Hadronic light-by-light contribution to $(g-2)_\mu$: a dispersive approach

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Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)

Setting up the stage:
   Gauge invariance and crossing symmetry
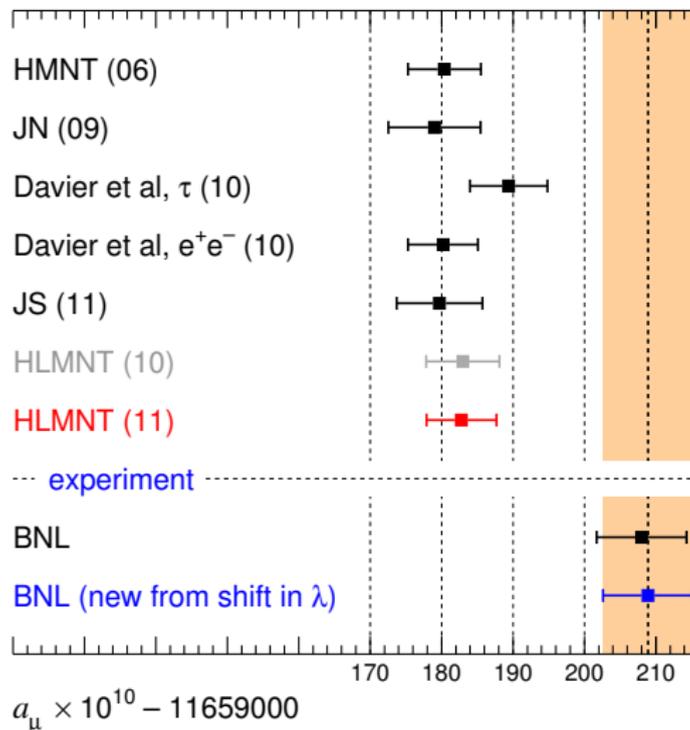   Master Formula

A dispersion relation for HLbL

Numerics
   - Pion box contribution
   - Pion rescattering contribution

Outlook and Conclusions

Status of $(g-2)_\mu$, experiment vs SM


Hagiwara et al. 2012

Status of $(g-2)_\mu$, experiment vs SM

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.95	0.04
electroweak, total	153.6	1.0
HVP (LO) [Hagiwara et al. 11]	6 949.	43.
HVP (NLO) [Hagiwara et al. 11]	−98.	1.
HLbL [Jegerlehner-Nyffeler 09]	116.	40.
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.	2.
theory	116 591 855.	59.

Hadronic light-by-light: irreducible uncertainty?

- ► Hadronic contributions responsible for most of the theory uncertainty

- ► Hadronic vacuum polarization (HVP) can be systematically improved
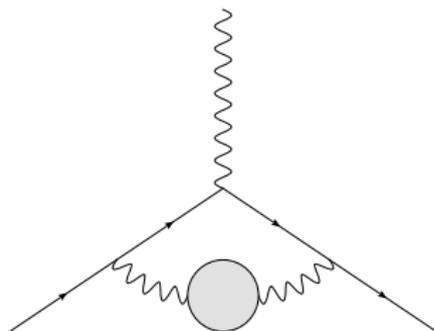
Hadronic light-by-light: irreducible uncertainty?

▶ Hadronic contributions responsible for most of the theory uncertainty

▶ Hadronic vacuum polarization (HVP) can be systematically improved → talk by Lehner

- ▶ basic principles: unitarity and analyticity
- ▶ direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- ▶ dedicated e^+e^- program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)
- ▶ alternative approach: lattice → talk by Lehner

Hadronic light-by-light: irreducible uncertainty?

- ▶ Hadronic contributions responsible for most of the theory uncertainty

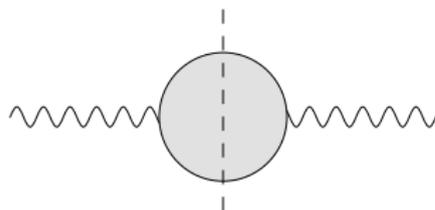- ▶ Hadronic vacuum polarization (HVP) can be systematically improved
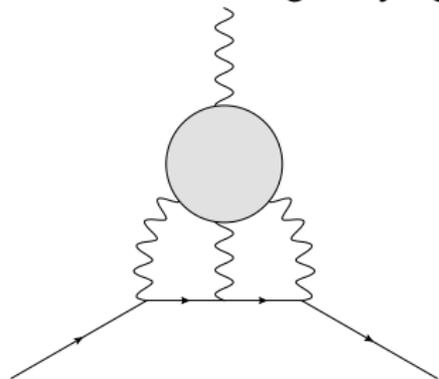 → talk by Lehner

