Dual simulation of the massless lattice Schwinger model with a topological term and non-zero chemical potential

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22.06.2017

Partition sum of the massless one-flavor model with staggered fermions and a topological term in conventional representation:

$$Z = \int \mathcal{D}[U] \int \mathcal{D}[\bar{\psi}, \psi] \; e^{-S_G[U] - i\theta Q[U] - S_\psi[U, \bar{\psi}, \psi]},$$

$S_G[U] \ldots$	Wilson Gauge action
$Q[U] \ldots$	Topological charge
$S_\psi[U, \bar{\psi}, \psi] \ldots$	Staggered Fermion action

$\rightarrow$ Sign problem for $\theta \neq 0$ or $\mu \neq 0$!

The partition sum in dual representation

Dual partition sum for one flavor and $\theta > 0$:

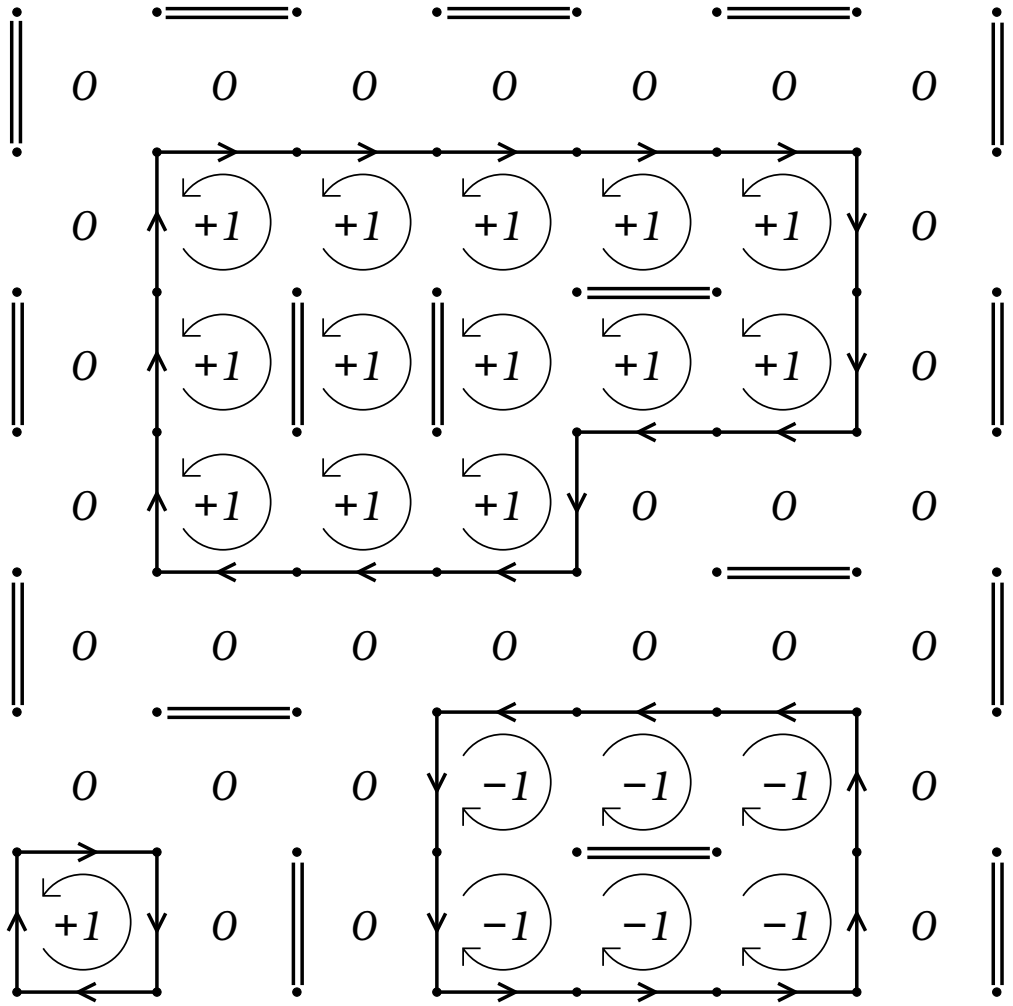
$$Z = \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} \prod_n \mathtt{I}_{|p(n)|}\left(2\sqrt{\eta\bar{\eta}}\right) \left(\sqrt{\frac{\eta}{\bar{\eta}}}\right)^{p(n)}$$

$$\eta = \frac{\beta}{2} - \frac{\theta}{4\pi}\,, \qquad \bar{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi}$$

$$l_{n,\nu} \in \{-1, 0, 1\}\,, \quad d_{n,\nu} \in \{0, 1\}\,, \quad p_n \in (-\infty, \infty)$$

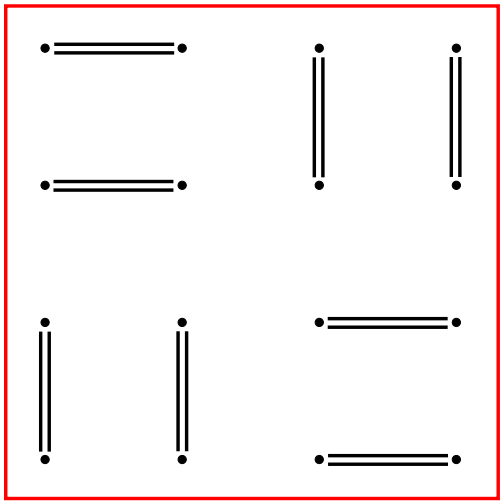
C. Gattringer, T. Kloiber, V. Sazonov, Solving the sign problems of the massless lattice Schwinger model with a dual formulation, Nucl.Phys. B897 (2015) 732-748 arXiv:1502.05479 [hep-lat]

Example of an admissible configuration

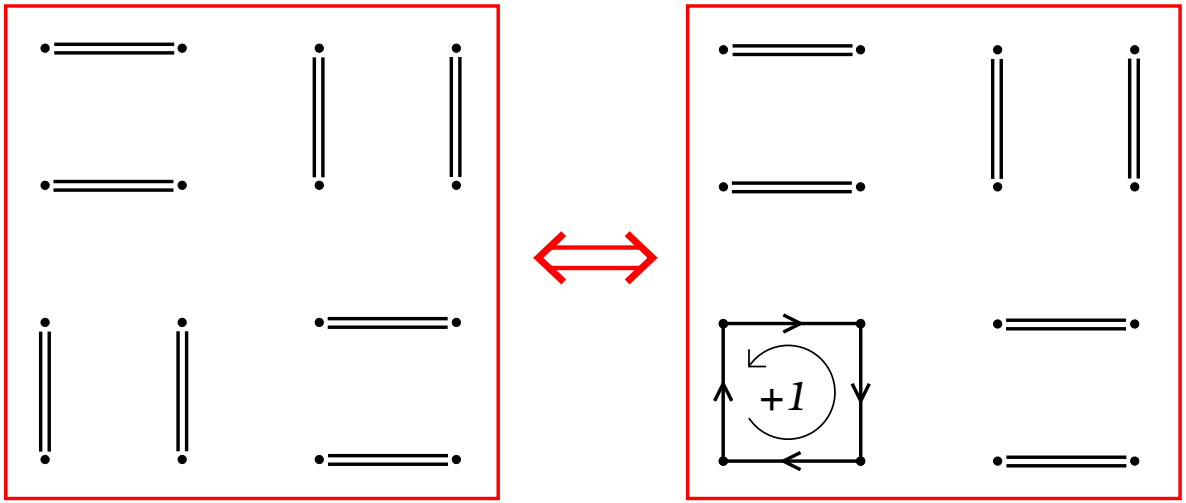


- **Pauli principle:** Each lattice point has to be occupied by a fermion exactly once.
- **Vanishing link flux:** $\Rightarrow$ Sheets of plaquettes bounded by loops.

