Finite volume study for muon g-2 light-by-light contribution

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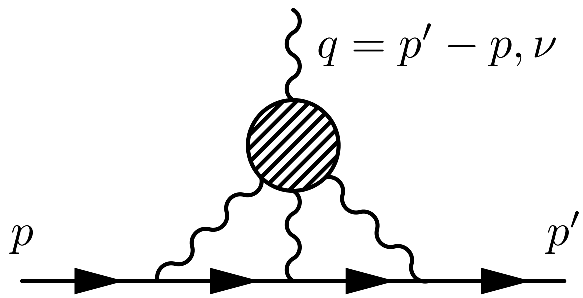
- Motivation
- Finite Volume Effect in QCD+QED
- QCD box in QED box
- Infinite Volume Photon and Lepton
- Current conservation and subtraction
- Results
- Discussion and Conclusion



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Introduction

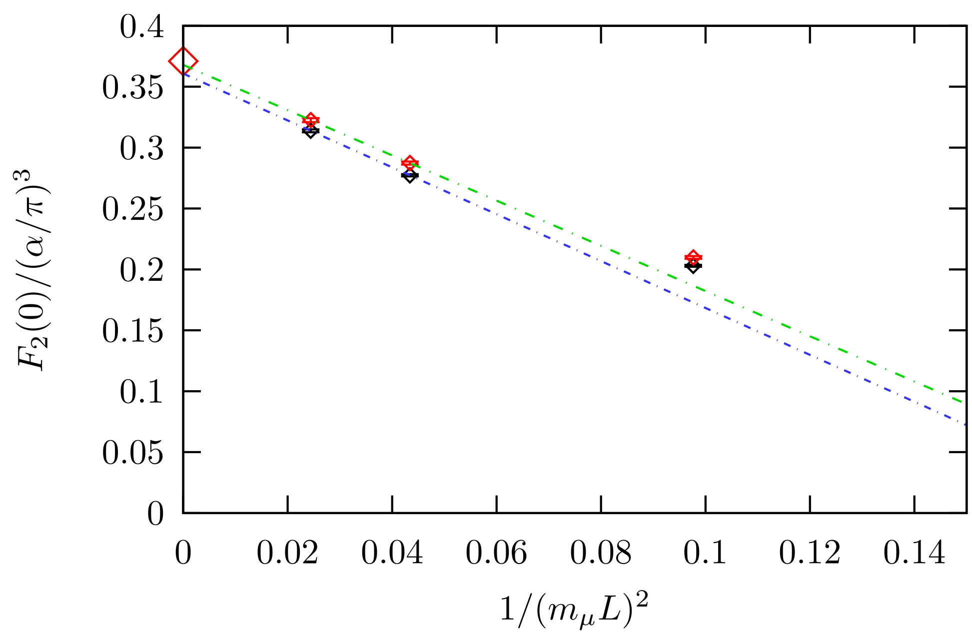
- Systematic errors of Hadronic contribution for $(g-2)_\mu$
  - Finite Volume : $1/L^2 \rightarrow \exp(m_\pi L)$
  - Discretization error : a^2 scaling from a^{-1}=1.73 GeV, 48^3 and a^{-1}=2.38 GeV 64^3
  - Disconnected diagram : SU(3)$_F$ leading diagram. We need to check size of other disconnected diagrams
  - Isospin breaking ?
  - ...
- c.f. In HVP case : Using Time Moment Representation (TMR), Finite Volume error : 2-7% at $m_\pi L \sim 4$ [E. Shintani, Thur. 17:10]

Finite Volume Error in QCD+QED

- Photon is massless. Muon, m_μ=106 MeV $< m_\pi$.
- We have used QED$_L$ [2008, Hayakawa Uno] for previous LbL calculation : $\mathcal{O}(1/L^2)$ error
- Lepton LbL (pure QED, loop mass is also m_μ) [L. Jin et al. 1510.07100] :