- ▶ Hadronic light-by-light (HLbL) is more problematic:

  - ▶ 4-point fct. of em currents in QCD
  - ▶ "*it cannot be expressed in terms of measurable quantities*"
  - ▶ up to now, only model calculations
  - ▶ lattice QCD is making fast progress
    RBC/UKQCD, Mainz, several parallel talks

Analytic Approaches to Hadronic light-by-light

- ▶ Model calculations

  - ▶ ENJL — Bijnens, Pallante, Prades (95-96)
  - ▶ NJL and hidden gauge — Hayakawa, Kinoshita, Sanda (95-96)
  - ▶ nonlocal χQM — Dorokhov, Broniowski (08)
  - ▶ AdS/CFT — Cappiello, Cata, D'Ambrosio (10)
  - ▶ Dyson-Schwinger — Goecke, Fischer, Williams (11)
  - ▶ constituent χQM — Greynat, de Rafael (12)
  - ▶ resonances in the narrow-width limit — Pauk, Vanderhaeghen (14)

- ▶ Impact of rigorously derived constraints

  - ▶ high-energy constraints taken into account in several models above
    addressed specifically by — Knecht, Nyffeler (01)
  - ▶ high-energy constraints related to the axial anomaly — Melnikov, Vainshtein (04) and Nyffeler (09)
  - ▶ sum rules for $\gamma^*\gamma \to X$ — Pascalutsa, Pauk, Vanderhaeghen (12)
    see also: workshop MesonNet (13)
  - ▶ low-energy constraints–pion polarizabilities — Engel, Ramsey-Musolf (13)

Different evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22±5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht
N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- ▶ large uncertainties (and differences among calculations) in individual contributions

- ▶ pseudoscalar pole contributions most important

- ▶ second most important: pion loop, *i.e.* two-pion cuts (*K*s are subdominant*)

- ▶ heavier single-particle poles decreasingly important (unless one models them to resum the high-energy tail)

Recent progress on the lattice

- a_μ^{HLbL} in finite-volume QCD and QED (QED$_{\mathrm{L}}$):   talks by Blum and Izubuchi
  - (RBC/UKQCD): Connected diagram with $m_\pi = 171$ MeV; $a_\mu^{\mathrm{HLbL}} = 13.21(68) \times 10^{-10}$ PRD93(2016)014503
  - (RBC/UKQCD): Connected and leading disconnected diagram with $m_\pi = 139$ MeV; $a_\mu^{\mathrm{HLbL}} = 5.35(1.35) \times 10^{-10}$
    (potentially large finite-volume systematics)   PRL118(2017)022005

- a_μ^{HLbL} in finite-volume QCD and infinite-volume QED: talk by Asmussen
  - (Mainz) Method proposed and successfully tested against the lepton-loop analytic result   arXiv:1510.08384 arXiv:1609.08454
  - (RBC/UKQCD) Similar method plus subtraction scheme to reduce systematic errors; successfully tested against lepton-loop analytic result   arXiv:1705.01067

- The pion pole contribution to a_μ^{HLbL}:   talk by Gerardin
  - (Mainz) Calculation of the pion transition form factor on the lattice gives: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = 6.50(83) \times 10^{-10}$ PRD94(2016)074507

Our approach to hadronic light-by-light

We address the calculation of the hadronic light-by-light tensor

- ▶ model independent ⇒ rely on dispersion relations

- ▶ as data-driven as possible

- ▶ takes into account high-energy constraints
  [OPE, perturbative QCD]
  (work in progress, not discussed here)

Alternative dispersive approach for the μ-FF   Pauk-Vanderhaeghen (14)

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Hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle = \left(q_\mu q_\nu - g_{\mu\nu} q^2 \right) \Pi(q^2)$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\bar{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\bar{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M_\pi^2}^\infty dt \frac{\mathrm{Im}\bar{\Pi}(t)}{t(t - q^2)}$$

Hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0| Tj_\mu(x)j_\nu(0)|0\rangle = \left(q_\mu q_\nu - g_{\mu\nu}q^2 \right) \Pi(q^2)$$

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Easy!

The HLbL tensor (much less easy...)