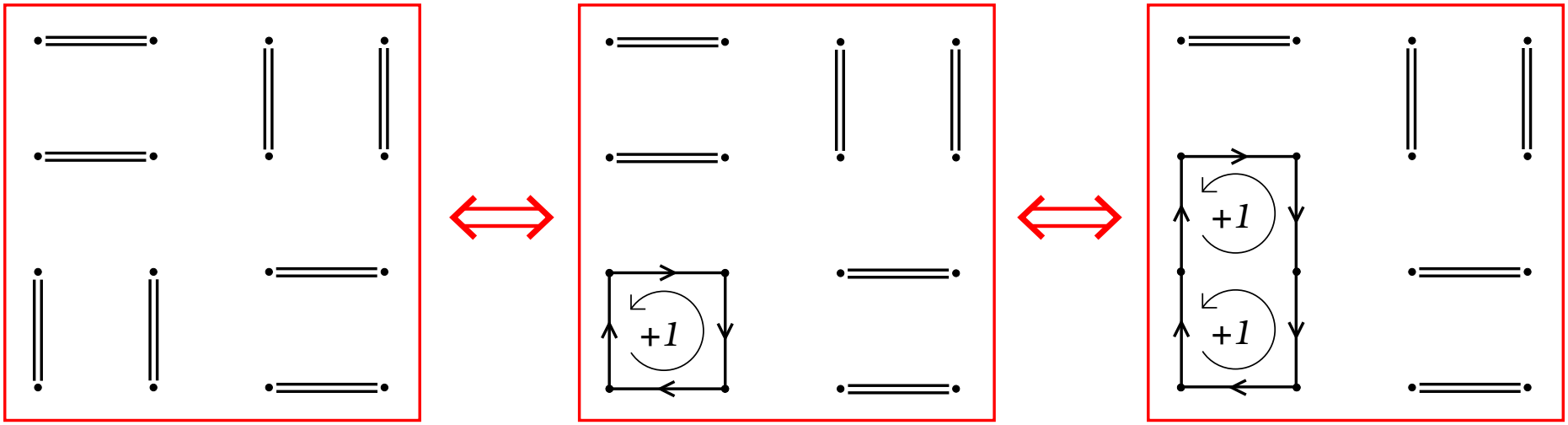
Local updates - accepted with Metropolis steps



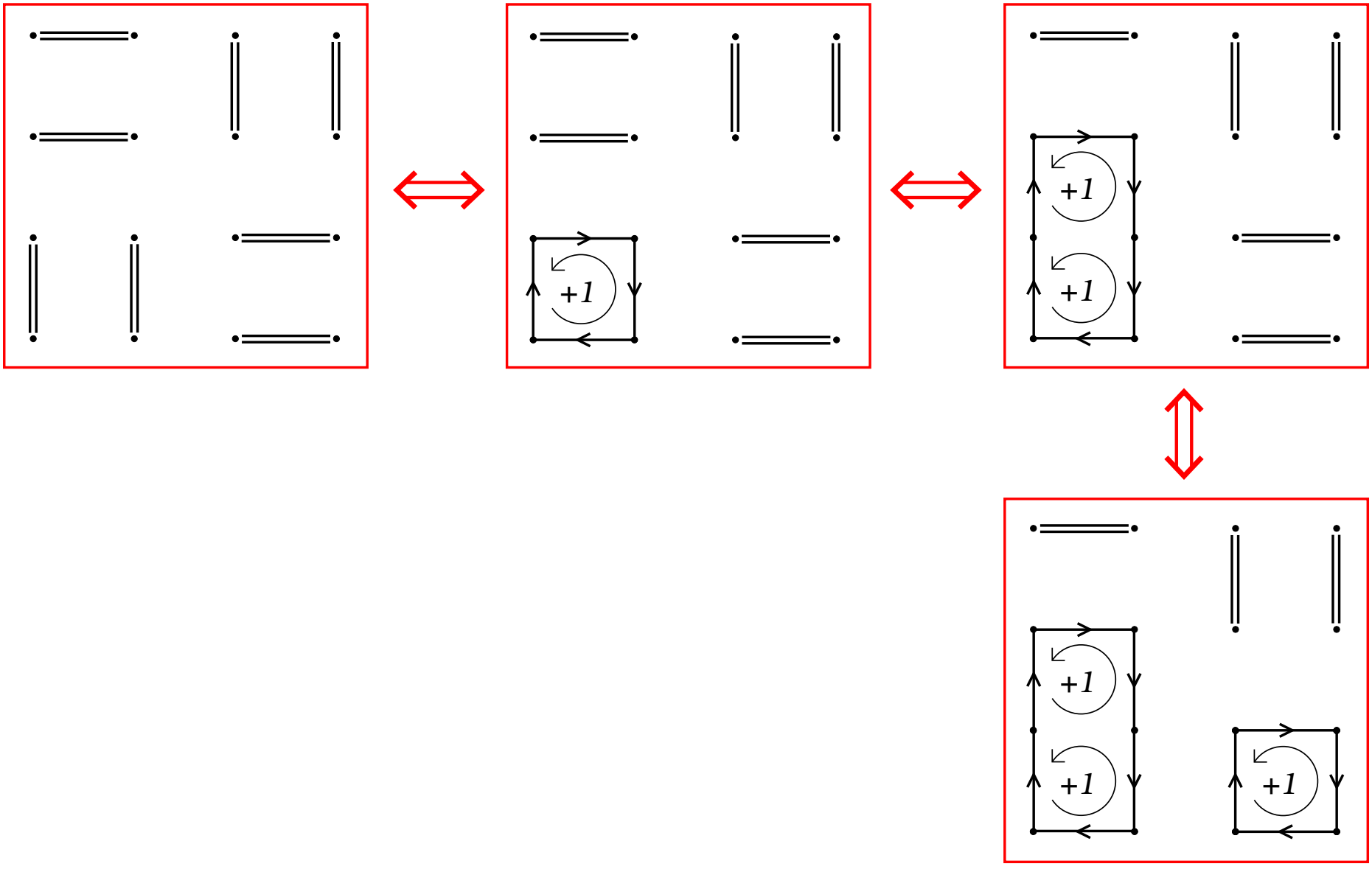
Local updates - accepted with Metropolis steps



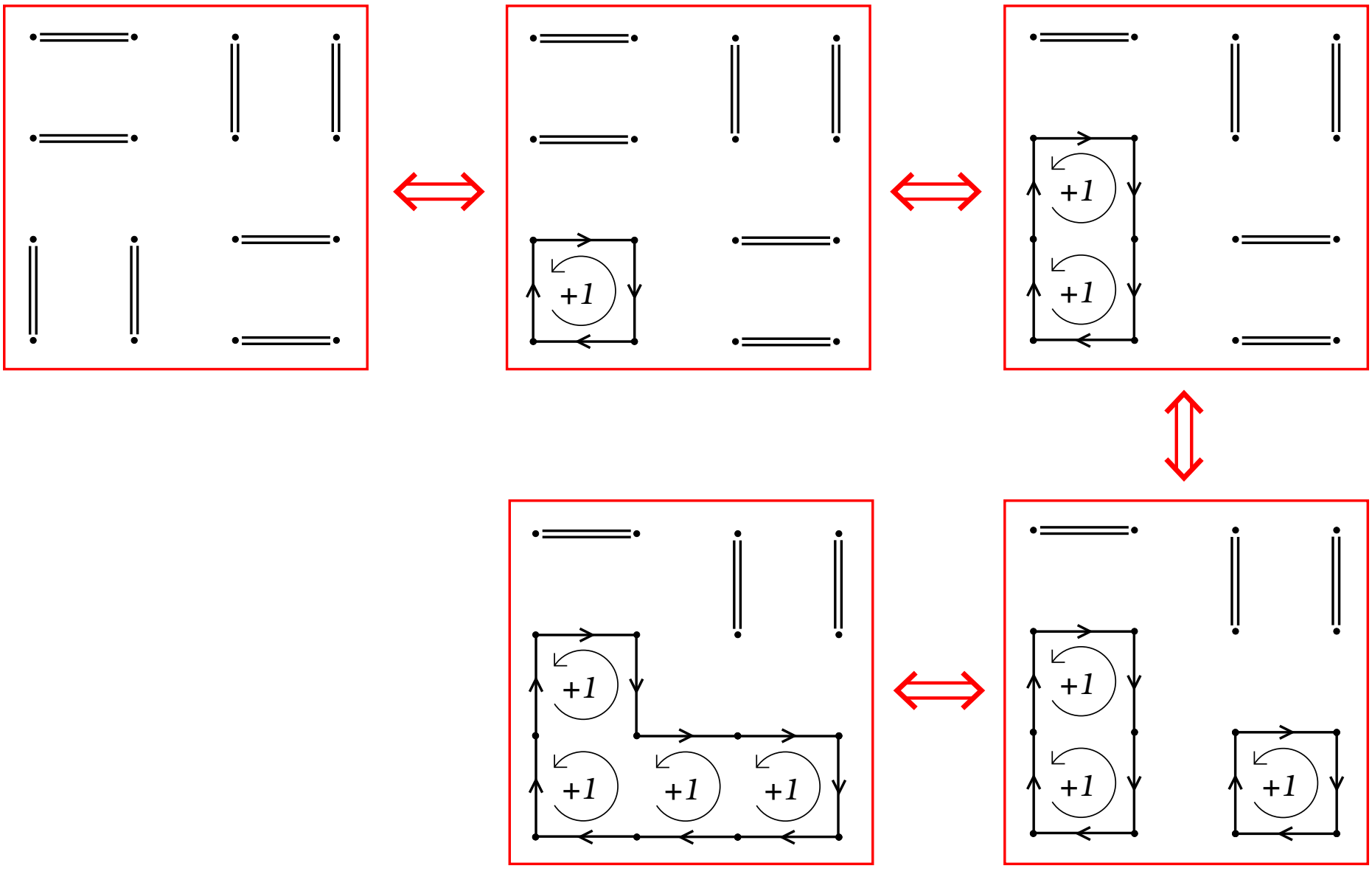
Local updates - accepted with Metropolis steps