$$[F_2(0)]_{\rm quad} / (\alpha/\pi)^3 = 0.3679(42) - 1.86(11)/(m_\mu L)^2,$$

$$[F_2(0)]_{\rm PT} / (\alpha/\pi)^3 = 0.3710052921,$$



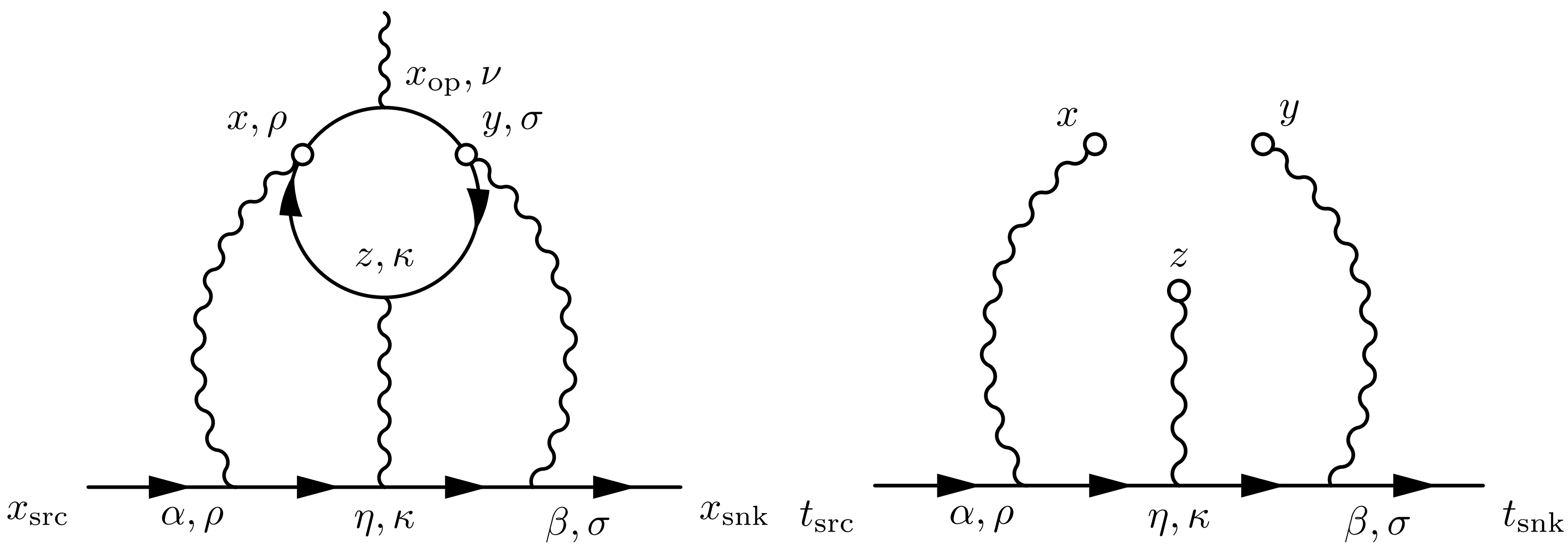
HLbL point source method [L. Jin et al. 1510.07100]

- Anomalous magnetic moment, $F_2(q^2)$ at $q^2 \to 0$ limit

$$\frac{F_2^{\mathrm{cHLbL}}(q^2=0)}{m}\frac{(\sigma_{s',s})_i}{2} = \frac{\sum_{x,y,z,x_{\mathrm{op}}}}{2VT}\epsilon_{i,j,k}\left(x_{\mathrm{op}} - x_{\mathrm{ref}}\right)_j \cdot i\bar{u}_{s'}(\vec{0})\mathcal{F}_k^C\left(x,y,z,x_{\mathrm{op}}\right)u_s(\vec{0})$$

- Stochastic sampling of x and y point pairs. Sum over x and z.

$$\mathcal{F}_\nu^C\left(x,y,z,x_{\mathrm{op}}\right) = (-ie)^6\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}),$$



Finite Volume error of HLbL in QED$_L$ [L. Jin 1509.08372]

- Massless photon propagator $\frac{1}{|x-y|^2}$
- One of three internal photons of $\mathcal{F}_\nu^C(x, y, z, x_{\text{op}})$:

$$G(\mathbf{k}, t_2, t_1) = \int \frac{\mathrm{d}p_0}{2\pi} e^{ip_0(t_2-t_1)} \frac{1}{p_0^2 + \mathbf{k}^2} = \frac{1}{2|\mathbf{k}|} \exp(-|\mathbf{k}||t_2 - t_1|).$$

- For a single internal photon, the behavior of the integrand in small $\mathbf{k}$ region is roughly

$$\int_{-\infty}^{\infty} dt_{\text{line}} \frac{1}{|\mathbf{k}|} \exp(-|\mathbf{k}||t_{\text{loop}} - t_{\text{line}}|)|\mathbf{k}| \sim \mathcal{O}\left(\frac{1}{|\mathbf{k}|}\right),$$

(t_{line} : time of EM vertex on external line, t_{loop} the location in time of the internal muon loop) The last factor of $|\mathbf{k}|$ comes from the fact that the photon has to couple to a neutral loop and the coupling at such a small momentum photon is suppressed by a factor of $|\mathbf{k}|$.

