Improvement of heavy-heavy current for calculation of $\bar{B} \to D^{(*)}\ell\bar{\nu}$ form factors using Oktay-Kronfeld heavy-quark

Jon A. Bailey, Yong-Chull Jang, Weonjong Lee, Jaehoon Leem

Lattice 2017, Granada

$\bar{B} \to D^* \ell \bar{\nu}$ semi-leptonic decay and $|V_{cb}|$

- Decay rate for $\bar{B} \to D^* \ell \bar{\nu}$ are proportional to $|V_{cb}|^2$.

$$\frac{d\Gamma}{dw}(\bar{B} \to D^{(*)} \ell \bar{\nu}) = \frac{G_F^2 M_B^3 |V_{cb}|^2}{4\pi^3} |\eta_{\text{EW}}|^2 (w^2 - 1)^{1/2} (w+1)^2 r^3 (1-r)^2 \times \left[1 + \frac{4w}{w+1} \frac{1 - 2wr + r^2}{(1-r)^2}\right] |\mathcal{F}_{D^*}(w)|^2$$

where $w = v_B \cdot v_{D^*}$ and $r = \dfrac{M_{D^*}}{M_B}$.

- Determination of $|V_{cb}|^2$ requires calculation of semi-leptonic form factor $|F_{D^*}(w)|^2$ using lattice QCD simulation.

Current Status of exclusive $|V_{cb}|$

- $\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil

$$|V_{cb}| = 38.71 \pm 0.47(\text{exp}) \pm 0.53(\text{theory}) \times 10^{-3}$$

(FNAL/MILC + HFAG)

from arXiv:1612.07233 (2016)

- $\bar{B} \to D \ell \nu$ at nonzero recoil

$$|V_{cb}| = 40.49 \pm 0.97 \times 10^{-3}$$

(FNAL/MILC + HPQCD + BABAR + Belle)

from PRD94, 094008 (2016)

Error of V_{cb} from lattice calculation

Table: Error budgets of FNAL/MILC form factor calculation (h_{A_1} at zero recoil) [PRD89, 114504 (2014)] and the expected error for the OK action

uncertainty	$h_{A_1}(w=1)$
statistics	0.4%
matching	0.4%
χPT	0.5%
$g_{D^* D\pi}$	0.3%
discretization	$(1.0\%)_{\mathsf{F}} \to (0.2\%)_{\mathsf{OK}}$
other	0.1%
total	$(1.4\%)_{\mathsf{F}} \to (0.8\%)_{\mathsf{OK}}$

- F : Femilab action
- OK : OK action

Improvement of heavy quark on lattice

- Masses of b-quark and c-quark are comparable with inverse of lattice spacing a^{-1}. (when $a = 0.05\text{fm} \sim 0.1\text{fm}$, $m_b a \simeq 1.1 \sim 2.2$ and $m_c a \simeq 0.33 \sim 0.66$.)
- In this case, any expansion over $m_q a$ in Symanzik improvement program fails.
- We need special care for heavy quark. Based on heavy quark symmetry, fermilab formalism gives a systematic way to control the heavy quark discretization error.

Improvement of heavy quark on lattice

Fermilab method : Wilson fermion + chromomagnetic term + chromoelectric term

- Lift the space-time axis interchange symmetry
- Do not allow time derivative to fermion (anti-fermion) for higher dimension operators.
- The coefficients in action are tuned as the explicit function of bare mass, and the coupling constants.
- Valid for arbitrary quark mass.

Fermilab method

- Dimension-four interactions. [PRD55, 3933]

$$S_0 = m_0 a^4 \sum_x \bar\psi(x)\psi(x)$$
$$+ a^4 \sum_x \bar\psi(x)\frac{1}{2}(1+\gamma_4)D_4^-\psi(x) - a^4 \sum_x \bar\psi(x)\frac{1}{2}(1-\gamma_4)D_4^+\psi(x)$$
$$+ \zeta a^4 \sum_x \bar\psi(x)\vec\gamma\cdot\vec D\psi(x) - \frac{1}{2}r_s\zeta a^5 \sum_x \bar\psi(x)\Delta^{(3)}\psi(x)$$

where $\zeta = \kappa_s/\kappa_t$, and the covariant derivatives are

$$D_4^+\psi = a^{-1}(T_4 - 1)\psi, \qquad D_4^-\psi = a^{-1}(1 - T_{-4})\psi,$$
$$D_i\psi = (2a)^{-1}(T_i - T_{-i})\psi, \qquad \Delta^{(3)}\psi = \sum_i a^{-2}(T_i + T_{-i} - 2)\psi,$$

with covariant translation

$$T_{\pm\mu}\psi(x) = U_{\pm\mu}(x)\psi(x \pm a\mu),$$

Fermilab method

- Dimension-five interactions.