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz \, e^{-i(x \cdot q_1 + y \cdot q_2 + z \cdot q_3)} \langle 0 | T\{j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \quad t = (q_1 + q_3)^2 \quad u = (q_2 + q_3)^2$$

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General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu}g^{\lambda\sigma}\Pi^1 + g^{\mu\lambda}g^{\nu\sigma}\Pi^2 + g^{\mu\sigma}g^{\nu\lambda}\Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi_{ijkl}^4 + \dots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \dots\}$, but in $d = 4$ only
136 are linearly independent   Eichmann *et al.* (14)

The HLbL tensor (much less easy...)

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Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

- ▶ 43 basis tensors (BT) $\qquad$ in $d = 4$: 41=no. of helicity amplitudes

- ▶ 11 additional ones (T) $\qquad$ to guarantee basis completeness everywhere

- ▶ of these 54 only 7 are distinct structures

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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$$
\begin{aligned}
T_1^{\mu\nu\lambda\sigma} &= \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta}q_{1\,\alpha}q_{2\,\beta}q_{3\,\gamma}q_{4\,\delta}, \\
T_4^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right), \\
T_7^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right), \\
T_{19}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right), \\
T_{31}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right), \\
T_{37}^{\mu\nu\lambda\sigma} &= \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\left(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right)\right. \\
&\qquad\qquad \left. + g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\right), \\
T_{49}^{\mu\nu\lambda\sigma} &= q_3^\sigma\left(q_1\cdot q_3 q_2\cdot q_4 q_4^\mu g^{\lambda\nu} - q_2\cdot q_3 q_1\cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu\left(q_1^\lambda q_2\cdot q_3 - q_2^\lambda q_1\cdot q_3\right)\right. \\
&\qquad \left. + q_1\cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2\cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1\cdot q_4 q_2\cdot q_4\left(q_3^\lambda g^{\mu\nu} - q_4^\mu g^{\lambda\nu}\right)\right) \\
&\quad - q_4^\lambda\left(q_1\cdot q_4 q_2\cdot q_3 q_3^\mu g^{\nu\sigma} - q_2\cdot q_4 q_1\cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu\left(q_1^\sigma q_2\cdot q_4 - q_2^\sigma q_1\cdot q_4\right)\right. \\
&\qquad \left. + q_1\cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2\cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1\cdot q_3 q_2\cdot q_3\left(q_4^\sigma g^{\mu\nu} - q_4^\mu g^{\nu\sigma}\right)\right) \\
&\quad + q_3\cdot q_4\left(\left(q_1^\lambda q_4^\mu - q_1\cdot q_4 g^{\lambda\mu}\right)\left(q_3^\nu q_2^\sigma - q_2\cdot q_3 g^{\nu\sigma}\right) - \left(q_2^\lambda q_4^\nu - q_2\cdot q_4 g^{\lambda\nu}\right)\left(q_3^\mu q_1^\sigma - q_1\cdot q_3 g^{\mu\sigma}\right)\right).
\end{aligned}
$$

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

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- 43 basis tensors (BT)   in $d=4$: 41=no. of helicity amplitudes
- 11 additional ones (T)   to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

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- ▶ the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

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$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions Π_i are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

HLbL contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2).$$

Contribution to a_μ: $\qquad\qquad\qquad\qquad\qquad\qquad m := m_\mu$

$$a_\mu = \frac{-1}{48m} \operatorname{Tr}\Big\{ (\not{p} + m)[\gamma^\rho, \gamma^\sigma](\not{p} + m) \Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p) \Big\}$$

$$\Gamma_{\rho\sigma} = e^6 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{\gamma^\mu (\not{p} + \not{q}_1 + m)\gamma^\lambda (\not{p} - \not{q}_2 + m)\gamma^\nu}{((p+q_1)^2 - m^2)((p-q_2)^2 - m^2)} \times$$

$$\times \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\Big|_{k=0}$$

BTT basis (no kin. singularities!) $\Rightarrow$ limit $k_\mu \to 0$ unproblematic

Master Formula

$$a_{\mu}^{\mathrm{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p)\hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_{\mu}^2][(p - q_2)^2 - m_{\mu}^2]}$$

- ▶ $\hat{T}_i$: known kernel functions
- ▶ $\hat{\Pi}_i$: linear combinations of the Π_i
- ▶ 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

