

Recent Progress in Applying Lattice QCD to Kaon Physics

Xu Feng



Plenary talk@Lattice 2017, Granada, 06/20/2017

“Standard” observables in Kaon physics

- f_{K^\pm}/f_{π^\pm} , $f_+(0)$, $\tau \rightarrow s$ inclusive decay and $|V_{us}|$
- B_K for SM and beyond

“Non-standard” observables in Kaon physics

- $K \rightarrow \pi\pi$ decays and direct CP violation
- ΔM_K and ϵ_K
- Rare Kaon decays

Thank you for sending me materials and communications!

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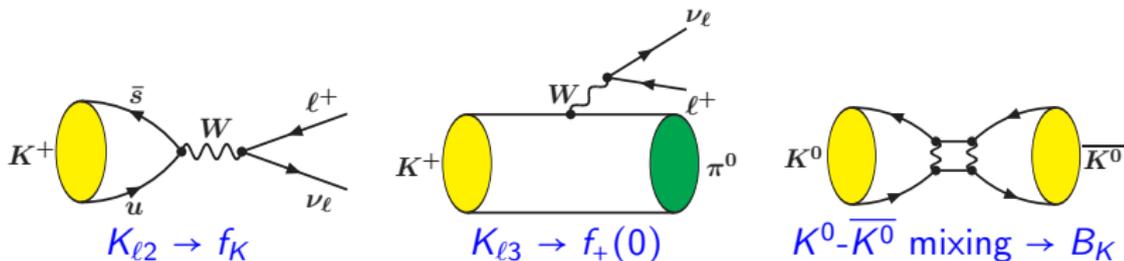
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Evaluate the hadronic matrix elements in Kaon physics

- Lattice QCD is powerful for “standard” hadronic matrix elements with



- ▶ single local operator insertion
- ▶ only single stable hadron or vacuum in the initial/final state
- ▶ spatial momenta carried by particles need to be small compared to $1/a$ (not a problem for Kaon physics, but essential for B decays)

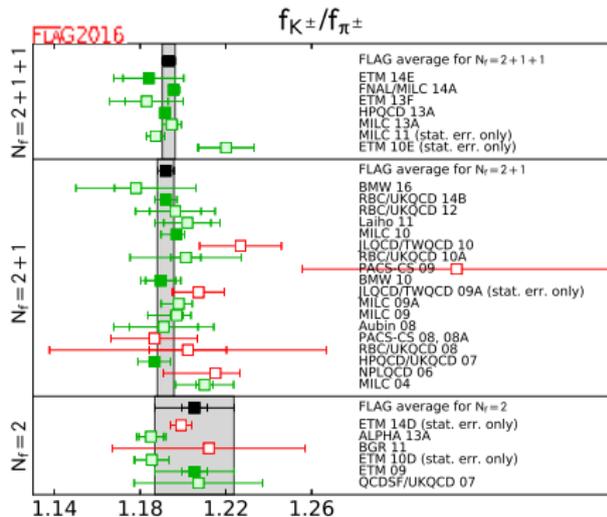
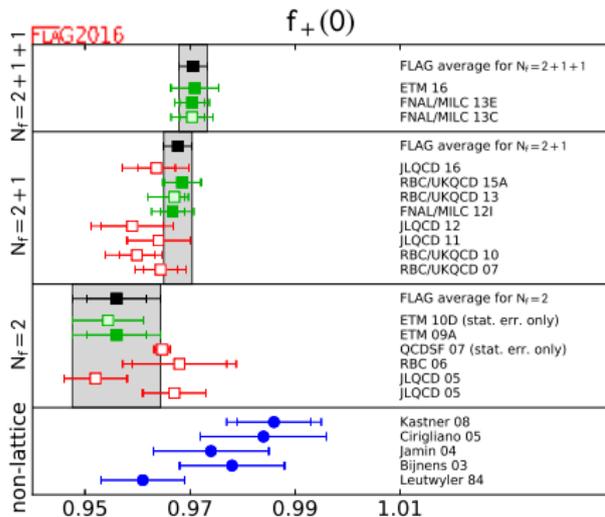
f_{K^\pm}/f_{π^\pm} , $f_+(0)$, $\tau \rightarrow s$ inclusive decay and $|V_{us}|$

“standard” quantities in Kaon physics: f_{K^\pm}/f_{π^\pm} and $f_+(0)$

Flavor Lattice Averaging Group (FLAG) average, updated in Nov. 2016

$$f_+^{K\pi}(0) = 0.9706(27) \Rightarrow 0.28\% \text{ error}$$

$$f_{K^\pm}/f_{\pi^\pm} = 1.1933(29) \Rightarrow 0.25\% \text{ error}$$

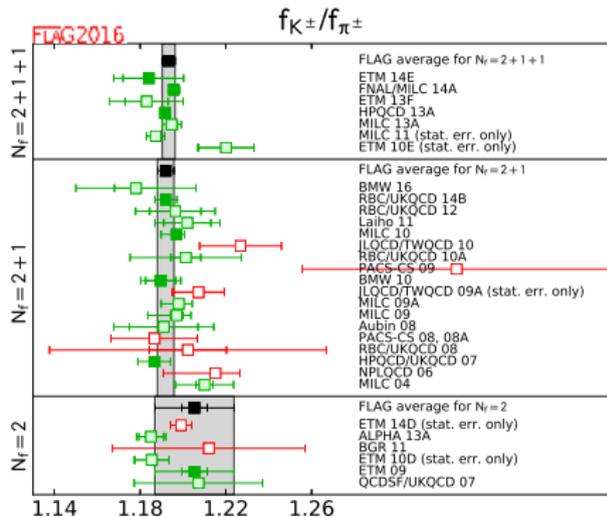
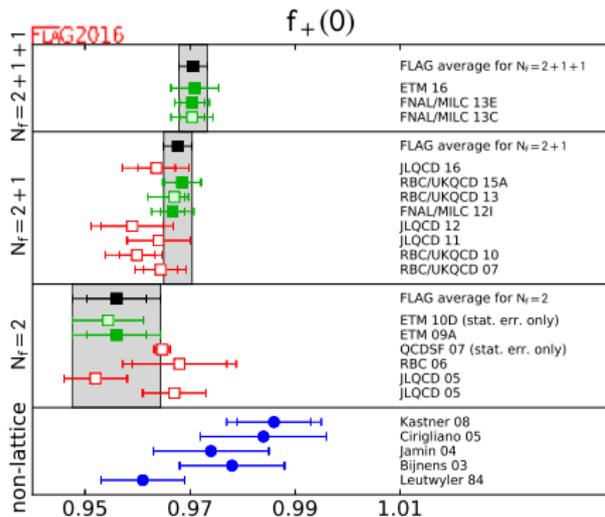


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Experimental information [arXiv:1411.5252, 1509.02220]

$$K_{\ell 3} \Rightarrow |V_{us}| f_+(0) = 0.2165(4) \Rightarrow |V_{us}| = 0.2231(7)$$

$$K_{\mu 2}/\pi_{\mu 2} \Rightarrow \left| \frac{V_{us}}{V_{ud}} \right| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2760(4) \Rightarrow \left| \frac{V_{us}}{V_{ud}} \right| = 0.2313(7)$$