$$S_B = -\frac{1}{2} c_B \zeta a^5 \sum_x \bar{\psi}(x) i\vec{\Sigma} \cdot \vec{B} \psi(x)$$

$$S_E = -\frac{1}{2} c_E \zeta a^5 \sum_x \bar{\psi}(x) \vec{\alpha} \cdot \vec{E} \psi(x)$$

- $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ and $E_i = F_{0i}$, where $F_{\mu\nu}$ is the field strength tensor given as,

$$F_{\mu\nu}(x) = \frac{1}{8a^2} \sum_{\bar{\mu}=\pm\mu, \bar{\nu}=\pm\nu} \mathrm{sign}(\bar{\mu}\bar{\nu}) T_{\bar{\mu}} T_{\bar{\nu}} T_{-\bar{\mu}} T_{-\bar{\nu}} - \mathrm{h.c.}$$

- $c_E = c_B$ for the Fermilab action.
- $c_E \neq c_B$ for the OK action.

OK action

Add dimension six and seven bilinears to the Fermilab action. [PRD78, 014504]

- Terms without time derivative to the fermion or anti-fermion field are allowed.
- On-shell improvement for dimension six and seven bilinears at tree-level.
- Estimate truncation error using HQET and NRQCD power counting. To control discretization effects below 1 percent (with lattice spacing $a = 0.045$fm ~ 0.125fm), one-loop improvement of c_B is required.

OK action

- $S_{\text{new}} = S_6 + S_7$

$$S_6 = c_1 a^6 \sum_x \bar{\psi}(x) \sum_i \gamma_i D_i \Delta_i \psi(x) + c_2 a^6 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}, \Delta^{(3)}\} \psi(x)$$
$$+ c_3 a^6 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}, i\vec{\Sigma} \cdot \vec{B}\} \psi(x) + c_{EE} a^6 \sum_x \bar{\psi}(x) \{\gamma_4 D_4, \vec{\alpha} \cdot \vec{E}\} \psi(x)$$
$$S_7 = a^7 \sum_x \bar{\psi}(x) \sum_i \left[c_4 \Delta_i^2 \psi(x) + c_5 \sum_{j \neq i} \{i\Sigma_i B_i, \Delta_j\} \right] \psi(x)$$

where

$$D_\mu \psi = (2a)^{-1}(T_\mu - T_{-\mu})\psi, \qquad \Delta_i \psi = a^{-2}(T_i + T_{-i} - 2)\psi,$$

Coefficients c_i are fixed by matching dispersion relation, one-gluon vertex, compton scattering amplitude.

Current improvement up to $\mathcal{O}(\lambda)$

- Fermilab formalism and the OK action give a systematic way to control the heavy quark discretization error up to λ^3 order.
- Our goal is to reduce the heavy quark discretization error of $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semi-leptonic form factors at the level of $\mathcal{O}(\lambda^3)$, where $\lambda \sim \Lambda_{QCD}/m_q, a\Lambda_{QCD}$.
- We must improve currents up to the λ^3 order $\to$ in the same level as the OK action.

Current improvement up to $\mathcal{O}(\lambda)$

- For $\mathcal{O}(\lambda)$ improvement in HQET, introducing improved quark field suffices. [PRD55, 3933]

$$J_\Gamma = \bar{\Psi}_{cI}\Gamma\Psi_{bI},$$

  where Γ indicates the Dirac structure. And the improved quark field is defined by,

$$\Psi_{fI} = e^{M_{1f}a/2}(1 + ad_{1f}\vec{\gamma}\cdot\vec{D})\psi_f, \quad f = b, c$$

  where $M_{1f} = \log(1 + m_{0f}a)/a$ is tree-level rest mass of f quark.
- The improvement parameter d_1 is determined by tree-level matching of two-quark matrix elements of flavor-changing currents.

Current improvement up to $\mathcal{O}(\lambda^3)$

- For $\mathcal{O}(\lambda^3)$ improvement, we introduce improved field ($a = 1$) [arxiv:1411.4227],

$$
\begin{aligned}
\Psi_I(x) =& e^{M_1/2}\Big[1 + d_1\vec{\gamma}\cdot\vec{D} + \frac{1}{2}d_2\Delta^{(3)} + \frac{1}{2}id_B\vec{\Sigma}\cdot\vec{B} + \frac{1}{2}d_E\vec{\alpha}\cdot\vec{E} \\
&+ d_{r_E}\{\vec{\gamma}\cdot\vec{D},\vec{\alpha}\cdot\vec{E}\} + d_{z_E}\gamma_4(\vec{D}\cdot\vec{E}-\vec{E}\cdot\vec{D}) \\
&+ \frac{1}{6}d_3\gamma_i D_i\Delta_i + \frac{1}{2}d_4\{\vec{\gamma}\cdot\vec{D},\Delta^{(3)}\} + d_5\{\vec{\gamma}\cdot\vec{D}, i\vec{\Sigma}\cdot\vec{B}\} \\
&+ d_{EE}\{\gamma_4 D_4,\vec{\alpha}\cdot\vec{E}\} + d_{z_3}\vec{\gamma}\cdot(\vec{D}\times\vec{B}+\vec{B}\times\vec{D})\Big]\psi(x)
\end{aligned}
$$

- By determining improved parameters d_i from matching condition, we obtain improved current.