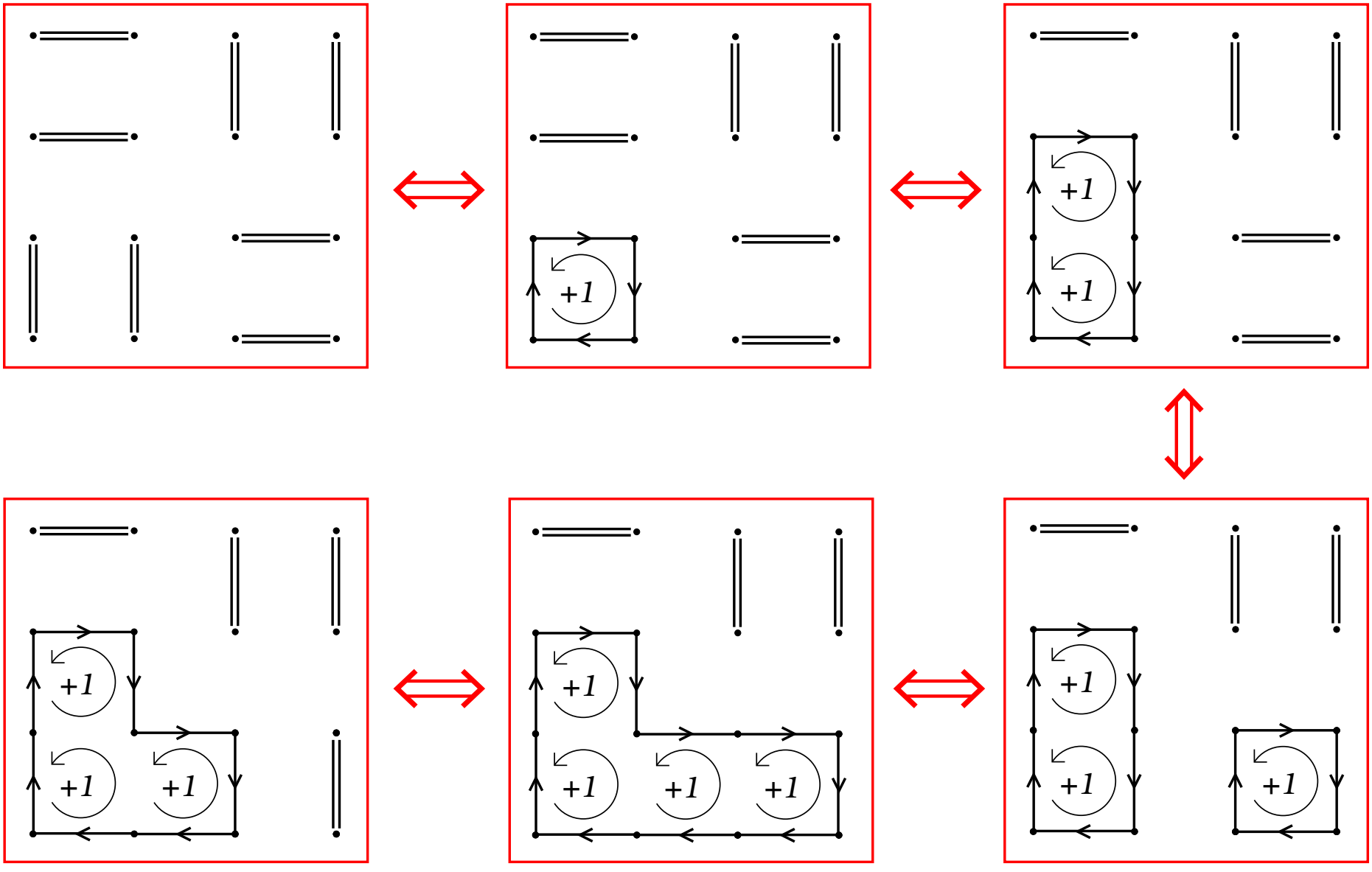
Local updates - accepted with Metropolis steps



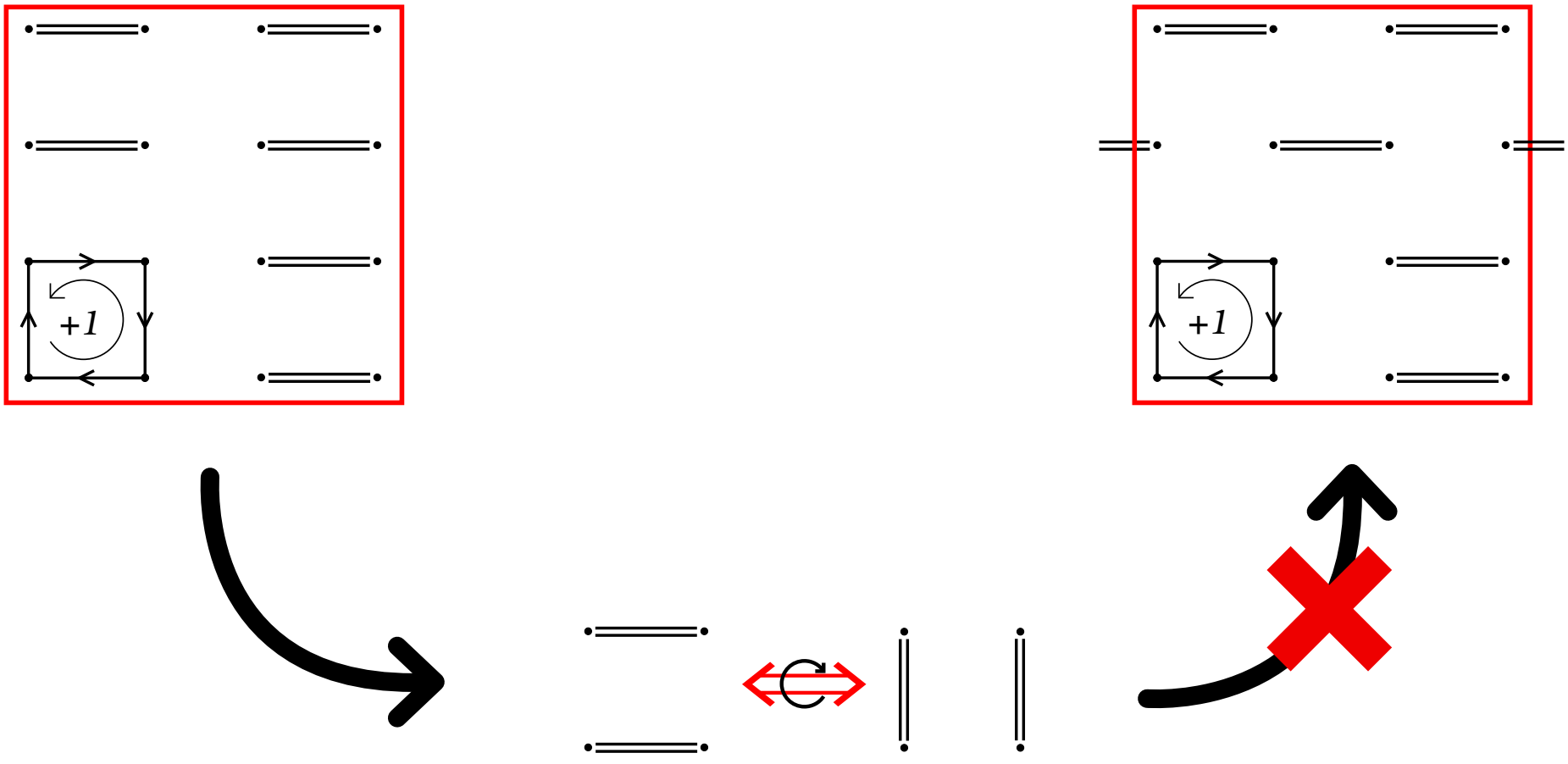
Local updates - accepted with Metropolis steps