Test the CKM unitarity

[S. Aoki et. al., FLAG report updated in Nov. 2016]

Most stringent test of CKM unitarity is given by the first row condition

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

- Use $|V_{us}|$ for $K_{\ell 3}$ + $|V_{us}/V_{ud}|$ for $K_{\ell 2}/\pi_{\ell 2}$ as input

$$|V_u|^2 = 0.9798(82) \quad \Rightarrow \quad 2.5\sigma \text{ deviation from } 1$$

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Most precise value of $|V_{ud}| = 0.97417(21)$ is from superallowed nuclear β decay

- Use $|V_{us}|$ for $K_{\ell 3}$ + $|V_{ud}|$ for β decay

$$|V_u|^2 = 0.9988(5) \quad \Rightarrow \quad \text{sharpen the test, still } 2.4\sigma \text{ deviation}$$

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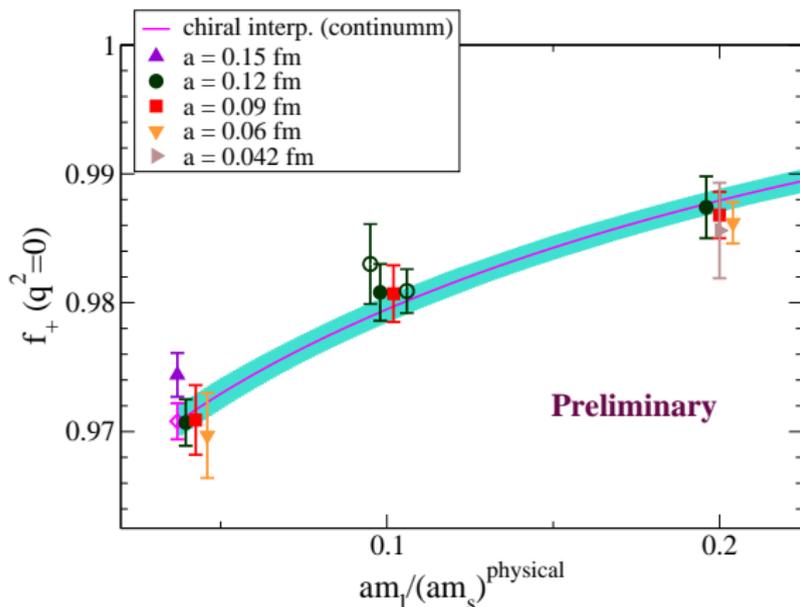
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Interesting to reduce the uncertainty from $f_+(0)$ and explore the $> 2\sigma$ deviation

$f_+(0)$: recent update from Fermilab Lattice-MILC Collaboration

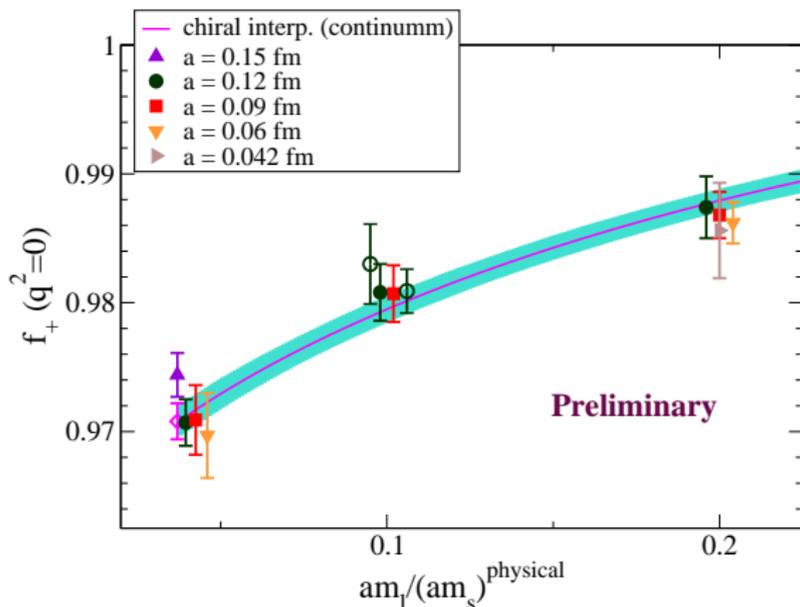
Use HISQ fermions on $N_f = 2 + 1 + 1$ MILC configurations [PoS LATTICE2016 286]



Plot, courtesy of E. Gámiz

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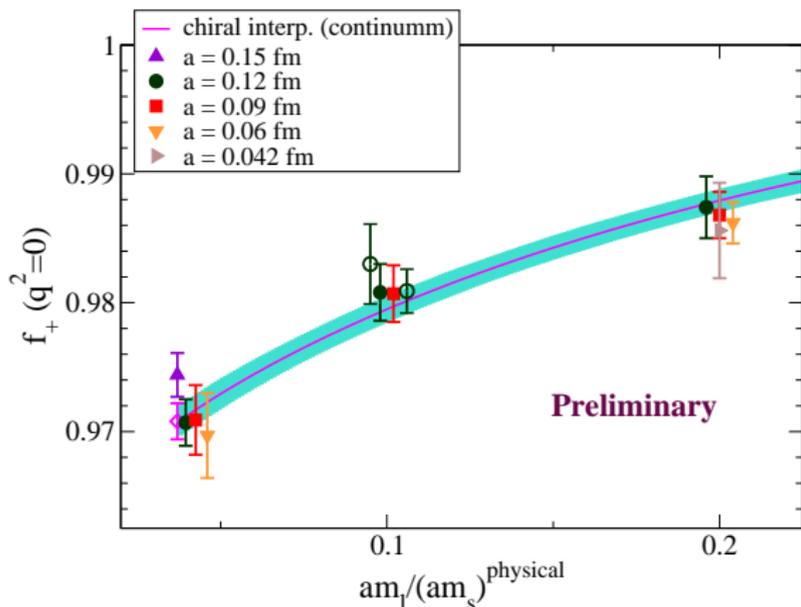
- 4 ensembles at physical m_π
- 2 ultra-fine lattice at $a = 0.06, 0.042$ fm



Stat. err reduces to
 $\sim 0.14\%$

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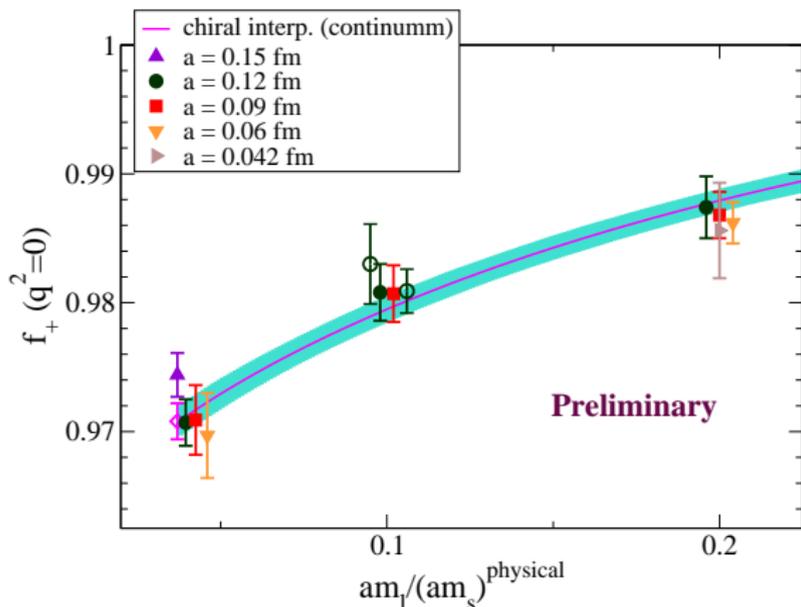