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Setting up the dispersive calculation
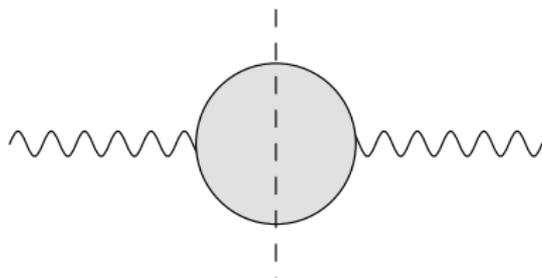
For HVP the unitarity relation is simple and looks the same for all possible intermediate states
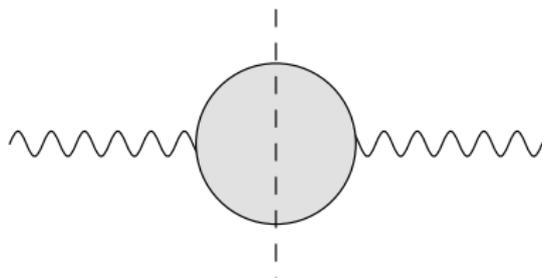
$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+ e^- \to \mathrm{hadrons})$$

Setting up the dispersive calculation

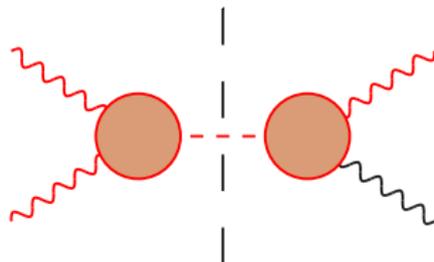For HVP the unitarity relation is simple and looks the same for all possible intermediate states

$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

For HLbL things are more complicated

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Pion pole: imaginary parts = δ-functions
Projection on the BTT basis: easy ✓
Our master formula=explicit expressions in the literature ✓
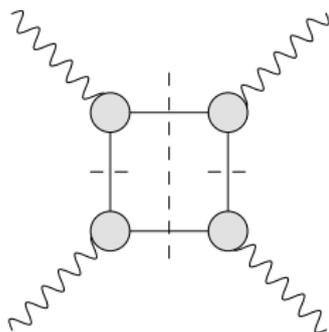Input: pion transition form factor Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)
First results of direct lattice calculations   Gerardin, Mayer, Nyffeler (2016)

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π-box with the BTT set:

– we have constructed a Mandelstam representation for the
  contribution of the 2-pion cut with LHC due to a pion pole
– we have explicitly checked that this is identical to sQED
  multiplied by $F_V^\pi(s)$ (FsQED)

Setting up the dispersive calculation
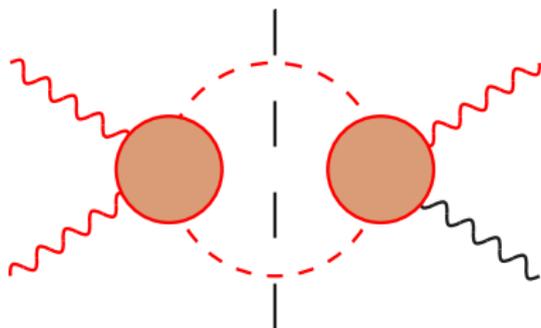
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$$\equiv F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)$$

$$\times \left[\vphantom{\sum} \quad + \quad + \quad \right]$$

Setting up the dispersive calculation

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The "rest" with 2π intermediate states has cuts only in one channel and will be
calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will not be discussed here

Partial wave expansion for 2π contributions

To complete the program of writing down a dispersion relation for two-pion contributions **is not easy**:

- unitarity relations are diagonal in a helicity amplitude basis;
- the helicity basis relevant for $(g-2)_\mu$ is the one with one on-shell photon, which has 27 elements;
- in the limit $q_4^2, q_4^\sigma \to 0$ of the HLbL tensor the number of independent elements of the BTT set drops from 41 to 27;
- there is freedom in the choice of this subset (singly-on-shell basis);
- the arbitrariness in the choice of the 27 elements of the singly-on-shell basis does not influence the final result because of sum rules
- these sum rules follow from the assumption that the HLbL tensor has a uniform behaviour at short distances

