d\hat{p}_2 \, \tilde{\Phi}_1(\hat{p}_1) \, \tilde{\Phi}_2(\hat{p}_2)\delta\left(\hat{p} - \hat{p}_1 - \hat{p}_2\right)$

  and $\hat{p}$ automatically satisfies the Leibniz rule:

  $\hat{p}\tilde{\Phi}_{12}(\hat{p}) = \left(\hat{p}_1\tilde{\Phi}_1(\hat{p}_1)\right)\tilde{\Phi}_2(\hat{p}_2) + \tilde{\Phi}_1(\hat{p}_1)\left(\hat{p}_2\tilde{\Phi}_2(\hat{p}_2)\right)$

- On the lattice the local product is replaced by the star product (additive $\Delta(p)$):

  $$\widetilde{\varphi_1 \star \varphi_2}(p_{12}) = \frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} dp_1 dp_2 \, V(p_{12}; p_1, p_2)\tilde{\varphi}_1(p_1)\tilde{\varphi}_2(p_2) \cdot$$
  $$\cdot \delta\left(\Delta(p_{12}) - \Delta(p_1) - \Delta(p_2)\right)$$

  $\Delta(p)$ satisfies the Leibniz rule with respect to the star product by construction.

The star product - 2

- The star product is non-local in coordinate representation.
- The star product is in commutative but in general is not associative.
- Associativity is needed for a consistent lattice formulation of some symmetries (gauge symmetry, not supersymmetry) and can be implemented if
  - the derivative operator $\Delta(p)$ takes all values from $-\infty$ to $+\infty$ so that for any p_1 and p_2 there is always a solution to $\Delta(p_1) + \Delta(p_2) = \Delta(p_{12})$.
  - All functions/fields are symmetric under $p_\mu \to \frac{2\pi}{a} - p_\mu$ (doublers identified)
  - the integration volume $V(p_{12}; p_1, p_2)$ satisfy some involved but in some cases solvable functional equations

  $\implies$ Associativity requires a $\Delta(p)$ that contains infinite powers of $\sin \frac{ap}{2} \implies$ non local derivative operator:

$$\hat{p}_\mu = \Delta(p_\mu) = \frac{2}{a} \sum_{k=1}^{\infty} c_k \left(\sin \frac{ap_\mu}{2} \right)^{2k+1}$$

Associative star product- 1

Choice for $\Delta(p_\mu)$:

$$\frac{a\hat{p}_\mu}{2} = \frac{a}{2}\Delta_G(p_\mu) = \frac{1}{2}\log\frac{1 + \sin\frac{ap_\mu}{2}}{1 - \sin\frac{ap_\mu}{2}}$$

The function at r.h.s. is the inverse Gudermannian function and it is a unique choice because:

- It's the least non-local, i.e. fastest decreasing c_k in the sine expansion.
- Any trigonometric function of $\frac{ap_\mu}{2}$ is an hyperbolic function of $\frac{a\hat{p}_\mu}{2}$ (most suitable for a lattice $\Leftrightarrow$ continuum momentum map). For instance

$$\cosh\frac{a\hat{p}_\mu}{2} = \frac{1}{\left|\cos\frac{ap_\mu}{2}\right|}, \qquad \tanh\frac{a\hat{p}_\mu}{2} = \sin\frac{ap_\mu}{2}$$

Associative star product- 2

- Δ_G is a non local derivative operator in coordinate representation:
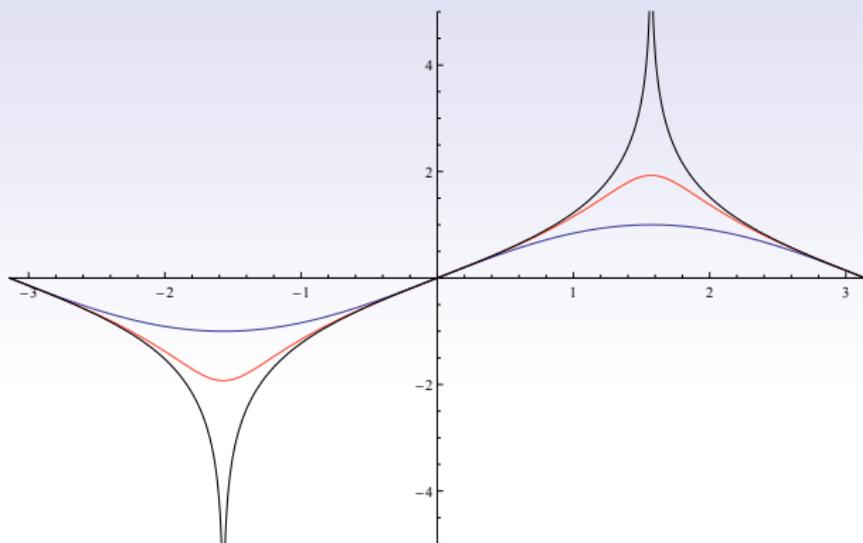
$$\Delta_G \varphi(x) = \frac{2}{a} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left[\varphi \left(x + \frac{(2k-1)a}{2} \right) - \varphi \left(x - \frac{(2k-1)a}{2} \right) \right]$$