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[C. Bernard et. al, JHEP 1703 (2017) 120]

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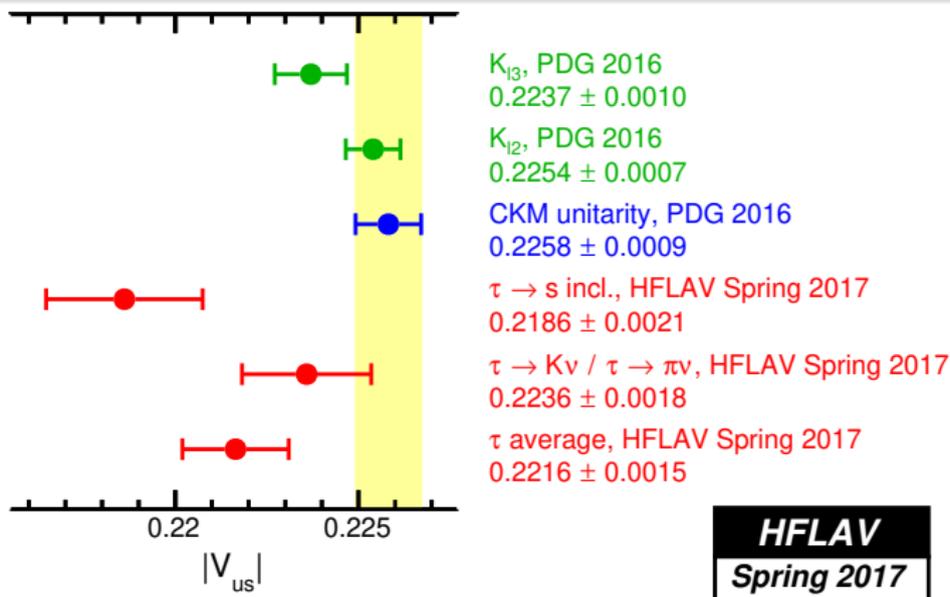


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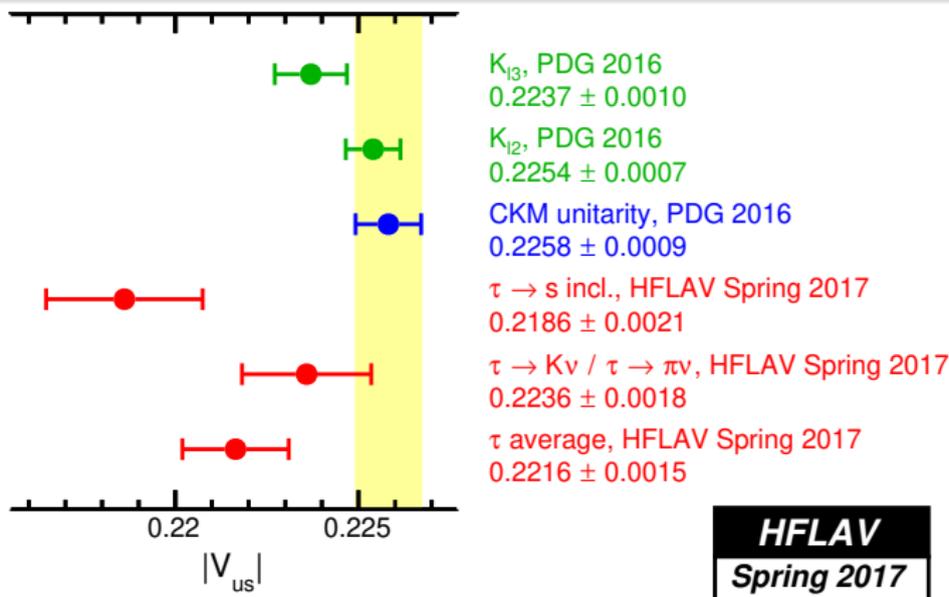
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- Chiral, continuum extrapolation + discretization uncertainty + FV corrections + NNLO isospin corrections + taste-violating effects + ...

Expect to have a final error of $\sim 0.2\%$



3.2 σ deviation between $\tau \rightarrow s$ inclusive decay and CKM unitarity



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$$R = \frac{\Gamma(\tau \rightarrow \text{strange-hadrons } \nu_\tau)}{\Gamma(\tau \rightarrow e \bar{\nu}_e \nu_\tau)}$$

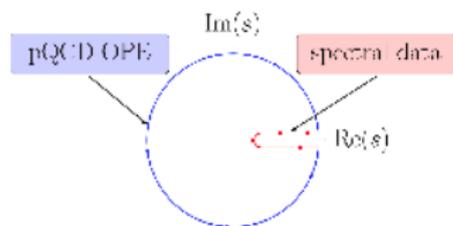
Optical theorem: Hadronic spectral func. of inclusive decay \Leftrightarrow imag. of HVP

$$\frac{dR}{ds} = \frac{12\pi |V_{us}|^2 S_{EW}}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im } \Pi^{(1)}(s) + \text{Im } \Pi^{(0)}(s) \right]$$

$\text{Im} \Pi^{(J)}(s)$ is generically non-perturbative at small s

- Conventional approach: use dispersion relation

[E. Braaten et. al., NPB373 (1992) 581; E. Gámiz et. al., PRL94 (2005) 011803]



$$\int_0^{s_0} ds W(s) \text{Im} \Pi(s) = \frac{i}{2} \oint_{|s|=s_0} ds W(s) \Pi(s)$$

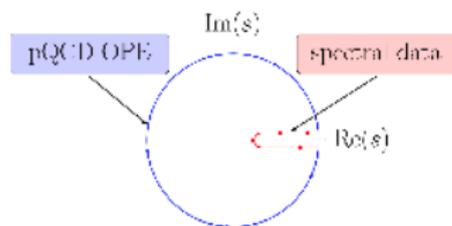
LHS given by $\frac{dR}{ds}$; RHS given by pQCD+OPE

Theoretical approaches to treat with inclusive τ decay

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- Study dependence on s_0 and $W(s)$ or use lattice data $\xrightarrow{\text{fit}}$ high-dim. OPE
[R. Hudspith et. al arXiv:1702.01767]

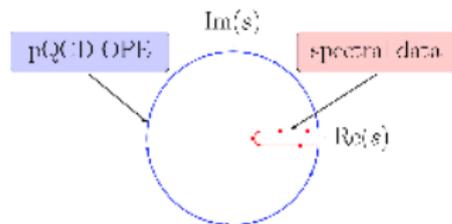
$$|V_{us}| = \begin{cases} 0.2229(22) & \text{using BaBar } \tau \rightarrow K\pi^0\nu_\tau, 3.2\sigma \rightarrow 1.2\sigma \\ 0.2204(23) & \text{using HFAG } \tau \rightarrow K\pi^0\nu_\tau, 3.2\sigma \rightarrow 2.2\sigma \end{cases}$$