S-wave 2π contributions

$$\hat{\Pi}_4^S = \frac{1}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s' - q_3^2)^2} \left(4s'\, \mathrm{Im} h^0_{++,++}(s') - (s' + q_1^2 - q_2^2)(s' - q_1^2 + q_2^2)\, \mathrm{Im} h^0_{00,++}(s') \right)$$

$$\hat{\Pi}_5^S = \frac{1}{\pi} \int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t' - q_2^2)^2} \left(4t'\, \mathrm{Im} h^0_{++,++}(t') - (t' + q_1^2 - q_3^2)(t' - q_1^2 + q_3^2)\, \mathrm{Im} h^0_{00,++}(t') \right)$$

$$\hat{\Pi}_6^S = \frac{1}{\pi} \int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u' - q_1^2)^2} \left(4u'\, \mathrm{Im} h^0_{++,++}(u') - (u' + q_2^2 - q_3^2)(u' - q_2^2 + q_3^2)\, \mathrm{Im} h^0_{00,++}(u') \right)$$

$$\hat{\Pi}_{11}^S = \frac{1}{\pi} \int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u' - q_1^2)^2} \left(2\, \mathrm{Im} h^0_{++,++}(u') - (u' - q_2^2 - q_3^2)\, \mathrm{Im} h^0_{00,++}(u') \right)$$

$$\hat{\Pi}_{16}^S = \frac{1}{\pi} \int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t' - q_2^2)^2} \left(2\, \mathrm{Im} h^0_{++,++}(t') - (t' - q_1^2 - q_3^2)\, \mathrm{Im} h^0_{00,++}(t') \right)$$

$$\hat{\Pi}_{17}^S = \frac{1}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s' - q_3^2)^2} \left(2\, \mathrm{Im} h^0_{++,++}(s') - (s' - q_1^2 - q_2^2)\, \mathrm{Im} h^0_{00,++}(s') \right)$$

Analogous expressions for the *D*, *G* and all higher waves have been derived but are too long to be shown

Outline

Introduction: $(g-2)_\mu$ and hadronic light-by-light (HLbL)

Setting up the stage:
   Gauge invariance and crossing symmetry
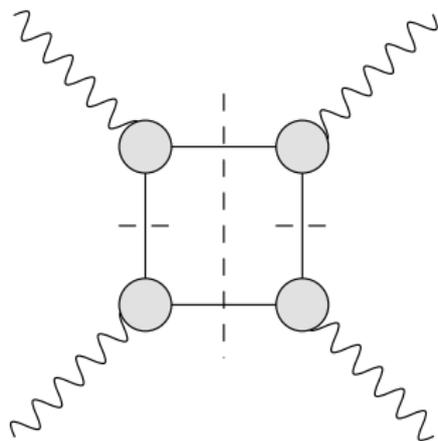   Master Formula

A dispersion relation for HLbL

Numerics
   - Pion box contribution
   - Pion rescattering contribution

Outlook and Conclusions

Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Pion box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \, I_i(x, y),$$

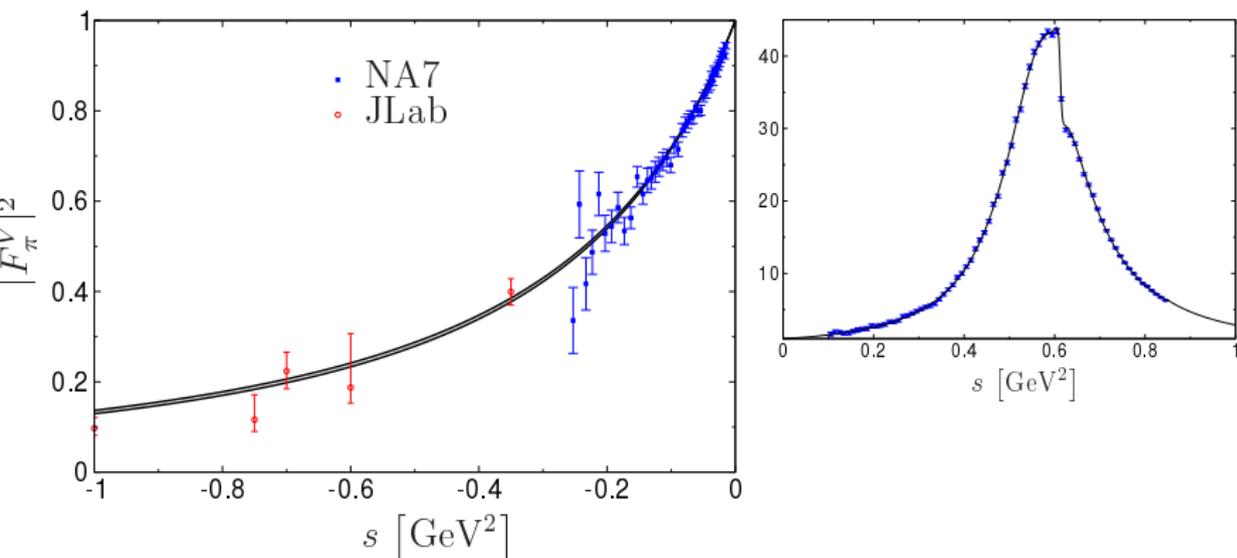where

$$I_1(x, y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xy q_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion box contribution

Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion box contribution

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
" " + subl. in N_c	–	–	–	0 ± 10	–	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3
total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Uncertainties are negligibly small:

$$a_\mu^{\mathrm{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion-box saturation with photon virtualities
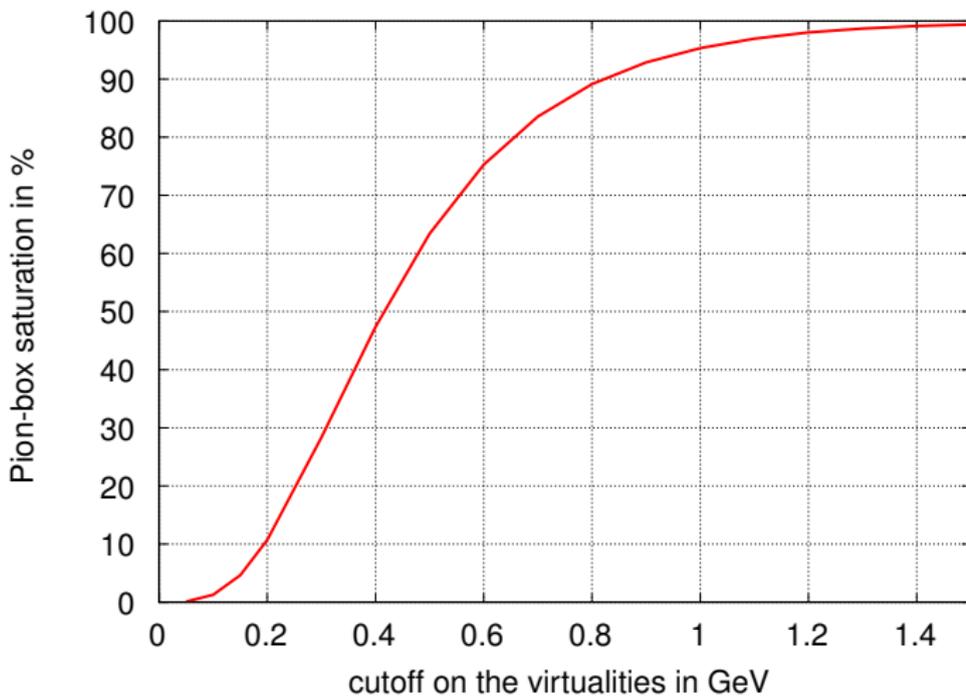
Check of the partial-wave formalism

Comparison partial-wave expansion of the pion-box vs. full result

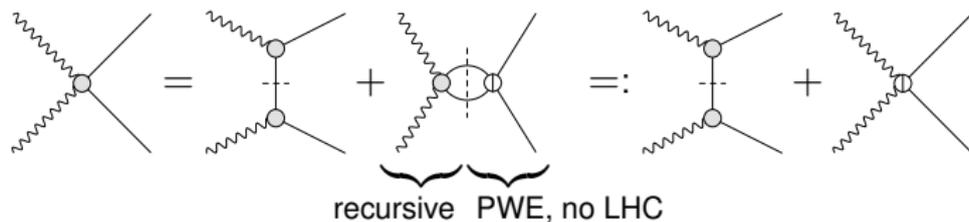| $J_{\max}$	$\delta_{J_{\max}}$	$\Delta_{J_{\max}}$
0	29.2%	55.4%
2	10.4%	20.9%
4	4.3%	11.0%
6	2.4%	6.2%
8	1.5%	3.7%
10	1.0%	2.4%
12	0.7%	1.6%
14	0.6%	1.1%

where

$$\delta_{J_{\max}} := 1 - \frac{a_{\mu, J_{\max}}^{\pi\text{-box, PW}}}{a_{\mu}^{\pi\text{-box}}} \qquad \Delta_{J_{\max}} := \frac{\left| \mathbf{a}_{\mu, J_{\max}}^{\pi\text{-box, PW}} - \mathbf{a}_{\mu}^{\pi\text{-box}} \right|}{\left| \mathbf{a}_{\mu}^{\pi\text{-box}} \right|}$$

