SU(3) Yang Mills theory at small distances and fine lattices

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Questions

Open boundary conditions (OBC) and small flow time:

Time translation invariance is destroyed close to open boundaries in time.

1. How large is the boundary region in YM theory with open boundary conditions?
   Is the extent of the boundary region dominated by discretisation effects or the continuum limit?
2. Can the continuum limit and zero flow-time limit be taken with accessible lattices?
   Both limits are required in the correct order for the small flow-time expansion.
   OBC as a test case for finite temperature [1].

Non perturbative running into the perturbative regime:

We compute the coupling $\alpha_{\rm qq}$ from the force $F(r)$ between static quarks for $r/r_0 \in [0.11, 1.3]$.

3. Where is the perturbative region?
   Comparison of the non perturbative running $\alpha_{\rm qq}(\mu)$ with the perturbative running at 4-loop order perturbation theory in the regime $\alpha_{\rm qq} \leq 0.3$
4. Determination of the Λ-parameter. ($\Lambda_{\overline{\rm MS}}$ with 3% error)

Basics

Gradient flow:

- Solution [2] of $\frac{{\rm d}B_\mu(t,x)}{{\rm d}t} = D_\nu G_{\nu\mu}(t,x)\,,\ B_\mu(0,x) = A_\mu(x)$ is a smoothing with RMS radius $\sqrt{8t}$, where t is the flow time.
- Can be used to introduce a scale $\sqrt{8t_0}$ via [2]: $t^2\,\langle E(t,x)\rangle|_{t=t_0} = 0.3$.
- **Small flow-time expansion:**
  Action density can be expressed as an asymptotic series [3]

$$E(t,x) = \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x) = c_1(t) + c_E(t) T_{\mu\mu}(x) + \mathcal{O}(t)\ ,$$

with the trace of the energy-momentum tensor $T_{\mu\mu}$ and coefficients [1]

$$c_1(t) = \left\langle E(t, x_0^{\rm plat})\right\rangle\ ,\ c_E(t) = \frac{1}{2b_0}\left(1 + 2b_0\bar{s}\bar{g}^2 + \mathcal{O}(\bar{g}^4)\right)\ ,\ \sqrt{20t_0} \lesssim x_0^{\rm plat} \lesssim T - \sqrt{20t_0}\,.$$

Flow times need to satisfy [4] $\quad a \ll \sqrt{8t} \ll$ relevant low energy scales .

Coupling in the qq-scheme:

- Force from Wilson loops

$$\bar{g}^2_{\rm qq}(r) = 3\pi r^2 F(r), \qquad \alpha_{\rm qq} = \bar{g}^2_{\rm qq}/(4\pi)$$

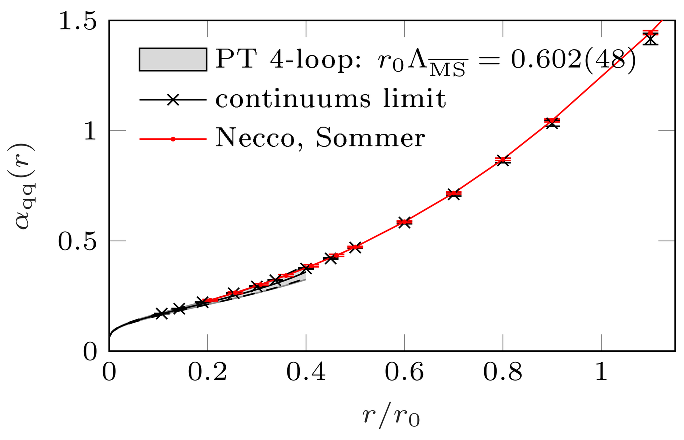
- Perturbative running up to 4-loop order [5, 6, 7, 8, 9]:

$$-r\frac{\rm d}{{\rm d}r}\bar{g}_{\rm qq}(r) = \beta(\bar{g}_{\rm qq})\,, \qquad \beta^{\rm qq}(\bar{g}_{\rm qq}) = -\bar{g}^3_{\rm qq}\left[b_0 + b_1\bar{g}^2_{\rm qq} + b_2\bar{g}^4_{\rm qq} + \left(b_3 + b_{3,\rm IR}\ln\left(C_{\rm A}\frac{\bar{g}^2_{\rm qq}}{8\pi}\right)\right)\bar{g}^6_{\rm qq}\right] + \mathcal{O}(\bar{g}^{11}_{\rm qq})$$