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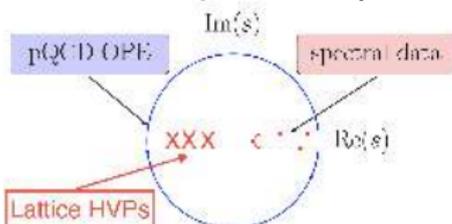
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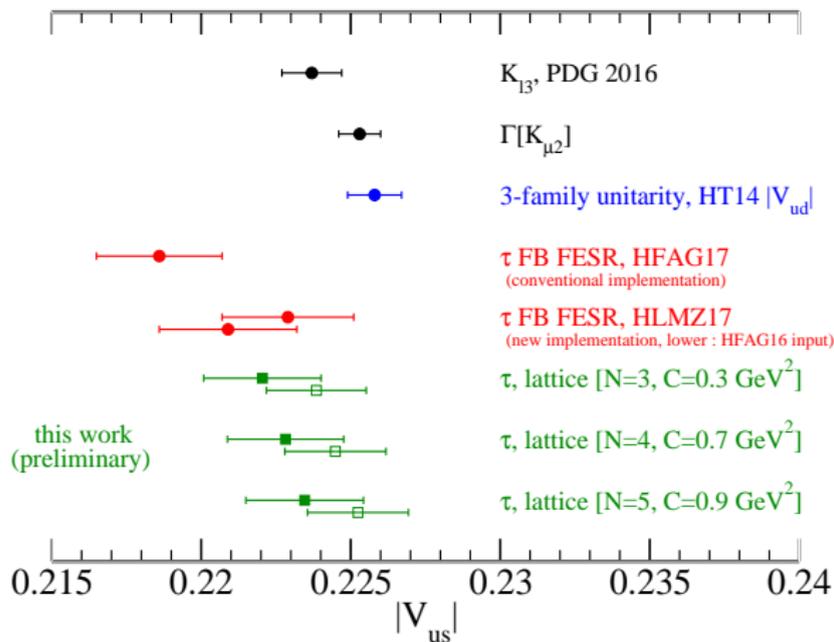
- Lattice QCD + dispersion relation [H. Ohki, Friday 17:50@Seminarios 6+7]



$$\text{use } W(s) = \prod_k \frac{1}{s + Q_k^2} \text{ and let } |s| = s_0 \rightarrow \infty$$

Residue at $s = -Q_k^2$ is given by Lattice HVPs_{10/48}

$|V_{us}|$ determined from inclusive τ decay + lattice HVPs



Plot, courtesy of T. Izubuchi & H. Ohki

[H. Ohki's talk, Friday 17:50]

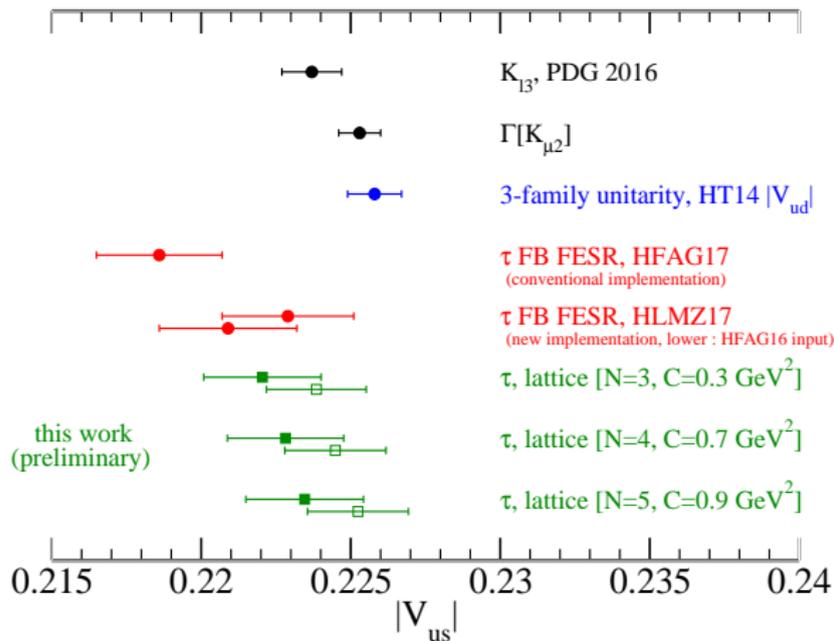
$N_f = 2 + 1$ Möbius DWF

- physical m_π
- $a^{-1} = 1.73, 2.36$ GeV
- $V = 5 \text{ fm}^3$
- Measurements:
88 × 48, 80 × 32

K -pole data:

$\tau \rightarrow K \nu_\tau$ (filled square)
 $K_{\mu 2}$ (open square)

$|V_{us}|$ determined from inclusive τ decay + lattice HVPs



Plot, courtesy of T. Izubuchi & H. Ohki

Choice of Q_k^2 : separated by a spacing $\Delta = \frac{0.2}{N-1}$ GeV² and $C = \frac{Q_{min}^2 + Q_{max}^2}{2}$

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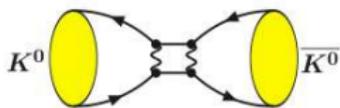
$\tau \rightarrow K \nu_\tau$ (filled square)
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- Not too large to suppress contribution from pQCD+OPE at $s > m_\tau^2$ and noisy experimental data at larger $s < m_\tau^2$
- Not too small to avoid large statistical error from lattice HVPs

B_K for SM and beyond

“standard” quantities in Kaon physics: B_K

Short distance dominance \Rightarrow OPE \Rightarrow Wilson coeff. $C(\mu) \times$ operator $Q^{\Delta S=2}(\mu)$



$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{G_F^2 M_W^2}{16\pi^2} C(\mu) Q^{\Delta S=2}(\mu)$$

- Serve as a dominant contribution to the indirect CP violation ϵ_K

$$\epsilon_K = \exp(i\phi_\epsilon) \sin(\phi_\epsilon) \left[\frac{\text{Im}[\langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{\Delta S=2} | K^0 \rangle]}{\Delta M_K} + \frac{\text{Im}[M_{00}^{\text{LD}}]}{\Delta M_K} + \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right]$$

- Within Standard Model, only single operator with $V - A$ structure

$$Q^{\Delta S=2} = [\bar{s}_a \gamma_\mu (1 - \gamma_5) d_a][\bar{s}_b \gamma_\mu (1 - \gamma_5) d_b]$$

- Beyond SM, 4 other operators possible

$$Q_2^{\Delta S=2} = [\bar{s}_a (1 - \gamma_5) d_a][\bar{s}_b (1 - \gamma_5) d_b]$$

$$Q_3^{\Delta S=2} = [\bar{s}_a (1 - \gamma_5) d_b][\bar{s}_b (1 - \gamma_5) d_a]$$