Matching condition

- Matching two-quark matrix elements : straightforward extension of calculation in [PRD55, 3933]. At tree-level, it gives constraint only on d_1, d_2, d_3, and d_4 [arxiv:1411.4227].
- To obtain d_i other than d_{1-4}, we must use matrix element with gluon exchange vertex. $\rightarrow$ We use four-quark matrix element for matching.

Matching four-quark matrix element

- We introduce following four-quark matrix element [arxiv:1612.09081],

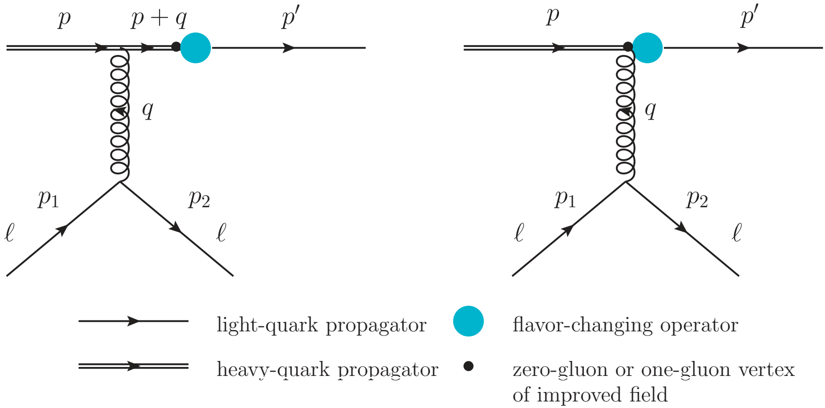
$$\langle \ell(\eta_2, p_2) u(\eta', p^{'}) | \bar{\psi}_u \Gamma \Psi_b | b(\eta, p) \ell(\eta_1, p_1) \rangle_{\text{lat}}$$

  where ℓ represents a light spectator quark, u and b indicate up quark and bottom quark, respectively.

- We consider heavy-light current instead of heavy-heavy current to simplify the calculation.

Matching four-quark matrix element

- The lattice Feynman diagrams with gluon exchange at the heavy-quark line are,



Matching four-quark matrix element

- The matching equation for the one-gluon exchange from b-quark line gives constraint on the improvement parameter d_i,

$$
\begin{aligned}
n_\mu(q)\Big[R_b^{(0)}(p+q)S^{\mathsf{lat}}(p+q)(-gt^a)\Lambda_\mu(p+q,p) \\
+(-gt^a)R_{b\mu}^{(1)}(p+q,p)\Big]\mathcal{N}_b(\vec{p})u_b^{\mathsf{lat}}(\eta,\vec{p}) \\
=\frac{m_b-i\gamma\cdot(p+q)}{m_b^2+(p+q)^2}\gamma_\mu\sqrt{\frac{m_b}{E_b}}u_b(\eta,\vec{p}).
\end{aligned}
$$

$\mathcal{N}_b$: lattice spinor normalization factor.
$n_\mu(q)$: wave-function factor for gluon.
$R_b^{(0)}$, $R_{b\mu}^{(1)}$: zero-gluon and one-gluon vertex for improved field which contains improvement parameters d_i.
Λ_μ : one-gluon vertex of OK-action.

Matching four-quark matrix element

- Expand the matching equation over $\vec{p}a$, qa, $\vec{p}/m_b$, and q/m_b.
- For each order in $\vec{p}$ and q, it gives constraints on improvement parameter d_i.
- The constraints are from temporal ($\mu = 4$), and spatial component ($\mu = i$) of the amplitudes.
- The constraints should be consistent.

Results

- We obtain all the eleven improvement parameters d_i, and two of them are zero. [arxiv:1612.09081]

$$d_{zE} = d_{z3} = 0,$$

- The results from matching four-quark matrix element are consistent with those from two-quark matrix element.

- However, there are remaining unresolved issues.