- Renormalization group invariant $\Lambda_{\rm qq} r = \varphi(\bar{g}_{\rm qq}(r))$

$$\varphi(\bar{g}_{\rm qq}(r)) = (b_0\bar{g}^2_{\rm qq})^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2_{\rm qq}}} \times \exp\left[-\int_0^{\bar{g}_{\rm qq}} {\rm d}x \left(\frac{1}{\beta^{\rm qq}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right)\right] \qquad (1)$$

Conversion to $\overline{\rm MS}$: $\quad \Lambda_{\overline{\rm MS}} = \Lambda_{\rm qq} \times \exp(35/66 - \gamma_{\rm E})$



Simulations

- Wilson plaquette action with OBC
  $(F_{0k}(0,\vec{x}) = F_{0k}(T,\vec{x}) = 0)$

$$S_{\rm G}[U] := \frac{\beta}{6}\sum_p w_p\,{\rm Re\,tr}\,[\mathbb{1} - U_p]$$

- **hybrid overrelaxation (HOR):**
  - Cabibbo-Marinari
  - $8 - 30$ overrelaxation sweeps per heat bath
- large volume $L = 2\,{\rm fm}$, $2 \leq T/L \leq 3.4$

β	a [fm]	r/r_0	$N_{\rm wl}$	$N_{\rm flow}$
6.0662	0.0834(4)	[0.42, 1.92]	121	511
6.2556	0.0624(4)	[0.31, 1.44]	101	361
6.3406	*0.0555(2)*	–	–	341
6.5619	0.0411(2)	[0.21, 1.04]	301	165
6.7859	0.0312(2)	[0.16, 1.22]	64	49
6.9606	*0.0250(3)*	–	–	76
7.1146	0.0206(2)	[0.10, 0.64]	64	–
7.3600	0.0152(2)	[0.07, 0.48]	93	–

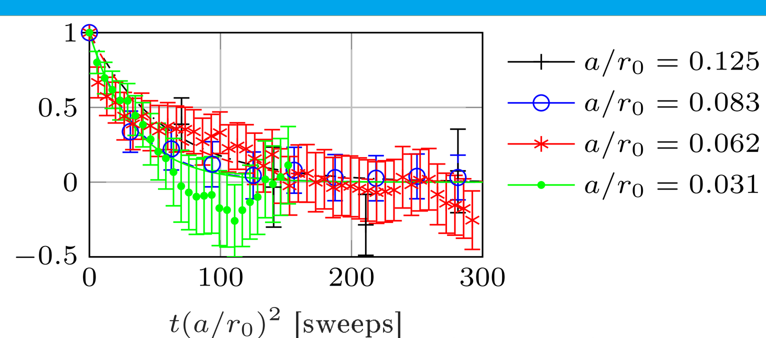
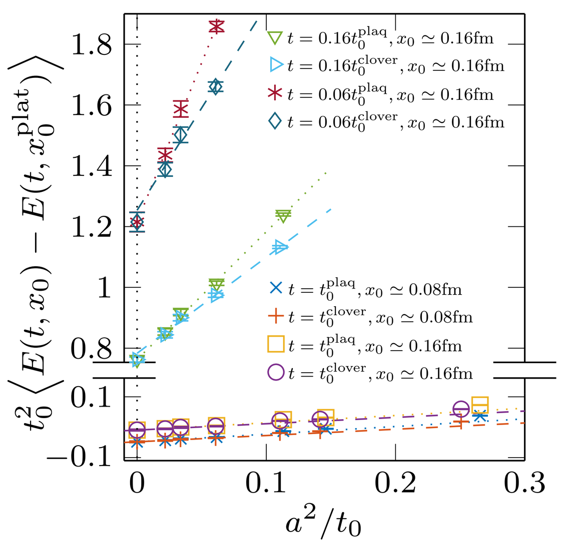