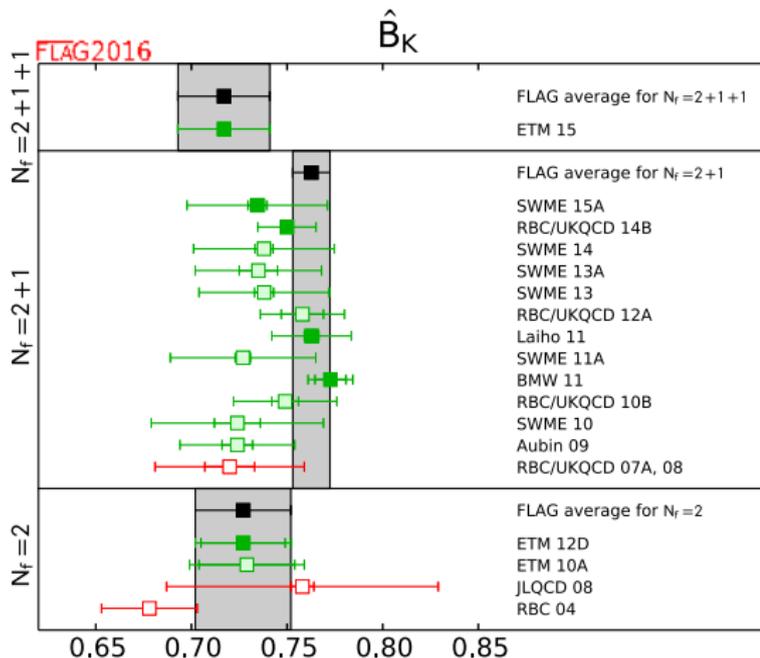
$$Q_4^{\Delta S=2} = [\bar{s}_a (1 - \gamma_5) d_a][\bar{s}_b (1 + \gamma_5) d_b]$$

$$Q_5^{\Delta S=2} = [\bar{s}_a (1 - \gamma_5) d_b][\bar{s}_b (1 + \gamma_5) d_a]$$

FLAG average for Standard Model B_K

- B_K in NDR- $\overline{\text{MS}}$ scheme: $B_K(\mu) = \frac{\langle K^0 | Q^{\Delta S=2}(\mu) | K^0 \rangle}{\frac{8}{3} f_K^2 m_K^2}$
- Renormalization group independent B parameter \hat{B}_K :

$$\hat{B}_K = \left(\frac{\bar{g}(\mu)^2}{4\pi} \right)^{-\gamma_0/(2\beta_0)} \exp \left\{ \int_0^{\bar{g}(\mu)} dg \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g} \right) \right\} B_K(\mu)$$

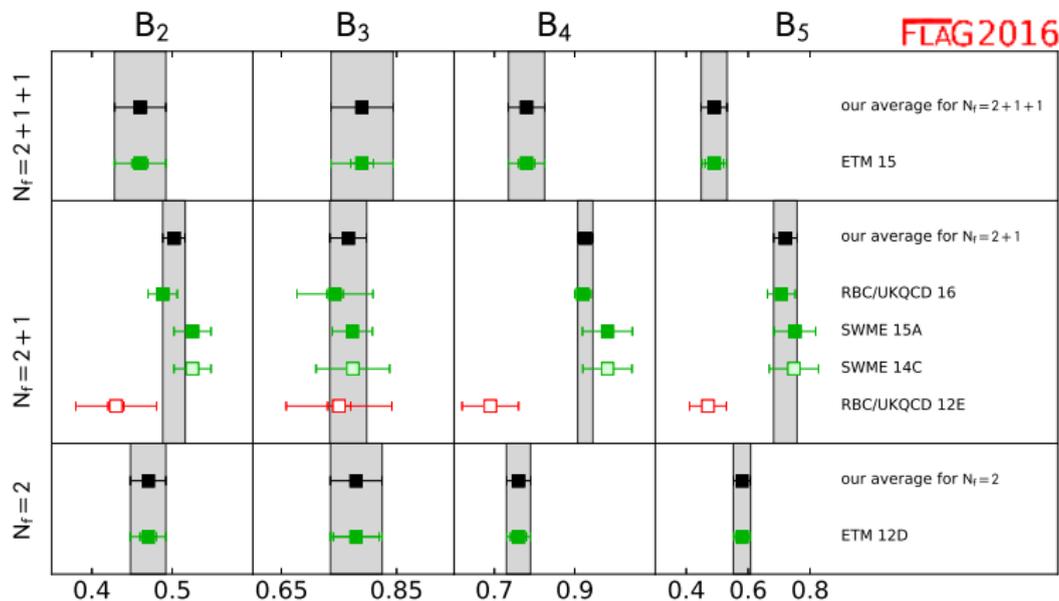


- $N_f = 2 + 1 + 1$:
 $\hat{B}_K = 0.717(24)$
- $N_f = 2 + 1$:
 $\hat{B}_K = 0.763(10)$
- $N_f = 2$:
 $\hat{B}_K = 0.727(25)$

FLAG average for BSM B_i , updated in Dec. 2016

$$B_i(\mu) = \frac{\langle \bar{K}^0 | Q_i(\mu) | K^0 \rangle}{N_i \langle \bar{K}^0 | \bar{s} \gamma_5 d | 0 \rangle \langle 0 | \bar{s} \gamma_5 d | K^0 \rangle}, \quad \{N_2, \dots, N_5\} = \{-5/3, 1/3, 2, 2/3\}$$

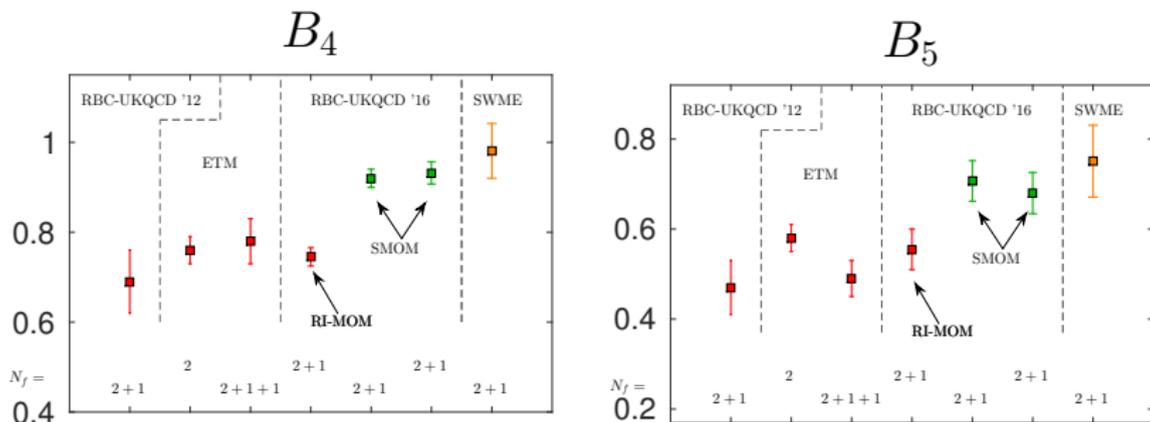
$B_i(\mu)$ at $\mu_{\overline{\text{MS}}} = 3 \text{ GeV}$