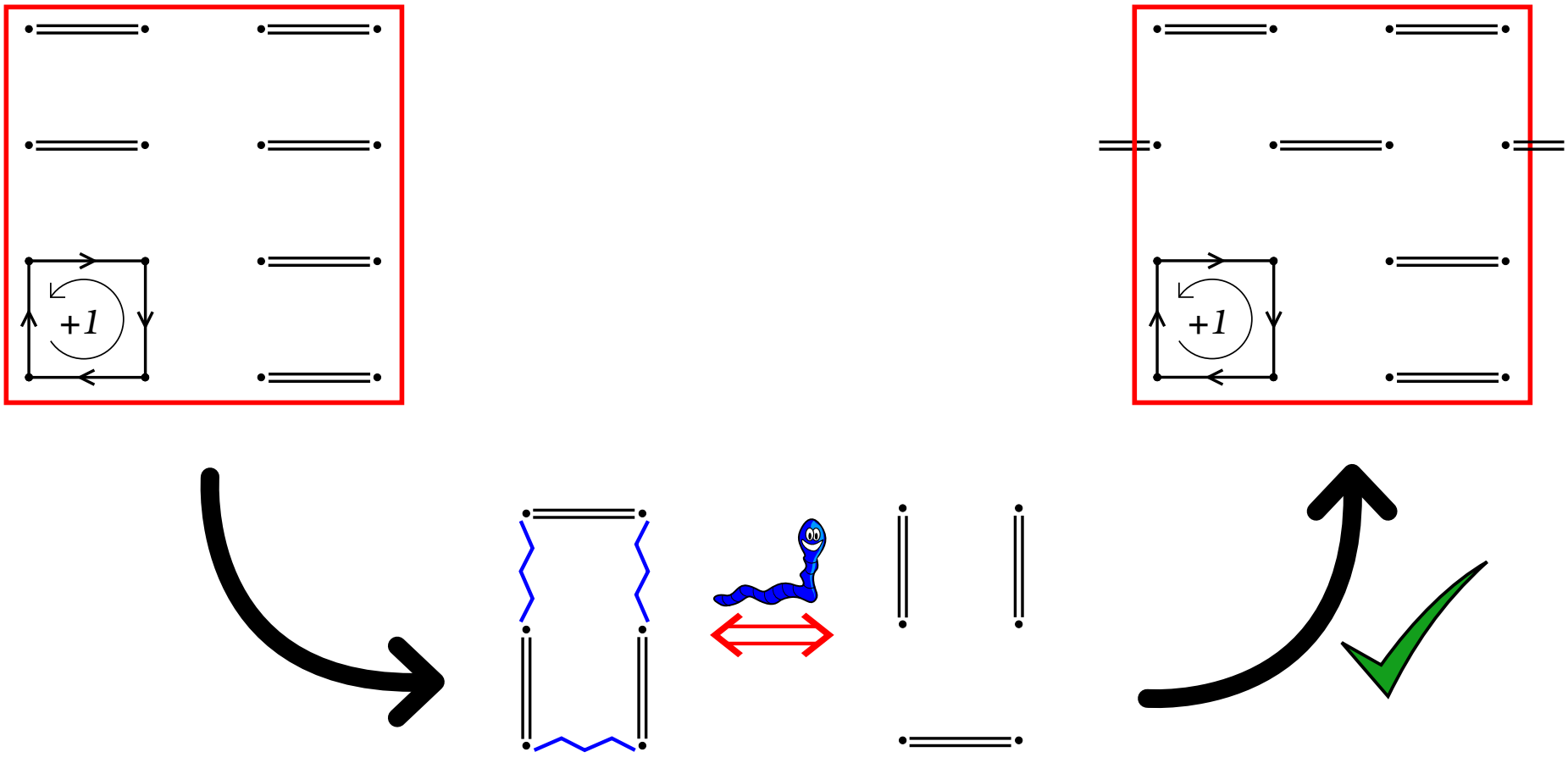
Local updates - accepted with Metropolis steps



Dimers



Worm update for dimers



Observables

- Plaquette expectation value and susceptibility:

$$\langle U_p \rangle = \frac{1}{V} \frac{\partial}{\partial \beta} ln(Z)$$

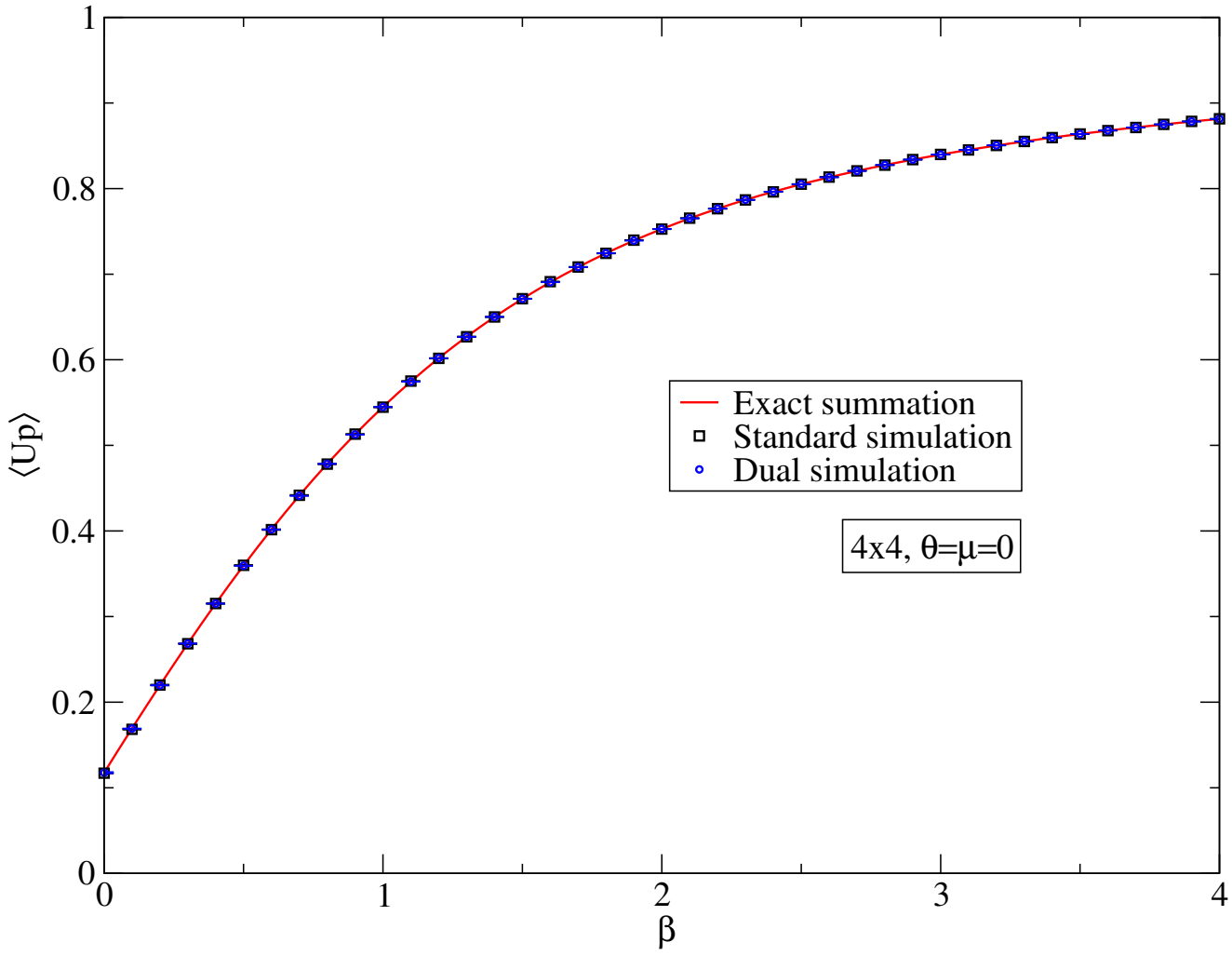
$$\chi_p = \frac{1}{V} \frac{\partial^2}{\partial \beta^2} ln(Z)$$