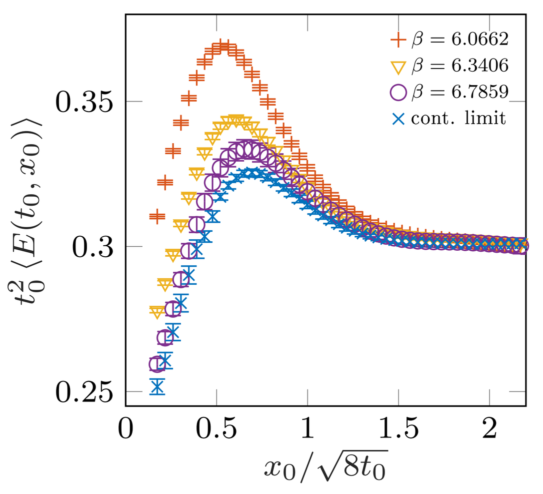
Fig. normalized autocorrelation function of the global topological charge scales roughly with a^2

Open boundaries and small flow time

- **Discretisation effects and continuum limit**
  - $t \gtrsim t_0$: moderate a-effects
    $t \in [0.06, 0.17]\,t_0$: more problematic
  - Continuum limit reached
    $t \simeq 0.06\,t_0$: with $a \in [0.025, 0.041]$ fm
    $t \simeq 0.16\,t_0$: with $a \in [0.025, 0.056]$ fm

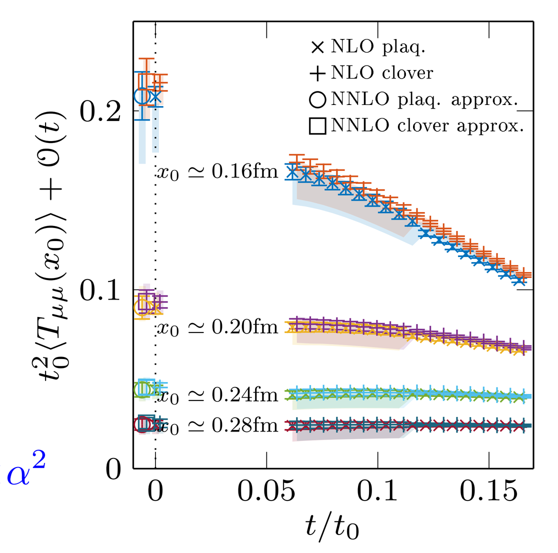


- **Width of boundary region:**
  $\sim \sqrt{20t_0}$
  - before and after continuum limit
  - $\Rightarrow$ improvements of any kind (boundary counterterms, improved flow, bulk Symanzik improvement) will not enlarge the extent of the boundary region significantly.
  - $\Rightarrow$ The boundary peak is a continuum phenomenon



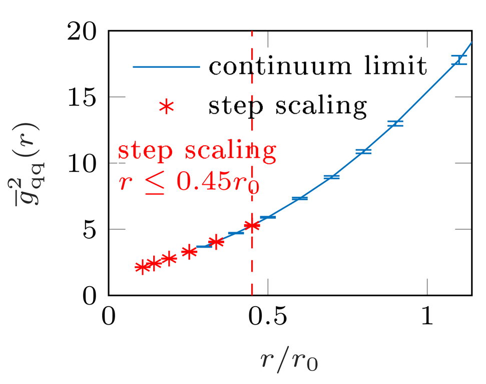
- **Zero flow-time limit**
  - possible with our $a \in [0.025, 0.056]$ fm
  - accuracy around 10 - 20% with our statistics
    naive linear extrapolation in t with first, central and last point
  - systematic effects of the continuum extrapolation seem under control
    Tested by excluding one lattice spacing (filled area).
  - stable with respect to a change in the perturbative $c_E(t)$ of

$$\Delta c_E(t) = \pm\frac{\bar{g}^4}{2b_0(4\pi)^2} = {\rm LO}\times\alpha^2$$



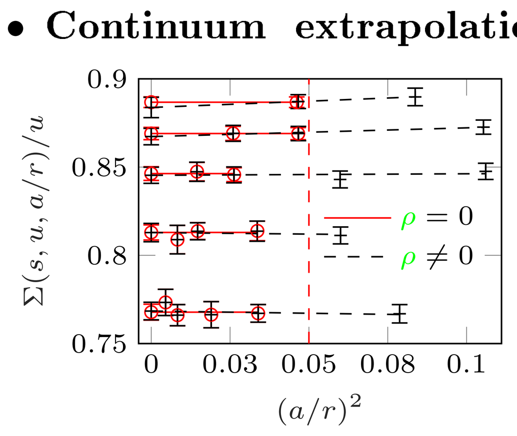
Step scaling for large volume and short distances