For $N_f = 2 + 1$, $B_2 = 0.502(14)$, $B_3 = 0.766(32)$, $B_4 = 0.926(19)$, $B_5 = 0.720(38)$

Resolution of the discrepancy for B_4 , B_5

$N_f = 2+1$ DWF, $a = 0.08, 0.11$ fm, $m_\pi = 300$ MeV [RBC-UKQCD, JHEP11(2016)001]



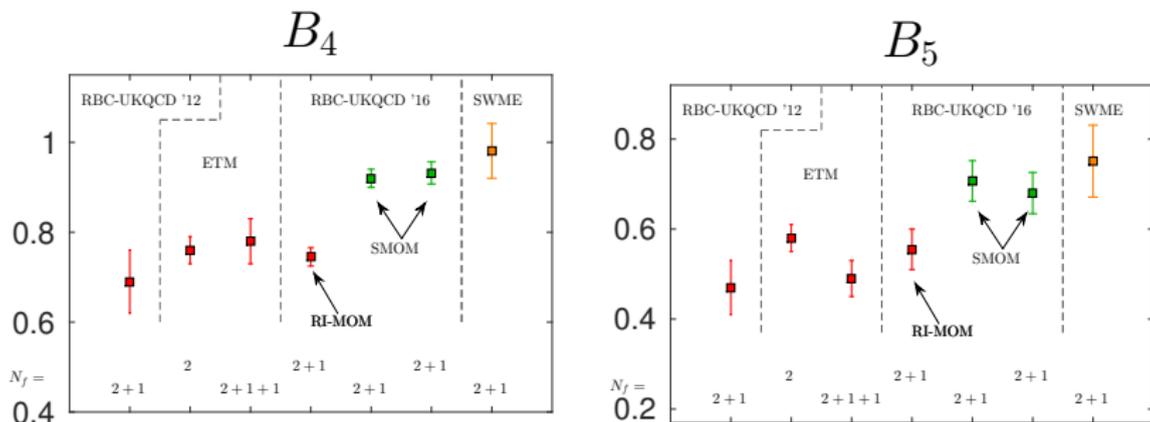
Plot, courtesy of N. Garron

- Use both RI/MOM and SMOM \Rightarrow the former is significantly smaller
- Use two RI/SMOM schemes, (\not{D}, \not{D}) and (γ_μ, γ_μ) \Rightarrow consistent results
- RI/(S)MOM result compatible with previous RI/(S)MOM calculation

Study suggests RI/MOM suffers from large IR artifacts \Rightarrow discrepancy

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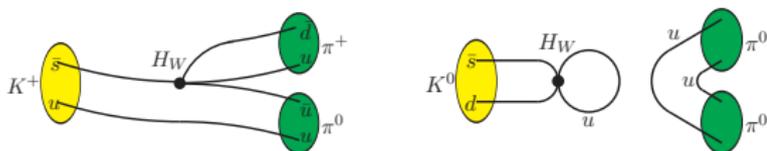
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On-going project: [J. Kettle's talk, Wednesday 11:30@Seminarios 6+7]

- 64^3 and 48^3 ensembles with physical m_π and finer lattice spacing

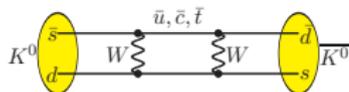
Go beyond “standard” quantities in lattice Kaon physics

- $K \rightarrow \pi\pi$ decays and direct CP violation

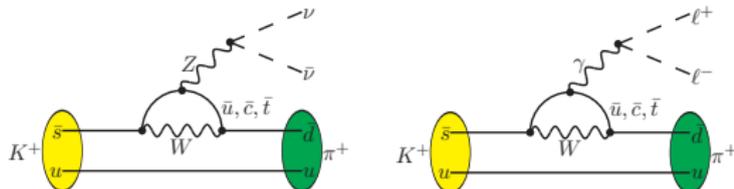


Final state involves $\pi\pi$ (multi-hadron system)

- Long-distance contributions to flavor changing processes
 - ΔM_K and ϵ_K



- Rare kaon decays: $K \rightarrow \pi\nu\bar{\nu}$ and $K \rightarrow \pi\ell^+\ell^-$



Hadronic matrix element for bilocal operators

$$\int d^4x \langle f | T[Q_1(x)Q_2(0)] | i \rangle$$

$K \rightarrow \pi\pi$ decays and direct CP violation

CP violation is first observed in neutral Kaon decays

- *CP* eigenstates
 - Under *CP* transform: $CP|K^0\rangle = -|\overline{K}^0\rangle$
 - Define *CP* eigenstates: $K_{\pm}^0 = (K^0 \mp \overline{K}^0)/\sqrt{2}$
- Weak eigenstates
 - $K_S \rightarrow 2\pi$ (*CP* = +)
 - $K_L \rightarrow 3\pi$ (*CP* = -)
- Neglecting *CP* violation, we have $K_S = K_+^0$ and $K_L = K_-^0$

1964, BNL discovered $K_L \rightarrow 2\pi \Rightarrow$ *CP* violation \Rightarrow Nobel prize (1980)

Direct and indirect CP violation

- $K_{L/S}$ are not CP eigenstates

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1 + \bar{\epsilon}^2}} (|K_{\mp}^0\rangle + \bar{\epsilon}|K_{\pm}^0\rangle)$$

- $K_L \rightarrow 2\pi$ ($CP = +$)
 - $K_{+}^0 \rightarrow 2\pi$ (indirect CP violation, ϵ or ϵ_K)
 - $K_{-}^0 \rightarrow 2\pi$ (direct CP violation, ϵ')

Direct and indirect CP violation

- $K_{L/S}$ are not CP eigenstates

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1+\bar{\epsilon}^2}} (|K_{\mp}^0\rangle + \bar{\epsilon}|K_{\pm}^0\rangle)$$

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 - $K_{+}^0 \rightarrow 2\pi$ (indirect CP violation, ϵ or ϵ_K)
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- Experimental measurement

$$\frac{A(K_L \rightarrow \pi^+\pi^-)}{A(K_S \rightarrow \pi^+\pi^-)} \equiv \eta_{+-} \equiv \epsilon + \epsilon'$$

$$\frac{A(K_L \rightarrow \pi^0\pi^0)}{A(K_S \rightarrow \pi^0\pi^0)} \equiv \eta_{00} \equiv \epsilon - 2\epsilon'$$

- Using $|\eta_{+-}|$ and $|\eta_{00}|$ as input, PDG quotes

$$|\epsilon| \approx \frac{1}{3} (2|\eta_{+-}| + |\eta_{00}|) = 2.228(11) \times 10^{-3}, \quad \text{Re}[\epsilon'/\epsilon] \approx \frac{1}{3} \left(1 - \frac{|\eta_{00}|}{|\eta_{+-}|}\right) = 1.66(23) \times 10^{-3}$$