- Finite Volume correction should be proportion to

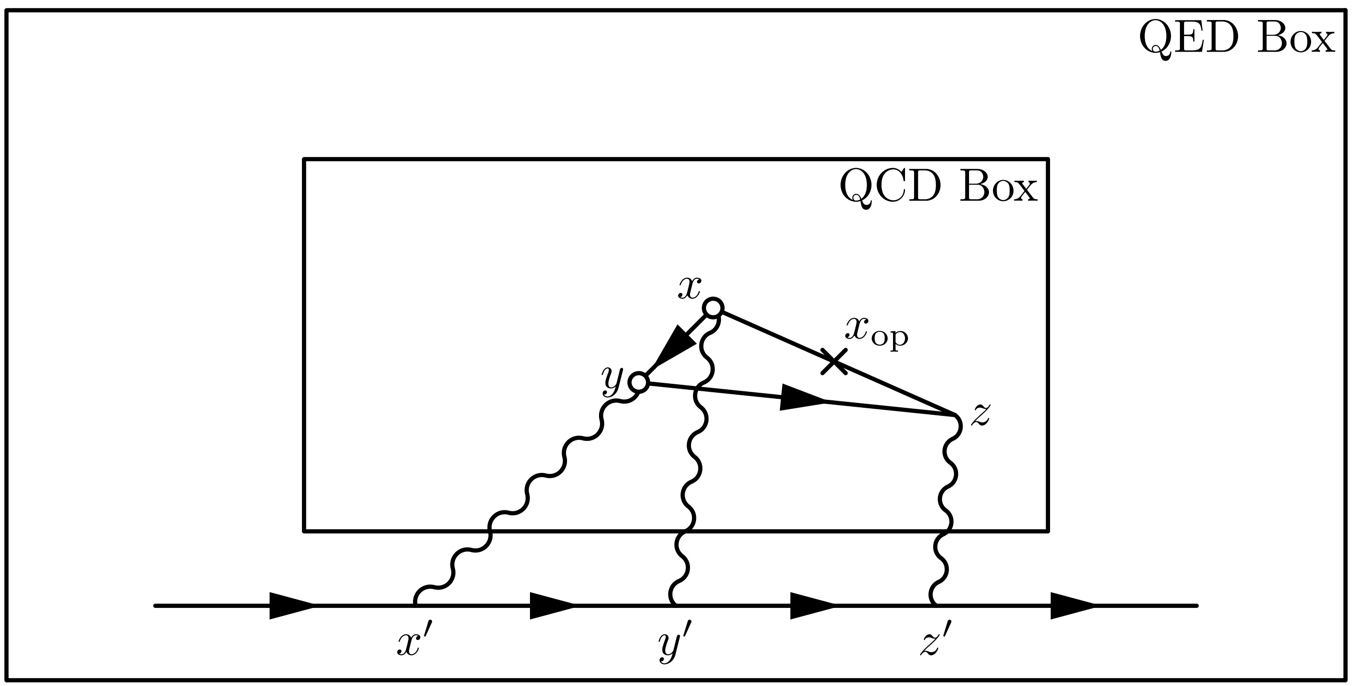
$$\int_0^{1/L} \mathcal{O}\left(\frac{1}{|\mathbf{k}|}\right) d^3k \sim \mathcal{O}\left(\frac{1}{L^2}\right).$$

QCD box inside a large QED box [L. Jin et al. 1511.05198]

- One could use a larger 4D QED box to compute photon+lepton part of the diagram $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$, and compute the hadron part $\mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ in a smaller QCD box.

- Hadron part could be recycled for different size of QED box, which introduces a beneficial correlation in taking $V \to \infty$ limit.

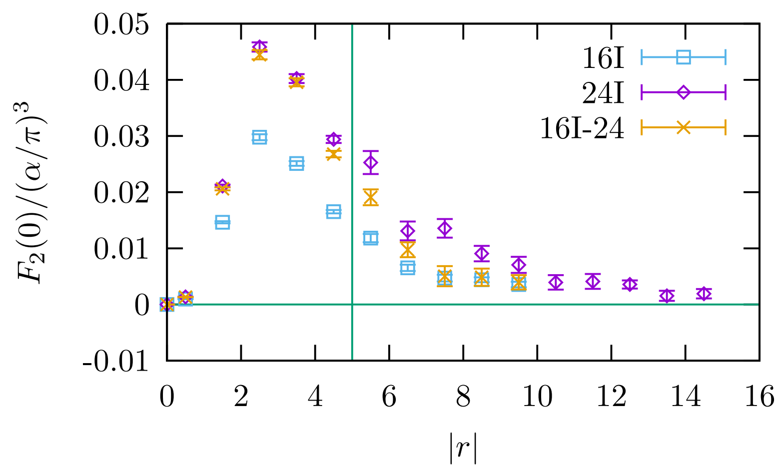
- As far as the integral outside of QCD box is small (from $\exp(-m_\pi r)$ suppression of hadron 4pt of size r), $\mathcal{O}(1/L^2)$ error would be largely suppressed.



QCD box inside a large QED box [L. Jin et al. 1511.05198]

- Result using $a^{-1} = 1.747\mathrm{GeV}$, $m_\pi = 423\mathrm{MeV}$, $m_\mu = 332\mathrm{MeV}$.

Ensemble	QCD Size	QED Size	$t_{\mathrm{snk}} - t_{\mathrm{src}}$	$\frac{F_2(q^2=0)}{(\alpha/\pi)^3}$
16I	$16^3 \times 32$	$16^3 \times 32$	16	0.1158(8)
24I	$24^3 \times 64$	$24^3 \times 64$	32	0.2144(27)
16I-24	$16^3 \times 32$	$24^3 \times 64$	32	0.1674(22)



- By using larger QED box, the major part of the finite-volume effects have been removed. But visible disagreement from QCD FV error on (1.8 fm)^3 volume.