Convergence for real helicity amplitudes should be much better

First evaluation of S- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:

recursive   PWE, no LHC

First evaluation of S- wave 2π-rescattering

Based on:

- ▶ taking the pion pole as the only left-hand singularity
- ▶ $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- ▶ $\pi\pi$ phases obtained with the inverse amplitude method

  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]

- ▶ numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

S-wave contributions: $\qquad\qquad a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

a_μ^{HLbL} in 10^{-11} units

cutoff	1 GeV	1.5 GeV	2 GeV	∞
$I = 0$	-9.2	-9.5	-9.3	-8.8
$I = 2$	2.0	1.3	1.1	0.9
sum	-7.3	-8.3	-8.3	-7.9

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Recall π-Box: $a_\mu^{\pi-\text{box}} = -15.9(2) \cdot 10^{-11}$

First evaluation of *S*- wave 2π-rescattering

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Our first numerical result

Two-pion contributions to HLbL:

pion box   rescattering contribution

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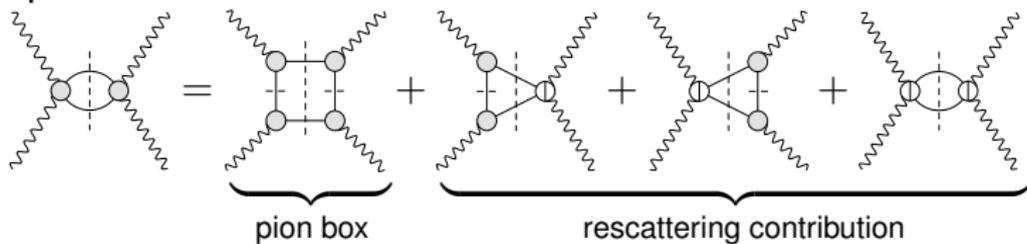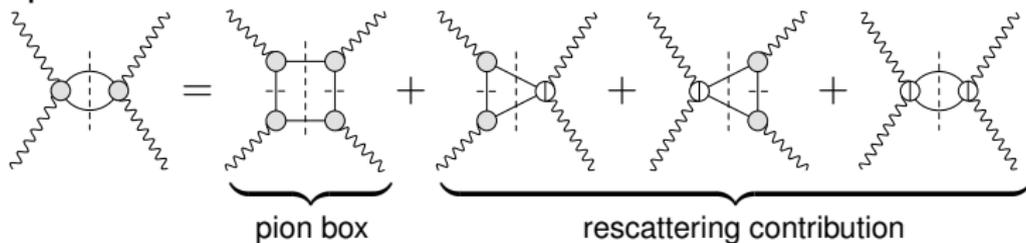
$$a_\mu^{\pi-\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1) \cdot 10^{-11}$$

Outline

Outlook and Conclusions

Conclusions

- ► The HLbL contribution to $(g - 2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach

- ► master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations

- ► the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - ► single-pion contribution: pion transition form factor
  - ► pion-box contribution: pion vector form factor
  - ► 2-pion rescattering: $\gamma^* \gamma^{(*)} \to \pi\pi$ helicity amplitudes

- ► I have presented results for the pion-box and the *S*-wave pion-rescattering contributions:
  model independence $=$ much reduced uncertainties

Outlook

- ▶ More work is needed to complete the evaluation of contributions of 2π intermediate states
    - ▶ take into account experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
    - ▶ estimate the dependence on the q^2 of the second photon (theoretically, there are no data on $\gamma^*\gamma^* \to \pi\pi$ – Lattice?)
    - ▶ $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$, including a full treatment of the LHC
- ▶ same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- ▶ short-distance constraints need to be derived and imposed