ϵ' is 1000 times smaller than the indirect CP violation ϵ

Thus direct CP violation ϵ' is very sensitive to New Physics

- Theoretically, Kaon decays into the isospin $I = 2$ and 0 $\pi\pi$ states

$$\Delta I = 3/2 \text{ transition: } \langle \pi\pi(I=2) | H_W | K^0 \rangle = A_2 e^{i\delta_2}$$

$$\Delta I = 1/2 \text{ transition: } \langle \pi\pi(I=0) | H_W | K^0 \rangle = A_0 e^{i\delta_0}$$

- If CP symmetry were protected $\Rightarrow A_2$ and A_0 are real amplitudes
- ϵ and ϵ' depend on the $K \rightarrow \pi\pi(I)$ amplitudes A_I

$$\epsilon = \bar{\epsilon} + i \left(\frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right)$$
$$\epsilon' = \frac{ie^{i(\delta_2 - \delta_0)}}{\sqrt{2}} \frac{\text{Re}[A_2]}{\text{Re}[A_0]} \left(\frac{\text{Im}[A_2]}{\text{Re}[A_2]} - \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right)$$

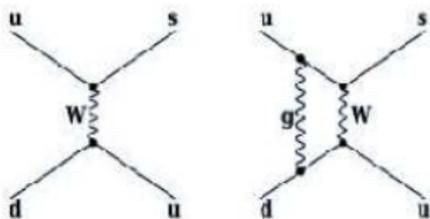
The target for lattice QCD is to calculate both amplitude A_2 and A_0

Weak Hamiltonian for $K \rightarrow \pi\pi$

Weak Hamiltonian is given by local four-quark operator

$$\mathcal{H}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ \sum_{i=1}^{10} [z_i(\mu) + \tau y_i(\mu)] Q_i \right\}, \quad \tau = -\frac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*}$$

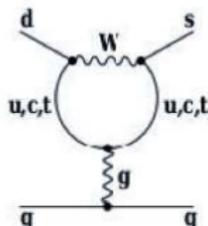
- $\tau = -\frac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*} = 1.543 + 0.635i$
- $z_i(\mu)$ and $y_i(\mu)$ are perturbative Wilson coefficients
- Q_i are local four-quark operator



Current-current operator

Q_1, Q_2

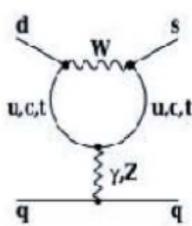
dominate $\text{Re}[A_0], \text{Re}[A_2]$



QCD penguin

$Q_3 - Q_6$

Q_6 dominate $\text{Im}[A_0]$



Electro-weak penguin

$Q_7 - Q_{10}$

Q_7, Q_8 dominate $\text{Im}[A_2]$

Recent results for $K \rightarrow \pi\pi (I = 2)$

Results for A_2 [RBC-UKQCD, PRD91 (2015) 074502]

- Use two ensembles (both at $m_\pi = 135$ MeV) for continuum extrapolation

$$48^3 \times 96, \quad a = 0.11 \text{ fm}, \quad L = 5.4 \text{ fm}, \quad N_{\text{conf}} = 76$$

$$64^3 \times 128, \quad a = 0.084 \text{ fm}, \quad L = 5.4 \text{ fm}, \quad N_{\text{conf}} = 40$$

- After continuum extrapolation:

$$\text{Re}[A_2] = 1.50(4)_{\text{stat}}(14)_{\text{syst}} \times 10^{-8} \text{ GeV}$$

$$\text{Im}[A_2] = -6.99(20)_{\text{stat}}(84)_{\text{syst}} \times 10^{-13} \text{ GeV}$$

- Experimental measurement:

$$\text{Re}[A_2] = 1.479(3) \times 10^{-8} \text{ GeV}$$

$\text{Im}[A_2]$ is unknown

- Scattering phase at $E_{\pi\pi} = M_K$

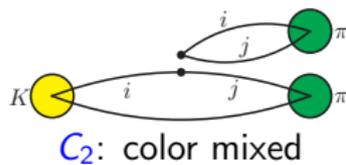
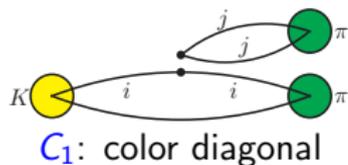
$$\delta_2 = -11.6(2.5)(1.2)^\circ$$

consistent with phenomenological analysis [Schenk, NPB363 (1991) 97]

Resolve the puzzle of $\Delta I = 1/2$ rule

$\Delta I = 1/2$ rule: $A_0 = 22.5 \times A_2 \Rightarrow a > 50$ year puzzle

- Wilson coefficient only contributes a factor of ~ 2
- $\text{Re}[A_2]$ and $\text{Re}[A_0]$ are dominated by diagrams C_1 and C_2



Color counting in LO PT $\Rightarrow C_2 = C_1/3$; Non-PT effects $\Rightarrow C_2 \approx -0.7C_1$

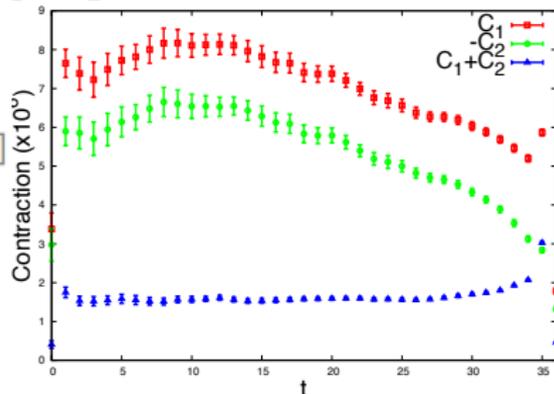
- $\text{Re}[A_2] \propto C_1 + C_2$, while $\text{Re}[A_0] \propto 2C_1 - C_2 \Rightarrow$ another factor of ~ 10

- ▶ Such cancellation is first observed in an earlier calculation

[RBC-UKQCD, PRL110 (2013) 152001]

- ▶ It is further confirmed in the latest calculation of A_2

[RBC-UKQCD, PRD91 (2015) 074502]

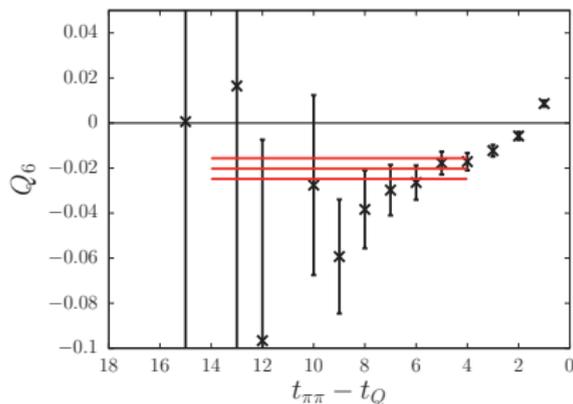
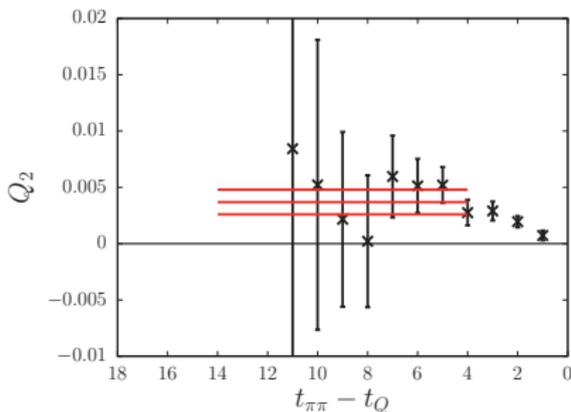