- Integrand plot ($|r| = |x - y|$). For small r, 16I and 16I-24 is very close, but visible deviation for $r > 5$, close to L/2=8.

Infinite Volume Photon and Lepton QED$_\infty$

[Feynman, Schwinger, Tomonaga]

- Instead of, or, in addition to, larger QED box, one could use infinite volume QED to compute $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$.

- Hadron part $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ has following features due to the mass gap :
  - ▷ *For large distance separation, the 4pt Green function is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\text{op}}) \sim \exp[-m_\pi \times \text{dist}(x,y,z,x_{\text{op}})]$
  - ▷ *For fixed* (x,y,z,x_{op})*, FV error (wraparound effect etc.) is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_V - \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_\infty \sim \exp[-m_\pi \times L]$

- By using QED$_\infty$ weight function $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$, which is not exponentially growing, asymptotic FV correction is exponentially suppressed

$$\Delta_V \left[\sum_{x,y,z,x_{\text{op}}} \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\text{op}})] \right] \sim \exp[-m_\pi L]$$

($x_{\text{ref}} = (x+y)/2$ is at middle of QCD box using transnational invariance)

- The idea of $(\text{QED})_\infty$ for suppressing FV error was discussed several times e.g. [C. Lehner, TI Lattice 2014] [L. Jin et al CIPAN15, 1509.08372] . More concrete proposals with numerical demonstration for π^0 exchange model by Mainz group [PRL 115(22):222003, 2015, LATTICE2016:164, 2016] .

- For g-2 HVP, infinite QED was used from the beginning [T. Blum, PRL91, 052001, 2003]

- Exponentially suppressed FV error using QED_∞ would be generic for Euclidean Green's function (e.g. QED correction to HVP)

- But not necessarily always the case for other quantities (e.g. QED mass shift is being checked [M. Hayakawa]).

$$\frac{\Delta_V M[\text{QED}_L]}{M} \sim -Q^2\alpha\frac{\kappa}{2ML}\left(1+\frac{2}{ML}\right)$$

- In addition to the point-source sampling method, QED_∞ may be implemented in the non-perturbative QCD+QED method, or RM123 method.

QED$_\infty$ weight function

$$i^3\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) \quad = \quad \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x) + \text{other 4 permutations}\,.$$

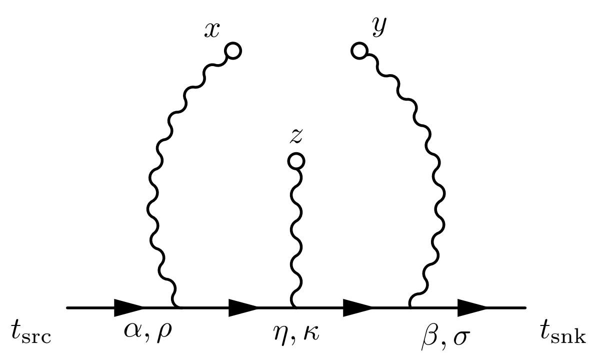
- Take hermitian part and using permutation,

$$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) \quad = \quad \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \frac{1}{2}[\mathfrak{G}_{\kappa,\sigma,\rho}(z,y,x)]^\dagger$$

  gives same F_2 but infrared finite.

- In $m_\mu = 1$ unit,

$$\mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y,z,x) \quad = \quad \frac{\gamma_0+1}{2} i\gamma_\sigma \left(-\not{\partial}_y + \gamma_0 + 1\right) i\gamma_\kappa \left(\not{\partial}_x + \gamma_0 + 1\right) i\gamma_\rho \frac{\gamma_0+1}{2}$$
$$\times \frac{1}{4\pi^2}\int d^4\eta \frac{1}{(\eta-z)^2} f(\eta-y) f(x-\eta).$$



Subtraction using current conservation

- From current conservation, $\partial_\rho V_\rho(x) = 0$, and mass gap, $\langle x V_\rho(x) \mathcal{O}(0)\rangle \sim |x|^n \exp(-m_\pi |x|)$

$$\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\sf op})\rangle = 0$$

$$\sum_z \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = 0$$

at $V \to \infty$ and $a \to 0$ limit (we use local currents).