Results

The results from four-quark matrix element matching are as follows, [arxiv:1612.09081]

$$d_B = d_1^2 - \frac{r_s\zeta}{2(1+m_0)},$$

$$d_E = \frac{1}{2m_b^2} - \frac{\zeta(1+m_0)(m_0^2+2m_0+2)}{[m_0(2+m_0)]^2} + \frac{\zeta(1+m_0)(1-c_E)}{m_0(2+m_0)},$$

$$d_{r_E} = -\frac{1}{8m_b^3} + \frac{r_s\zeta}{24(1+m_0)} + \frac{\zeta c_{EE}(2+2m_0+m_0^2)}{2m_0(1+m_0)(2+m_0)} + \frac{\zeta^2 c_E(2+2m_0+m_0^2)}{[2m_0(2+m_0)]^2}$$
$$+ \frac{\zeta^2(12+24m_0+16m_0^2+4m_0^3+m_0^4)}{12m_0^3(2+m_0)^3} - d_1\frac{\zeta(1+m_0)[2+m_0(2+m_0)c_E]}{2m_0^2(2+m_0)^2},$$

Results

$$d_{EE} = \frac{1+m_0}{(m_0^2+2m_0+2)}\Big[-\frac{1}{4m_b^3}+\frac{\zeta(1+m_0)(m_0^2+2m_0+2)}{[m_0(2+m_0)]^3}$$
$$+\frac{\zeta c_E(1+m_0)}{[m_0(2+m_0)]^2}+\frac{(2+2m_0+m_0^2)c_{EE}}{m_0(2+m_0)}\Big],$$

$$d_5 = -\frac{1}{16m_b^3}+\frac{\zeta c_B[(1+m_0)(4+6m_0+3m_0^2)\zeta-m_0^2(2+m_0)^2 d_1]}{8(1+m_0)[m_0(2+m_0)]^2}$$
$$+\frac{c_3(1+m_0)}{m_0(2+m_0)}+\frac{\zeta^2 c_E[d_1 m_0(2+m_0)-\zeta(1+m_0)]}{4[m_0(2+m_0)]^2}$$
$$+\frac{-\zeta^2 d_1+\zeta(1+m_0)\Big[\frac{(2+2m_0+m_0^2)\zeta^2}{[m_0(2+m_0)]^2}+d_B\Big]}{4m_0(2+m_0)}.$$

Redundancy of operators

- Matching condition determines d_{r_E} uniquely, however, the same operator in the action belongs to redundant operator.

- The isospectral transformation which brings a change in d_{r_E} is,

$$\Psi_I(x) \to e^{J_{r_E}} \Psi_I(x), \quad J_{r_E} = a^3 \delta_{r_E} \{\boldsymbol{\gamma} \cdot \boldsymbol{D}, \boldsymbol{\alpha} \cdot \boldsymbol{E}\}. \tag{1}$$

  Then, d_{r_E} changes to $d_{r_E} + \delta_{r_E}$.

- At the same time, the transformation yields a change along the following dimension six operator in the action,

$$\delta S = 2\delta_{r_E} a^6 (m_0 a) \bar{\psi}(x) \sum_x \{\boldsymbol{\gamma} \cdot \boldsymbol{D}, \boldsymbol{\alpha} \cdot \boldsymbol{E}\} \psi(x), \tag{2}$$

Redundancy of operators

- However, the above transformation in the action is used to set the coefficient of operator to zero ($r_E = 0$). Hence, there is no degree of freedom left to fix d_{r_E}.
- Therefore, unique determination of d_{r_E} is natural.

Remaining issue I

- Several of our results (d_4, d_{EE}, d_{rE}, and d_5) diverge as $m_0 \to 0$. For example d_4 is, (set $r_s = \zeta = 1$, $m_b = M_2$, in the limit of $am_0 \to 0$)

$$d_4 = \frac{1}{8}\frac{1}{am_0} - \frac{3}{32} - \frac{5}{32}(am_0) + \cdots$$

- Superficially, this behavior appears to violate the principle that Fermilab formulation works both in the chiral limit and in the heavy quark limit simultaneously.

- However, the discretization effects in the matrix element vanish as $a \to 0$ with fixed quark mass ($m_0 \neq 0$), and everything is regular in this limit.
  - In the definition of improved field, d_4, d_{EE}, d_{r_E}, d_5 are multiplied by a^3.
  - For fixed non-zero m_0, there is no divergence in the matrix element.

Remaining issue II

- Although our results for d_1, d_2, and d_B agree with those in [PRD55, 3933], our result for d_E differs. (set $r_s = \zeta = 1$, and $m_b = M_2$)

$$d_E(\text{SWME}) - d_E(\text{FNAL}) = -\frac{1}{2 + m_0},$$

- d_E(FNAL) is obtained for heavy-heavy system using NRQCD power counting.
- d_E(SWME) is obtained for heavy-light system using HQET power counting.
- This issue is under investigation.

Remaining issue III

- We have not proved yet that the improved quark field is sufficient for the current improvement.
- The proof can be achieved through the HQET analysis.
- This issue is under investigation.