- Topological charge density and susceptibility:

$$\langle q \rangle = -\frac{1}{V} \frac{\partial}{\partial \theta} ln(Z)$$

$$\chi_q = -\frac{1}{V} \frac{\partial^2}{\partial \theta^2} ln(Z)$$

Testing the algorithm



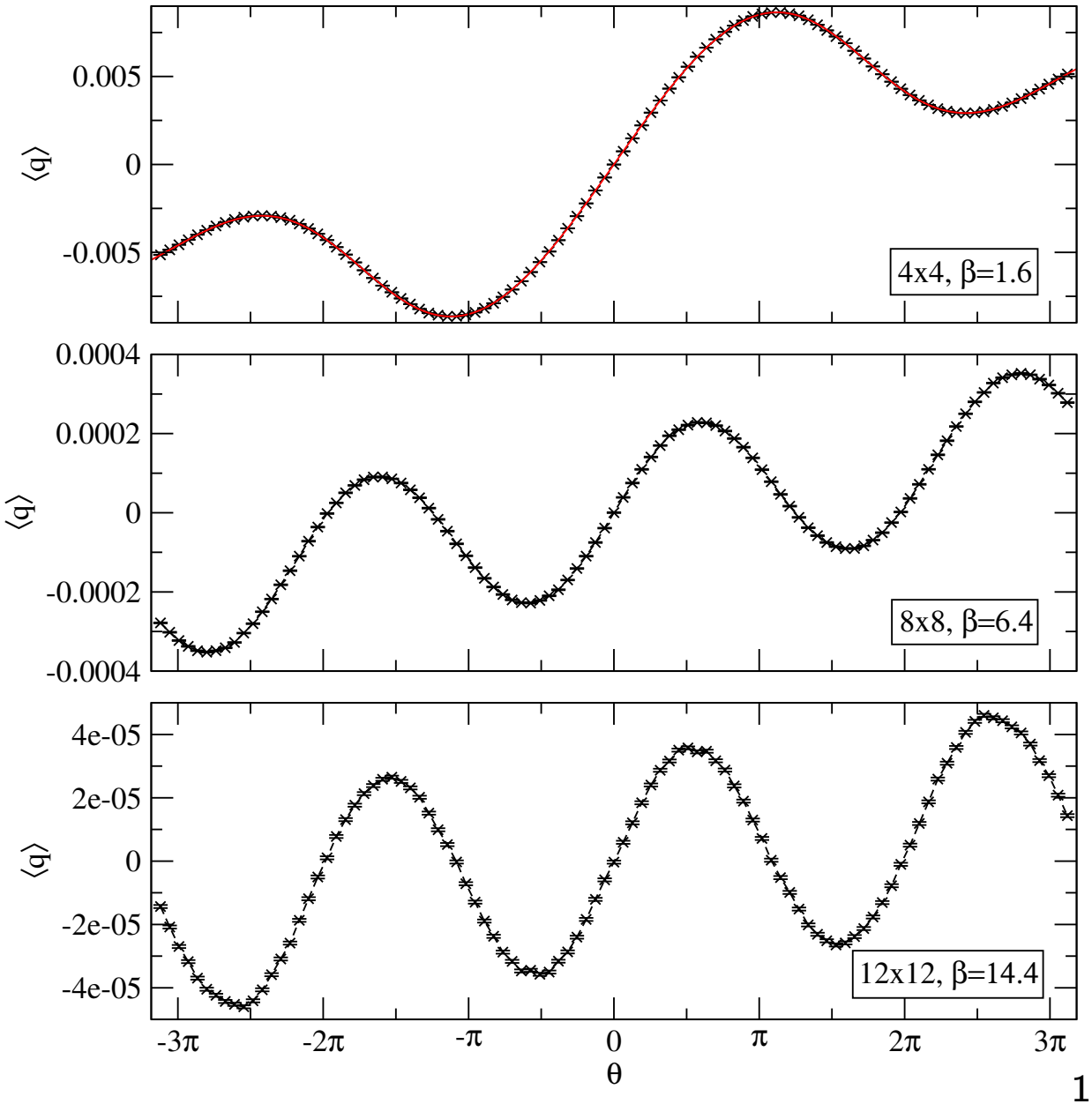
Testing the algorithm with standard simulation at $\theta = 0$ and exact summation on a small lattice.

Emerging 2π periodicity

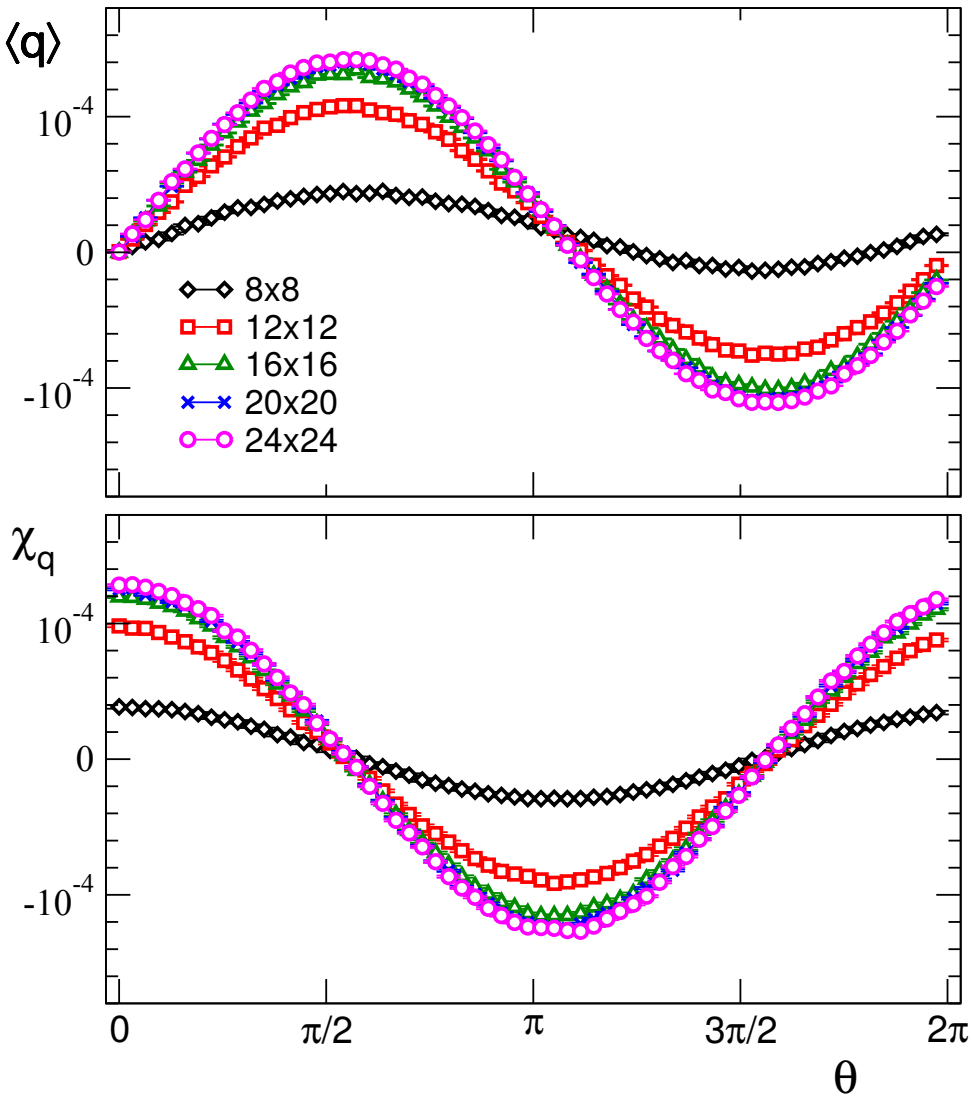
Topological charge density:

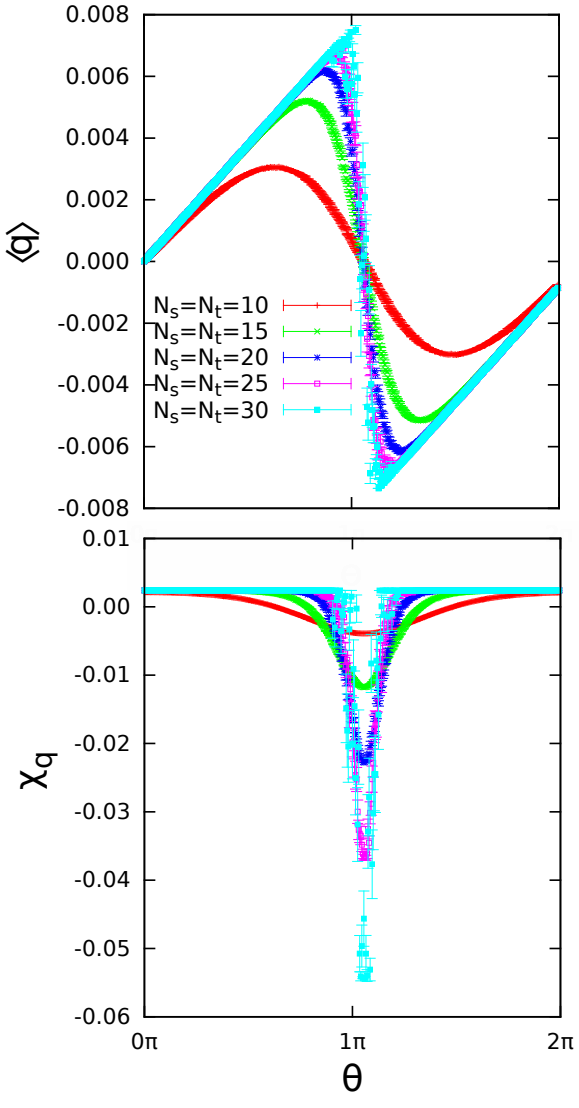
Approach to the continuum limit with $\frac{\beta}{N_s N_t} = 0.1 = const$

Equivalent behavior for $\langle U_p \rangle$.



Behavior near $\theta = \pi$





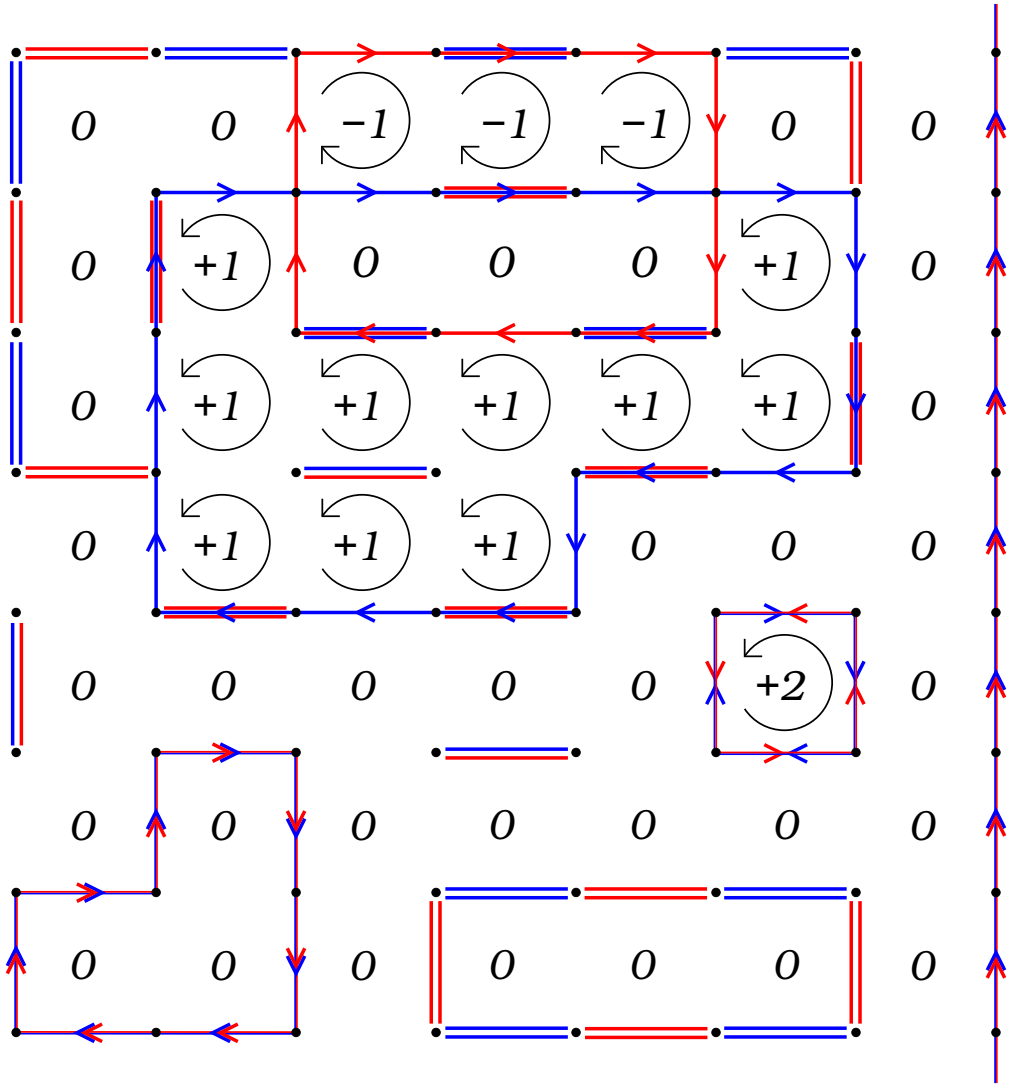
Comparison to scalar QED_2: *C. Gattringer et al.: Phys. Rev. D* **92** *(2015)*