- **Step scaling** $\bar{g}^2_{\rm qq}(sr) = \sigma(s, \bar{g}^2_{\rm qq}(r))$ with $s = 0.75$ for $r \leq 0.45$
  - **Benefits:**
    - + distances are kept small at fine lattice spacings a
    - + less computational resources and still high precision



$$u_0 = \bar{g}^2_{\rm qq}(r_*)$$
$$u_1 = \bar{g}^2_{\rm qq}(sr_*) = \sigma(s, u_0)$$
$$u_2 = \bar{g}^2_{\rm qq}(s^2 r_*) = \sigma(s, u_1)$$
$$\vdots$$
$$u_5 = \bar{g}^2_{\rm qq}(s^5 r_*) = \sigma(s, u_4)$$

- **Continuum extrapolation** in every step scaling iteration:



$$\bar{g}^2_{\rm qq}(sr, a) = \Sigma(s, u, a/r)\,|_{\bar{g}^2_{\rm qq}(r,a) = u}$$

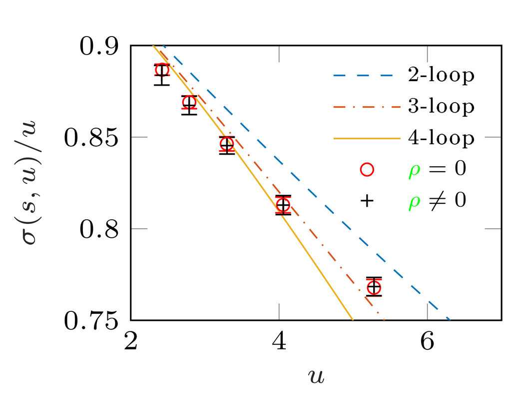
with and without slope ρ

$$\sigma(s, u) = \lim_{a/r\to 0}\Sigma(s, u, a/r)$$

with fit

$$\Sigma(s, u, a/r) = \sigma(s, u)\{1 + \rho\,(a/r)^2\}$$

- **Comparison $\sigma(s,u)$** - non perturbative (NP) vs. perturbation theory (PT)



PT: $\quad \ln(s) = -\int_{\sqrt{u}}^{\sqrt{\sigma(s,u)}} \frac{1}{\beta^{\rm qq}(g)}\,{\rm d}g$

NP results cross 4-loop PT at $\alpha_{\rm qq} \approx 1/3$

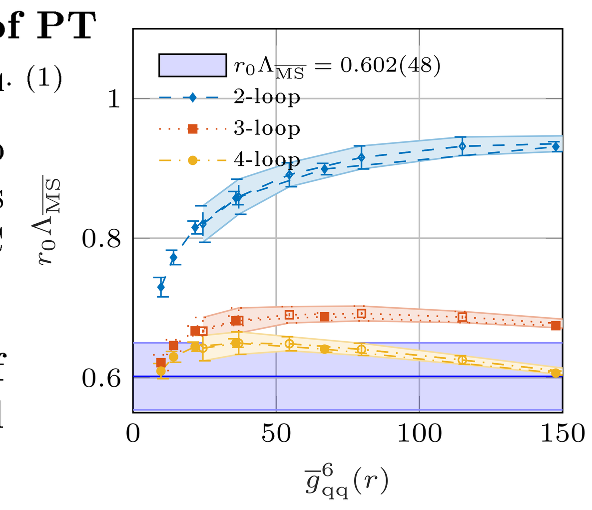
Conclusions

Open BC allow for simulations at fine lattice spacings:

- + short ($\sim$ perturbative) distances in the continuum limit
  $\Rightarrow$ PT at $\alpha_{\rm qq} \approx 0.24$ is not very accurate
- + small flow time in the continuum limit
- – the boundary region remains large (≈ 0.8 fm) in the continuum limit