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)
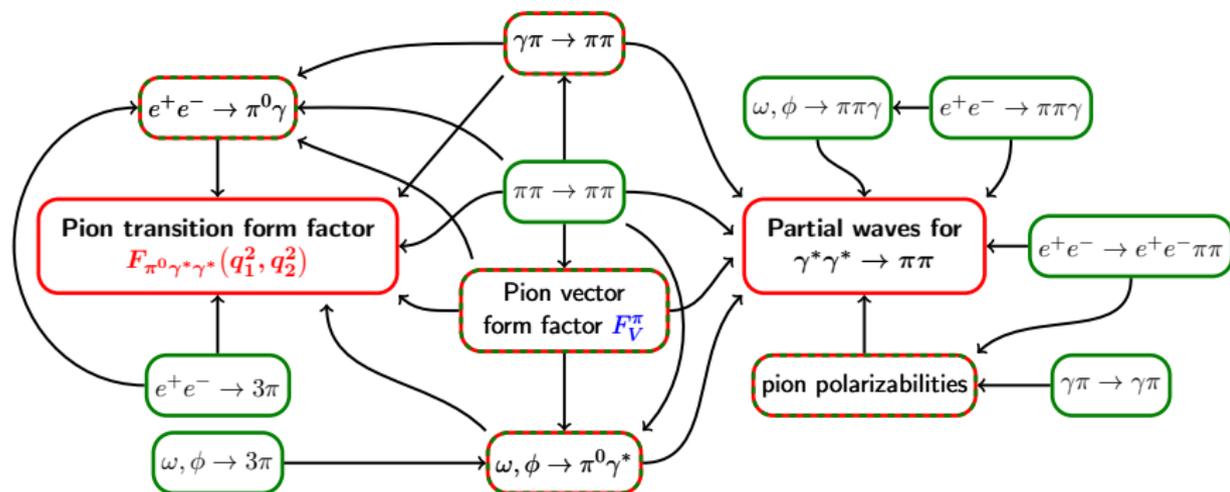
Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions
by and a collaboration among (lattice) theorists and experimentalists

Backup Slides

Consider $\gamma^*(q_1, \lambda_1)\gamma^*(q_2, \lambda_2) \to \pi^a(p_1)\pi^b(p_2)$:

$$W_{ab}^{\mu\nu}(p_1, p_2, q_1) = i \int d^4x \, e^{-iq_1 \cdot x} \langle \pi^a(p_1)\pi^b(p_2)| T\{j_{\text{em}}^\mu(x)j_{\text{em}}^\nu(0)\}|0\rangle$$

General tensor decomposition (q_i, $i = 1, \ldots, 3$, $q_3 = p_2 - p_1$):

$$W^{\mu\nu} = g^{\mu\nu} W_1 + \sum_{i,j} q_i^\mu q_j^\nu W_2^{ij}$$

gives ten independent scalar functions.
Gauge invariance requires:

$$q_1^\mu W_{\mu\nu} = q_2^\nu W_{\mu\nu} = 0$$

Gauge invariance: Bardeen-Tung-Tarrach approach

Consider the projector

$$I^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}$$

which satisfies

$$I_\mu{}^\lambda W_{\lambda\nu} = W_{\mu\lambda} I^\lambda{}_\nu = W_{\mu\nu}, \qquad q_1^\mu I_{\mu\nu} = q_2^\nu I_{\mu\nu} = 0$$

and contract it twice with $W_{\mu\nu}$, leaving it invariant:

$$W_{\mu\nu} = I_{\mu\mu'} I_{\nu'\nu} W^{\mu'\nu'} = \sum_{i=1}^{5} \bar{T}_{\mu\nu}^i \bar{A}_i = \sum_{i=1}^{5} T_{\mu\nu}^i A_i$$

The $\bar{A}_i$ are free of kinematic singularities, but have zeros. To remove the zeros from the $\bar{A}_i \Rightarrow$ remove the poles from the $\bar{T}_i^{\mu\nu}$

Gauge invariance: Bardeen-Tung-Tarrach approach

$$T_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu,$$

$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu,$$

$$T_3^{\mu\nu} = q_1^2 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_1^\mu q_1^\nu,$$

$$T_4^{\mu\nu} = q_2^2 q_1 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu - q_1 \cdot q_3 q_2^\mu q_2^\nu,$$

$$T_5^{\mu\nu} = q_1 \cdot q_3 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_3^\nu - q_1 \cdot q_3 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_3^\mu q_1^\nu,$$

This is a basis of gauge-invariant tensors, but for $q_1 \cdot q_2 = 0$ it becomes degenerate: need one more structure: Tarrach (75)

$$T_6^{\mu\nu} = \left(q_1^2 q_3^\mu - q_1 \cdot q_3 q_1^\mu\right)\left(q_2^2 q_3^\nu - q_2 \cdot q_3 q_2^\nu\right)$$

Inverse-amplitude method's input