- Continuum limit $a \to 0$ not well defined in coordinate representation.

- Regularized derivative operator $\Delta_G^{(z)}(p_\mu)$ defined as

$$\Delta_G^{(z)}(p_\mu) = \frac{1}{a\hat{z}} \log \frac{1 + \hat{z} \sin \frac{ap_\mu}{2}}{1 - \hat{z} \sin \frac{ap_\mu}{2}}$$

interpolates between the sine function ($\hat{z} = 0$) and the inverse gudermannian function ($\hat{z} = 1$).

The inverse gudermannian function

- Red line: "Regularized inverse Gudemannian"
  $\frac{1}{2\hat{z}} \log \frac{1+\hat{z}\sin x}{1-\hat{z}\sin x}$ at $\hat{z} = 0.95$
- Blue line: Same at $z = 0 \Rightarrow \sin x$
- Black line: Same at $z = 1 \Rightarrow \frac{1}{2} \log \frac{1+\sin x}{1-\sin x}$

Associative star product - 3

- Final form of the associative star product after solving eqn.s for the integration volume and modulo a possible rescaling of all fields in momentum representation:

$$\cos\frac{ap_{12}}{2}\widetilde{\varphi_1 \star \varphi_2}(p_{12}) \quad = \frac{2}{\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dp_1\, dp_2\, \tilde{\varphi}_1(p_1)\tilde{\varphi}_2(p_2)\cdot$$
$$\cdot\delta\left(\Delta_G(p_{12}) - \Delta_G(p_1) - \Delta_G(p_2)\right),$$

- If we choose $\hat{p} = \Delta_G(p)$ as integration variable and define:

$$\tilde{\varphi}(p) = \cosh\frac{a\hat{p}}{2}\tilde{\Phi}(\hat{p})$$

the associative star product takes the form od an ordinary local product in momentum representation:

$$\widetilde{\Phi_1 \star \Phi_2}(\hat{p}_{12}) \quad = \frac{2}{\pi}\int_{-\infty}^{+\infty} d\hat{p}_1\, d\hat{p}_2\, \tilde{\Phi}_1(\hat{p}_1)\tilde{\Phi}_2(\hat{p}_2)\cdot$$
$$\cdot\delta\left(\hat{p}_{12} - \hat{p}_1 - \hat{p}_2\right)$$

Continuum ⇔ lattice "blocking" transformations

The relation above between $\tilde{\varphi}(p)$ and $\tilde{\Phi}(\hat{p})$ can be used to define a lattice action from one in the continuum through a kind of blocking transformation:

$$e^{-S_{latt}(\tilde{\varphi})} = \int \mathcal{D}\tilde{\Phi}_A \prod_{p_\mu, A} \delta \left(\tilde{\Phi}_A(\Delta_G(p)) - \prod_\mu \cos \frac{ap_\mu}{2} \tilde{\varphi}_A(p) \right) e^{-S_{cl}(\tilde{\Phi})}$$

The above blocking transformation has the following properties:

- It is invertible: there is no loss of information going from the continuum to the lattice, just a reshuffling of the degrees of freedom.

- Hence all symmetries of the continuum theory are preserved on the lattice.

- However the lattice theory has the same ultraviolet divergences as the continuum one and needs regularazing. The lattice spacing a does not act as a regulator.

- Replacing $\Delta_G(p)$ with its regularized version $\Delta_G^{(z)}(p)$ introduces a momentum cutoff and thus regularizes the ultraviolet divergences. The cutoff depends on the lattice constant a and the parameter z and can be sent to infinity by either $a \to 0$ or $z \to 1$. The first case corresponds to the standard continuum limit, while in the second the infinite cutoff limit is taken preserving the lattice structure

- The momentum cutoff preserves supersymmetry, but breaks gauge invariance. A gauge invariant regularization that keeps the lattice structure has not yet been found.

- This is the main problem/limitation of this formalism at the present stage.