Λ-parameter of YM theory

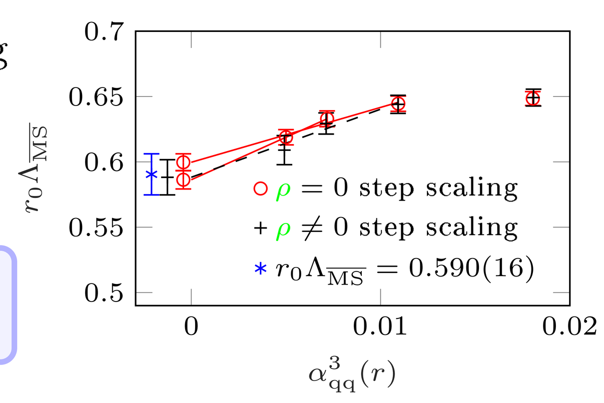
- **$\Lambda_{\overline{\rm MS}}$ for different orders of PT**
  $(r, \alpha_{\rm qq}(r))$ inserted into $\varphi(\bar{g}_{\rm qq}(r))$ eq. (1)
  - comparison with Necco (shaded error bars) [10] shows no evidence for impact of BC (open/periodic)
  - error is smaller because of step scaling and additional lattice spacings



- **Extrapolation of the Λ-parameter**
  - from last three stepscaling steps at 4-loop:

$$r_0\Lambda_{\rm qq} = r_0 \lim_{\substack{r\to 0 \\ (\bar{g}_{\rm qq}\to 0)}} \varphi(\bar{g}_{\rm qq}(r))/r$$

$$\Rightarrow r_0\Lambda_{\overline{\rm MS}} = 0.590(16)$$



extrapolations: 2 pts. + 3 pts. with $\rho = 0$ and 3 pts. $\rho \neq 0$

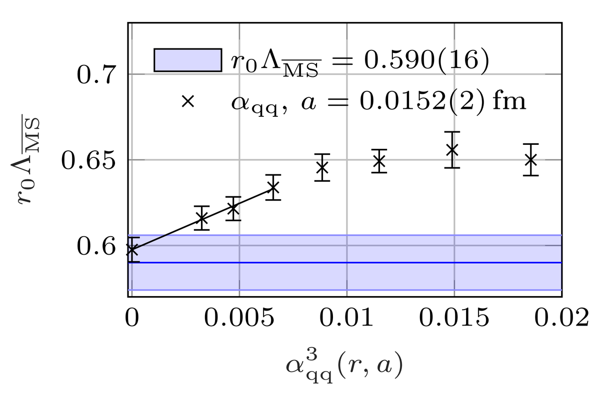
- **Where is the perturbative region?**
  - running described by perturbative $\beta^{\rm qq}$ function at 4-loop deviates already at $\alpha_{\rm qq} \approx 0.24$ significantly from the NP running, similar to the SF-coupling at $\alpha = 0.19$ [11]

at $\alpha_{\rm qq} = 0.246$:

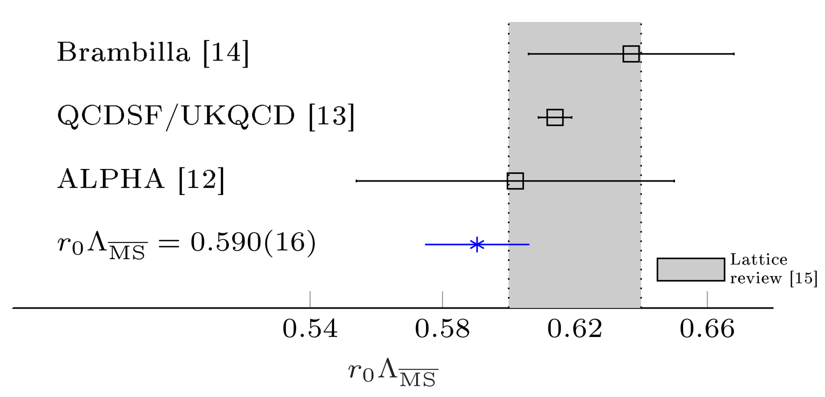
$$\frac{\Lambda^{4\text{-loop}}_{\overline{\rm MS}} - \Lambda_{\overline{\rm MS}}}{\Lambda_{\overline{\rm MS}}} \approx 5\alpha^3_{\rm qq}$$

at $\alpha_{\rm qq} < 0.22$

$$\frac{\Lambda^{4\text{-loop}}_{\overline{\rm MS}} - \Lambda_{\overline{\rm MS}}}{\Lambda_{\overline{\rm MS}}} = 7.0(5)\alpha^3_{\rm qq}$$



- **Comparison to other results**



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