Two flavor model with chemical potential

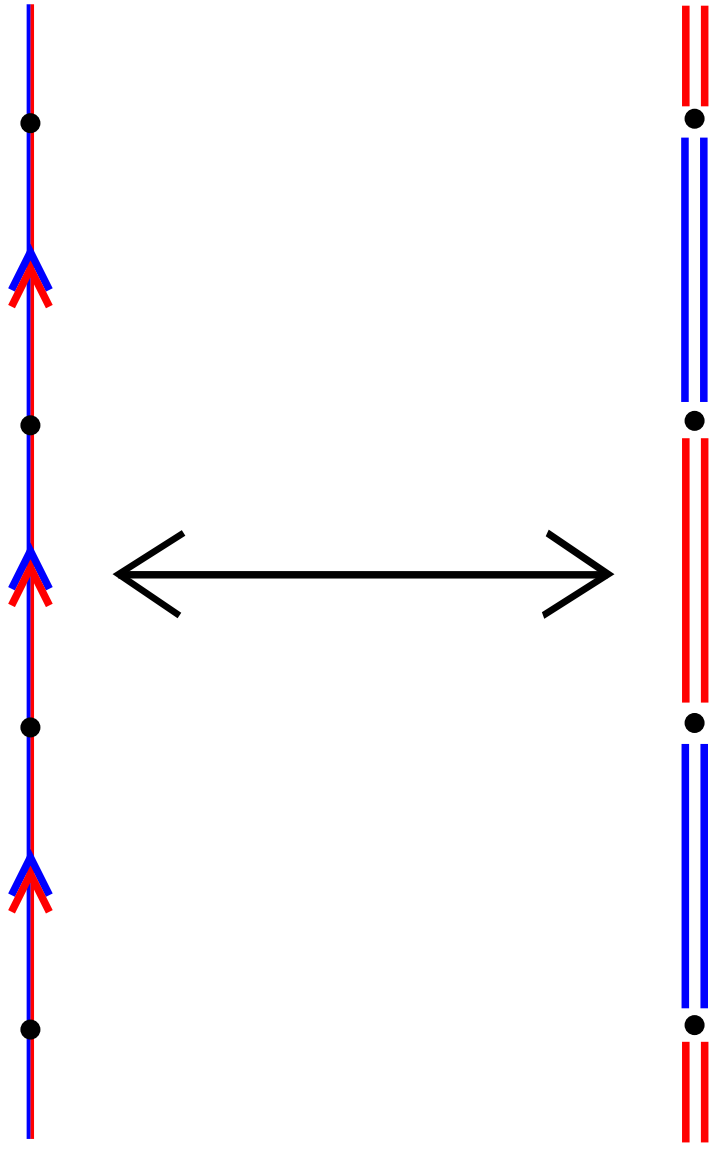
$$Z = \left(\frac{1}{2}\right)^{2V} \sum_{\{l,d,\bar{l},\bar{d},p\}} e^{\mu_\psi N_T W(l)} e^{\mu_\chi N_T W(\bar{l})} \prod_n \mathtt{I}_{|p(n)|}\left(2\sqrt{\eta\bar{\eta}}\right) \left(\sqrt{\frac{\eta}{\bar{\eta}}}\right)^{p(n)}$$

$$l_{n,\nu}, \bar{l}_{n,\nu} \in \{-1, 0, 1\}, \quad d_{n,\nu}, \bar{d}_{n,\nu} \in \{0, 1\}, \quad p_n \in (-\infty, \infty)$$

Example of an admissible configuration

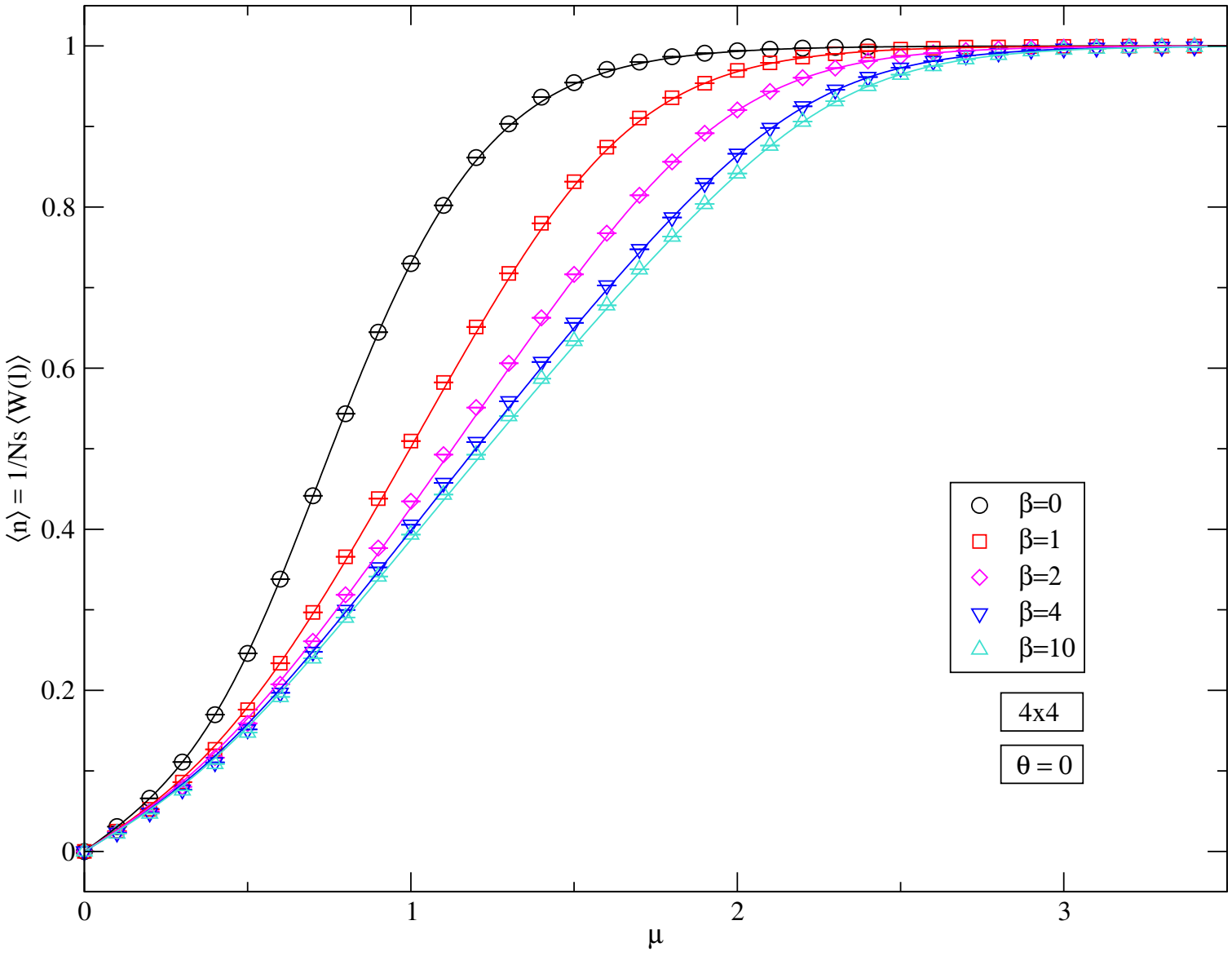


Additional updates for the winding numbers



- The weight only changes for temporally winding structurs.
- Challange: topological nature of states at large μ makes the system very stiff.

Testing the updates against exact summation



Summary

- First example of a dual simulation of a lattice field theory with gauge fields and fermions at arbitrary couplings, finite chemical potential and non-zero vacuum angle.
- Fermionic systems pose a challange due to additional constraints from the Pauli principle.
- Mix of local and worm updates works very well for $\theta > 0$.
- Remaining θ dependence due to staggered fermions.
- For large μ, two-dimensional worldline models can be quite stiff due to topologically stabilized configurations.

Thank you for your attention! :-)

Massless quantum electrodynamics in 2D with a topological term:

$$S[A,\bar{\psi},\psi] = S_G[A] + S_\psi[A,\bar{\psi},\psi] + i\theta Q[A]$$

$$S_G[A] = \frac{\beta}{4}\int d^2x\ F_{\mu\nu}(x)F_{\mu\nu}(x)$$

$$S_\psi[A,\bar{\psi},\psi] = \int d^2x\ \bar{\psi}(x)\left[\gamma_\mu\left(\partial_\mu + iA_\mu(x)\right)\right]\psi(x)$$

$$Q[A] = \frac{1}{4\pi}\int d^2x\ \epsilon_{\mu\nu}F_{\mu\nu}(x)$$

Partition sum of the massless one-flavor model with staggered fermions and a topological term in conventional representation:

$$Z = \int \mathcal{D}[U] \int \mathcal{D}[\bar{\psi}, \psi] \; e^{-S_G[U] - i\theta Q[U] - S_\psi[U, \bar{\psi}, \psi]},$$

$$S_\psi[U, \bar{\psi}, \psi] = \frac{1}{2} \sum_{n,\nu} \left[\gamma_\nu(n) U_\nu(n) \bar{\psi}(n) \psi(n + \hat{\nu}) \right.$$
$$\left. - \gamma_\nu(n) U_\nu(n - \hat{\nu})^{-1} \bar{\psi}(n) \psi(n - \hat{\nu}) \right],$$