- We could further change QED weight

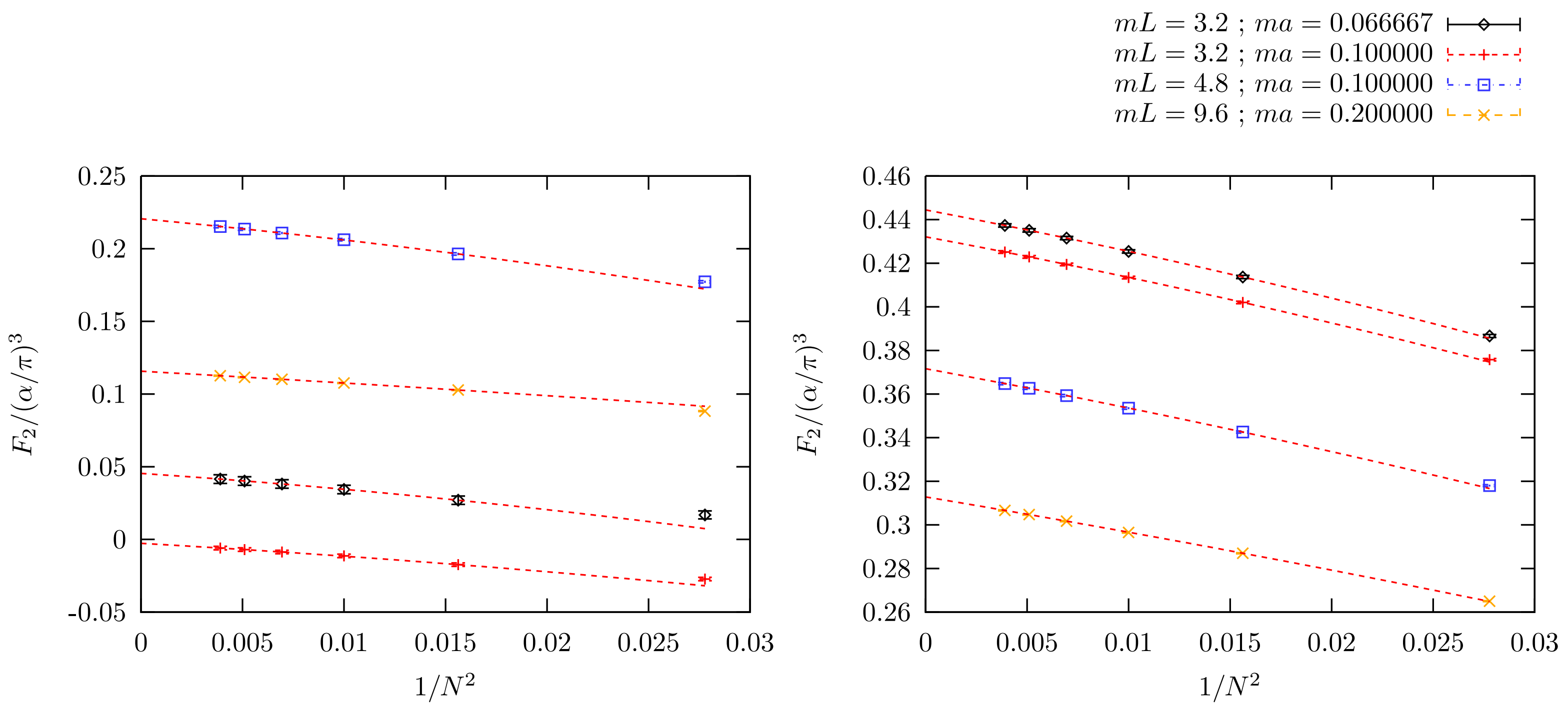
$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$$

without changing sum $\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$.

- Subtraction changes discretization error and finite volume error.
- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.
- Also now $\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(z,z,x) = \mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,z) = 0$, so short distance $\mathcal{O}(a^2)$ is suppressed.
- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. (x,y,z) is represented by 5 parameters, compute on N^5 grid points and interpolates. ($|x-y| < 11$ fm).

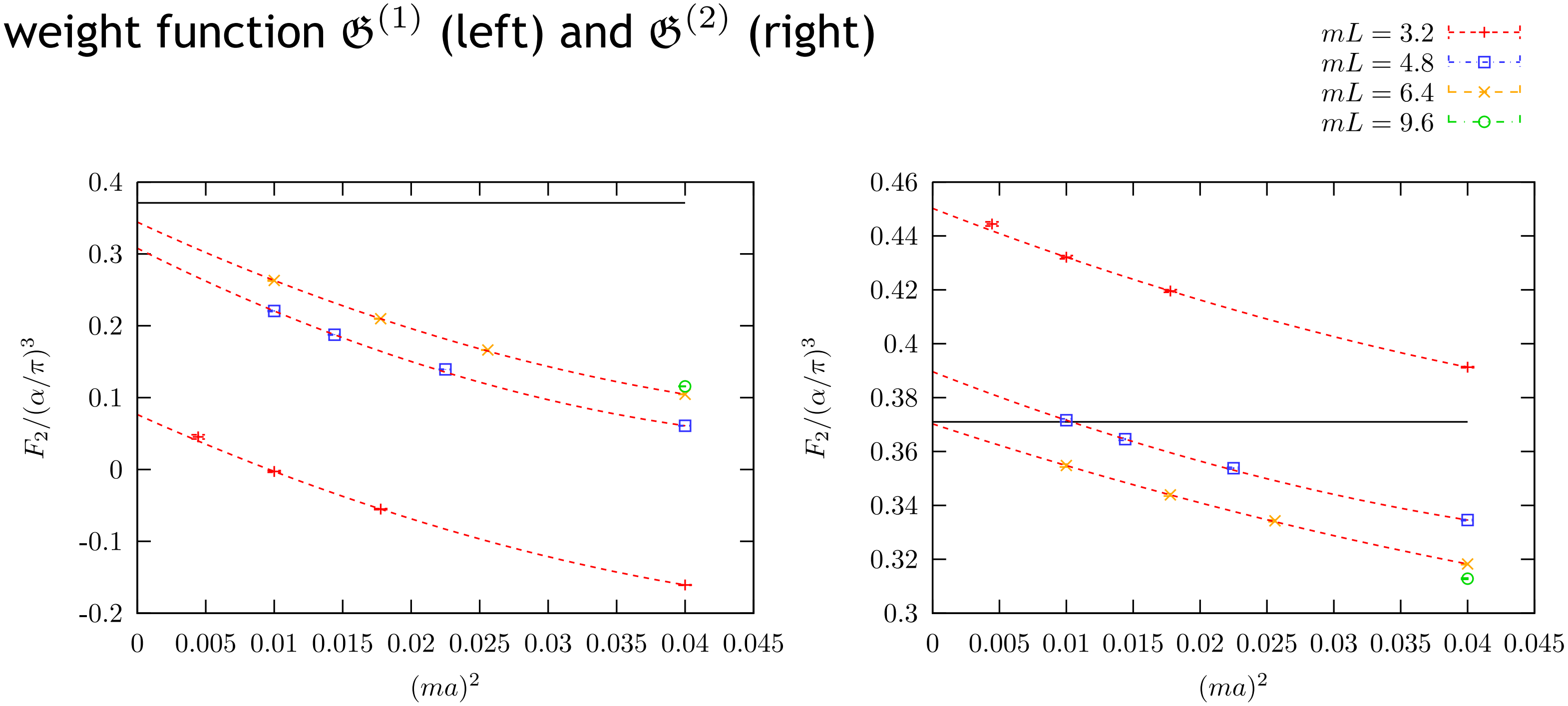
Results, QED case, Grid interpolation

- $m_{\text{loop}} = m_\mu$
- QED weight function without and with current-conservation subtraction $\mathfrak{G}^{(1)}$ (left) and $\mathfrak{G}^{(2)}$ (right)
- Grid points for (x, y, z) interpolation : $N = 6, 8, 10, 12, 14, 16$, second-order fits to $1/N^2$



Results, QED case, Discretization error

- QED weight function $\mathfrak{G}^{(1)}$ (left) and $\mathfrak{G}^{(2)}$ (right)