$$S_G[U] = \beta \sum_n \mathrm{Re}\left[1 - U_p(n)\right],$$

$$Q[U] = \frac{1}{i4\pi} \sum_n \left[U_p(n) - U_p(n)^{-1} \right],$$

$$\gamma_1(n) = 1, \quad \gamma_2(n) = (-1)^{n_1},$$

$$U_p(n) = U_1(n) \; U_2(n + \hat{1}) \; U_1(n + \hat{2})^{-1} \; U_2(n)^{-1}.$$

Expansion in β

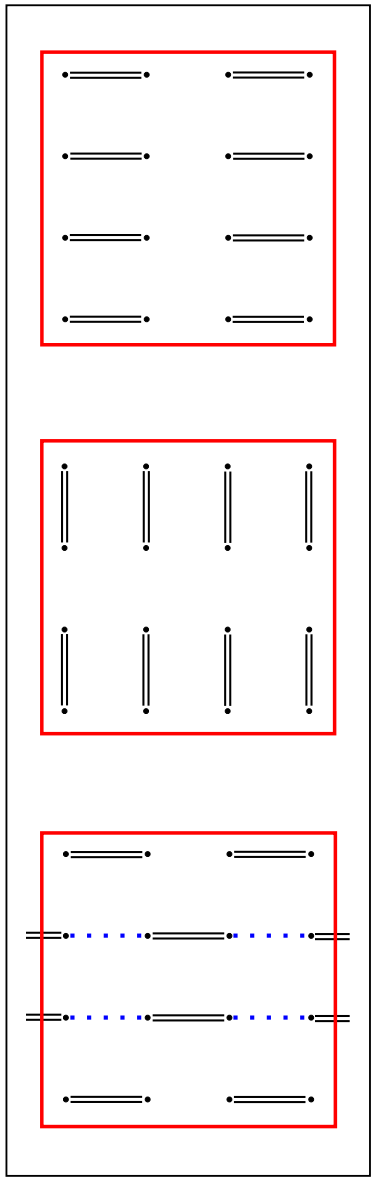
For the case of a vanishing vacuum angle the partition sum in dual representation becomes

$$Z = \sum_{\{l,d,p\}} \prod_{n} \mathtt{I}_{|p(n)|}(\beta)$$

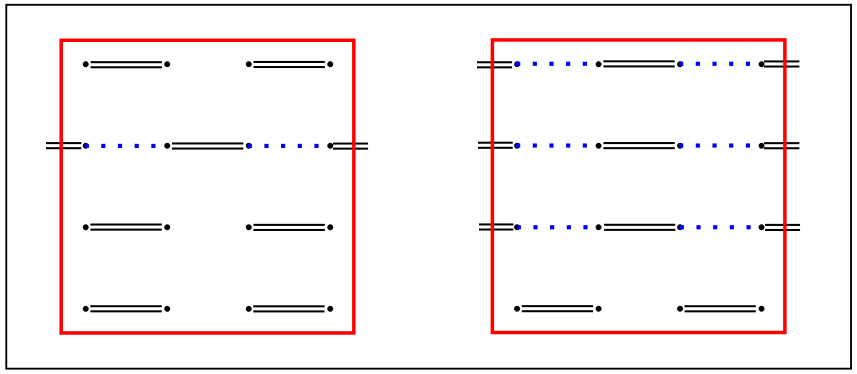
$\Rightarrow$ We expand in $\beta \mathrel{\hat{=}}$ expansion in the number of plaquettes

Worm updates

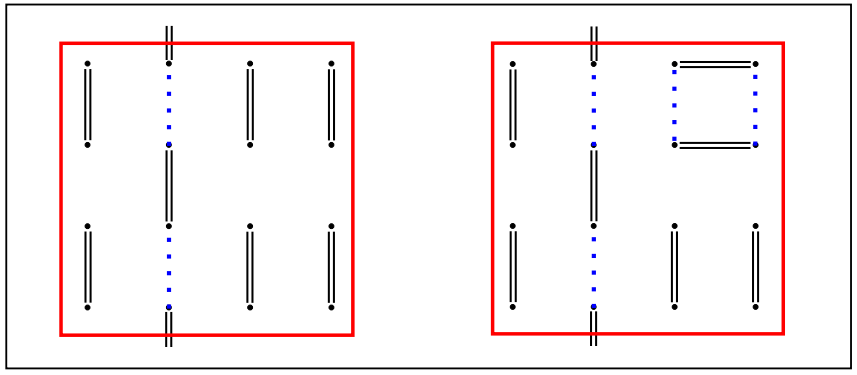
(e,e):



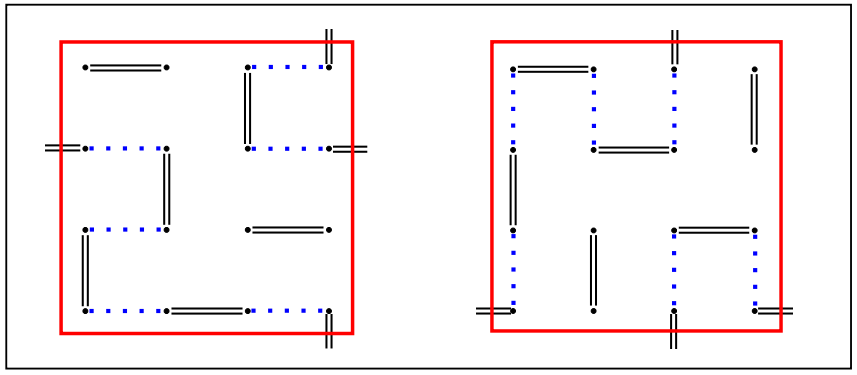
(o,e):



(e,o):



(o,o):



The worm updates

$\Longrightarrow$ We can test dimer (worm) algorithm by explicitely counting the number of configurations in the respective classes:

Class	*Number of configurations*	*Analytical ratio*	*Worm update*
(e,e)	136	0.5	0.49979 ± 0.000498781
(e,o)	64	0.235294	0.235595 ± 0.000426918
(o,e)	64	0.235294	0.235127 ± 0.00043796
(o,o)	8	0.0294118	0.029488 ± 0.000171351

(a) Comparison of the ratios on a 4×4 lattice.

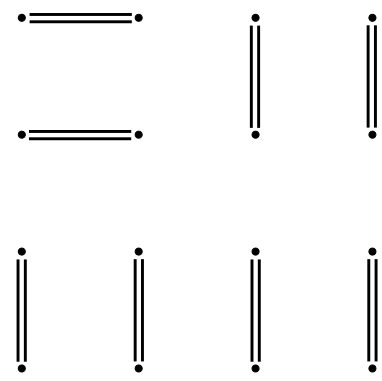
Class	*Number of configurations*	*Analytical ratio*	*Worm update*
(e,e)	155926656	0.5	0.50041 ± 0.000566953
(e,o)	67108864	0.215194	0.214472 ± 0.000467769
(o,e)	67108864	0.215194	0.215176 ± 0.000483039
(o,o)	21708928	0.0696126	0.069942 ± 0.000280876

(c) Comparison of the ratios on a 8×8 lattice.

Zero-th order in β:

272 dimer configurations

$\rightarrow$ 272 $\mathtt{I}_0(\beta)^{16}$



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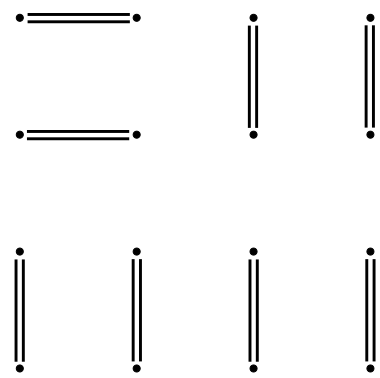

First order in β:

32 dimer configurations

16 placements of the plaquette

2 orientations of the loop

$\rightarrow$ 1024 $\mathtt{I}_0(\beta)^{15}\ \mathtt{I}_1(\beta)$