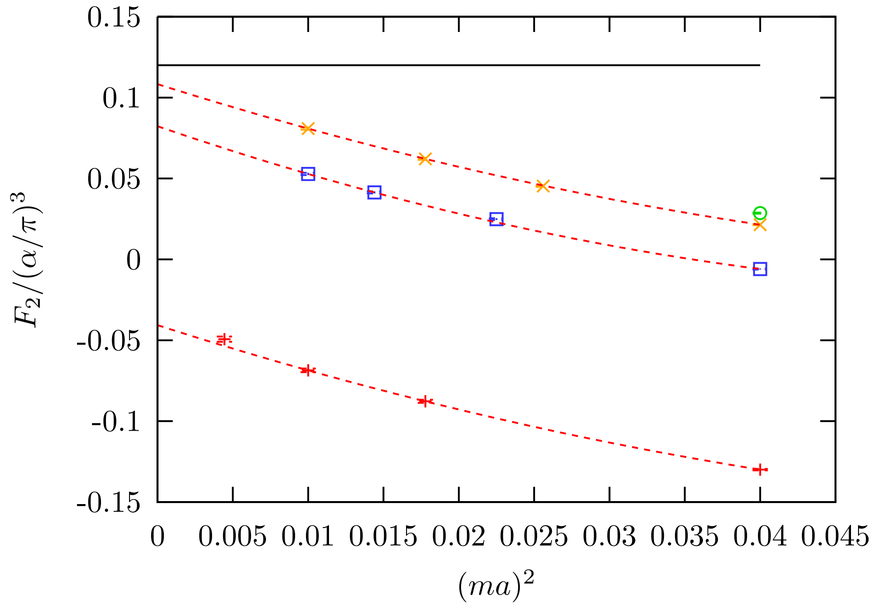


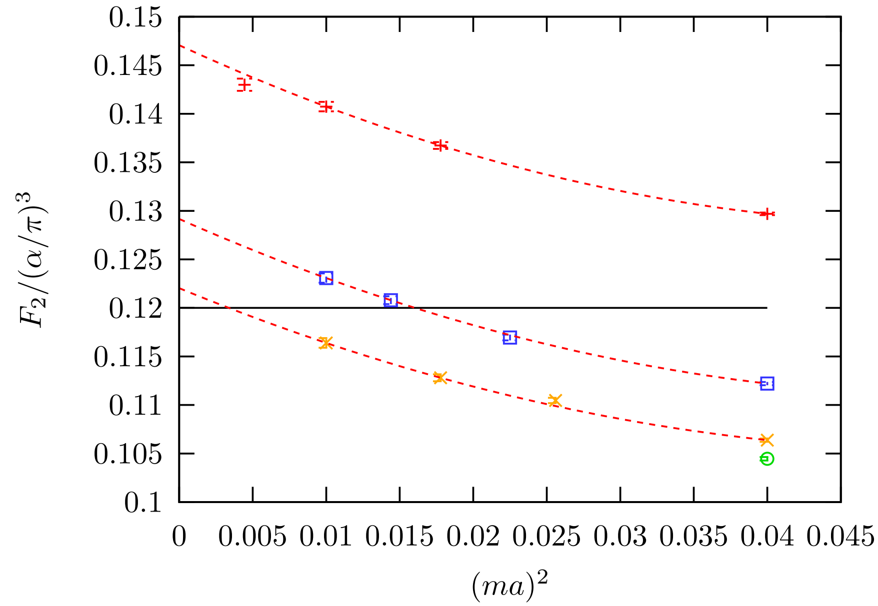
- Discretization error becomes almost independent of volume when $mL \geqslant 4.8$;
- The finite volume effect becomes negligible for $mL = 9.6$.
- Fit data with $mL \geqslant 4.8$, (and 3rd order also for systematic error)

$$F_2(L,a) \quad = \quad F_2(L) + k_1 a^2 + k_2 a^4.$$

Results, QED case, Discretization error, $m_{\mathbf{loop}} = 2m_\mu$

- QED weight function without and with current-conservation subtraction $\mathfrak{G}^{(1)}$ (left) and $\mathfrak{G}^{(2)}$ (right)





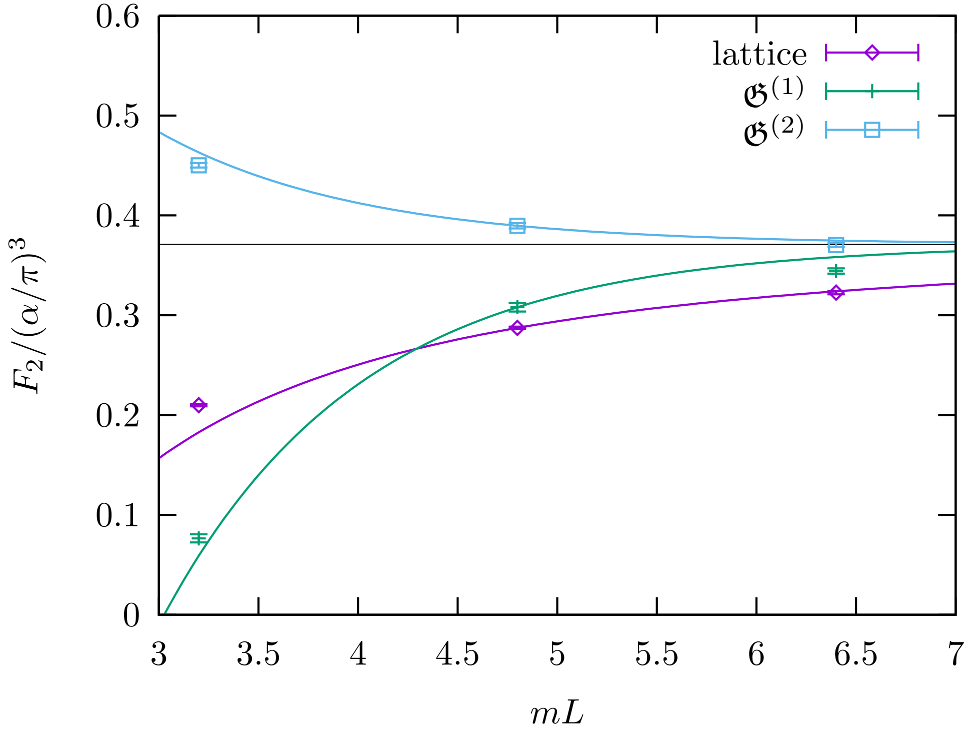
- $V \to \infty$ values are taken from mL =9.6 case : $\lim_{L\to\infty} F_2(L) \approx F_2(9.6/m)$.

- For $m_{\mathsf{loop}}/m_\mu = 1$ and 2,

$$F_2/(\alpha/\pi)^3 = 0.3686(37)(35), 0.1232(30)(28),$$

QED perturbation results : 0.371, 0.120

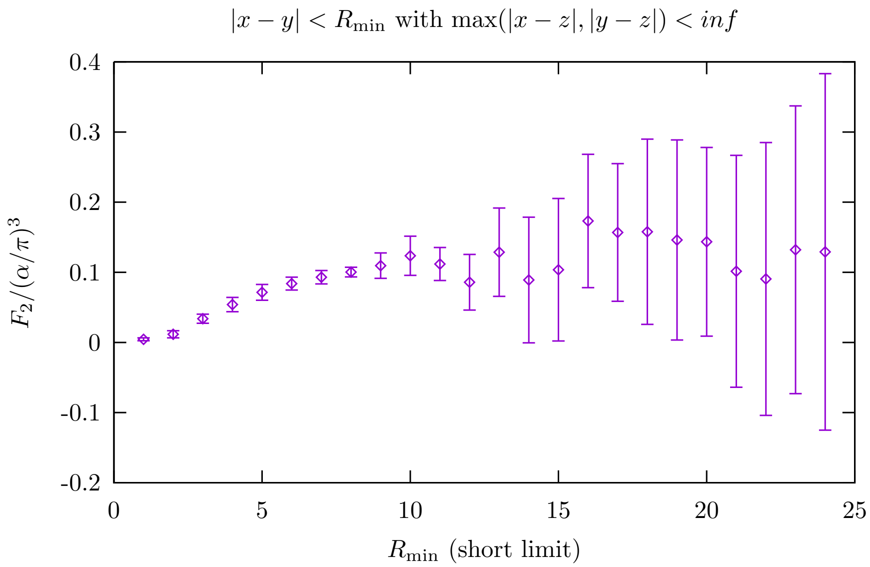
Results, QED case, Finite Volume Error

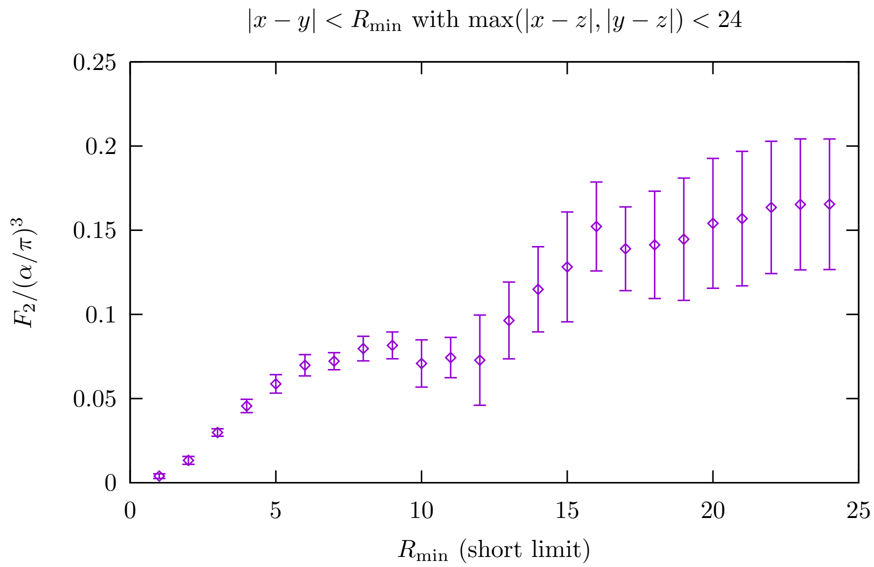


- QED weight : QED$_L$ (purple diamond), QED$_\infty$ without subtraction (green plus), with subtraction (blue square)
- Curves correspond to expected finite volume scaling $(0.371 + k/L^2)$ and infinite volume scaling $(0.371 + ke^{-mL})$, where the coefficient k is chosen to match the data at $mL = 4.8$.
- The right most point for the finite volume weighting function lies a bit off its scaling curve because the discretization error has not been completely removed, and the coefficient k does not contain any possible volume dependence.

Preliminary results, QCD case

- QCD case with physical point quark mass,
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.





- c.f. QED_L case, $\left.\frac{g\mu-2}{2}\right|_{\text{cHLbL}} = (0.0926 \pm 0.0077)\left(\frac{\alpha}{\pi}\right)^3$

Conclusion and Discussion

- Finite Volume Error in HLbL calculation is investigated.
- QED$_\infty$ weight function, infinite volume/continuum limit diagram, is shown to be advantageous in suppressing finite volume error (improved from $\mathcal{O}(1/L^2)$).

$$F_2(L, a) = F_2 + \mathcal{O}(e^{-mL}) + \mathcal{O}((ma)^2).$$

- Exponential suppression of FV would be a generic feature of Euclidean Green's function using QED$_\infty$.
- QED weight function with subtraction, $\mathfrak{G}^{(2)}$, using EM current conservation, is useful for (similar method already used in HVP) leads smaller finite volume and discretization error error. (similar observation for FV in HVP case).
- Very preliminary QCD results with $\mathfrak{G}^{(2)}$ is shown.
- Further study for $\mathfrak{G}^{(2)}$ effect in π^0 exchange model would be interesting to forecast QCD's finite size effect (on going)
- $1/a = 1$ GeV coarse DWF ensemble at physical point with 3 volumes, (4.8 fm)^3, (6.4 fm)^3, and (9.6 fm)^3 will be very useful. [R. Mawhinney's talk Tuesday]