Puzzle of $\Delta I = 1/2$ rule is resolved from first principles

Recent results for $K \rightarrow \pi\pi (I = 0)$

Results for A_0 [RBC-UKQCD, PRL115 (2015) 212001]

- Use a $32^3 \times 64$ ensemble, $N_{\text{conf}} = 216$, $a = 0.14$ fm, $L = 4.53$ fm
 $M_\pi = 143.1(2.0)$ MeV, $M_K = 490(2.2)$ MeV, $E_{\pi\pi} = 498(11)$ MeV
- G-boundary condition is used: non-trivial to tune the volume $\Rightarrow M_K = E_{\pi\pi}$
- The largest contributions to $\text{Re}[A_0]$ and $\text{Im}[A_0]$ come from Q_2 (current-current) and Q_6 (QCD penguin) operator



- Scattering phase at $E_{\pi\pi} = M_K$: $\delta_0 = 23.8(4.9)(1.2)^\circ$
 - ▶ somewhat smaller than phenomenological expectation $\delta_0 = 38.0(1.3)^\circ$

- Determine the $K \rightarrow \pi\pi(I = 0)$ amplitude A_0

- Lattice results

$$\text{Re}[A_0] = 4.66(1.00)_{\text{stat}}(1.26)_{\text{syst}} \times 10^{-7} \text{ GeV}$$

$$\text{Im}[A_0] = -1.90(1.23)_{\text{stat}}(1.08)_{\text{syst}} \times 10^{-11} \text{ GeV}$$

- Experimental measurement

$$\text{Re}[A_0] = 3.3201(18) \times 10^{-7} \text{ GeV}$$

$\text{Im}[A_0]$ is unknown

- Determine the direct CP violation $\text{Re}[\epsilon'/\epsilon]$

$$\text{Re}[\epsilon'/\epsilon] = 0.14(52)_{\text{stat}}(46)_{\text{syst}} \times 10^{-3} \quad \text{Lattice}$$

$$\text{Re}[\epsilon'/\epsilon] = 1.66(23) \times 10^{-3} \quad \text{Experiment}$$

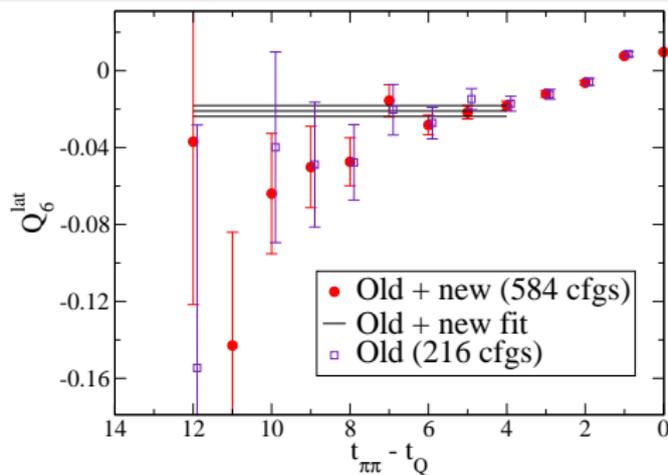
2.1 σ deviation \Rightarrow require more accurate lattice results

Improve both statistics and systematics

Efforts for statistics improvement

[C. Kelly, Wed. 13:10@Seminarios 6+7]

- Statistics increased: 216 \rightarrow 584
- Aim to reduce stat. error by a factor of 2 within the next year



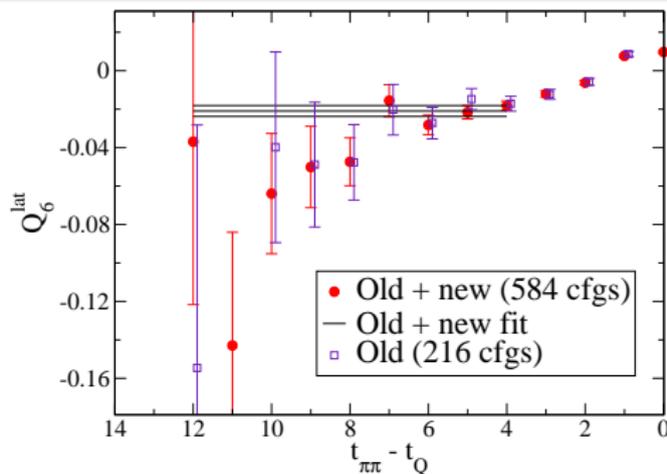
Plot, courtesy of C. Kelly

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Plot, courtesy of C. Kelly

Efforts for systematic improvement

- Add the σ field to study $\sigma \rightarrow \pi\pi$ in the $l = 0$ channel [T. Wang @Columbia]
- Include EM in $K \rightarrow \pi\pi$ [N. Christ's talk, Wed. 12:10@Seminarios 6+7]
 - $\Delta l = 1/2$ rule may make the $O(\alpha_e)$ EM effect on A_2 20 times larger
- Calculate Wilson coefficients non-perturbatively [M. Bruno's talk, Friday 17:30@Seminarios 6+7]
 - currently use unphysically light W -boson around 2 GeV

Long-distance contributions to flavor changing processes

ΔM_K and ϵ_K

Weak interaction causes the mixing between $K^0-\bar{K}^0$



- Time evolution of the $K^0-\bar{K}^0$ mixing system

$$i \frac{d}{dt} \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = \left[\begin{pmatrix} M_{00} & M_{0\bar{0}} \\ M_{\bar{0}0} & M_{\bar{0}\bar{0}} \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{00} & \Gamma_{0\bar{0}} \\ \Gamma_{\bar{0}0} & \Gamma_{\bar{0}\bar{0}} \end{pmatrix} \right] \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix}$$

- 2×2 M and Γ matrices are calculated to 2nd-order in H_W

$$M_{ij} = M_K \delta_{ij} + \langle i | H_W | j \rangle + \mathcal{P} \int_{\alpha} \frac{\langle i | H_W | \alpha \rangle \langle \alpha | H_W | j \rangle}{M_K - E_{\alpha}}$$

$$\Gamma_{ij} = 2\pi \int_{\alpha} \langle i | H_W | \alpha \rangle \langle \alpha | H_W | j \rangle \delta(E_{\alpha} - M_K)$$

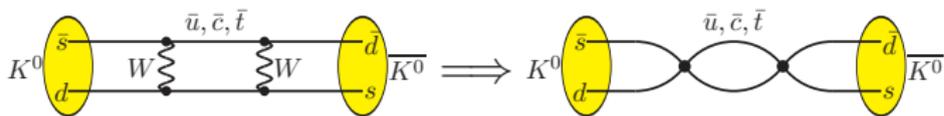
- ΔM_K and ϵ_K are related to $\text{Re}[M_{0\bar{0}}]$ and $\text{Im}[M_{0\bar{0}}]$, respectively

$$\Delta M_K = M_{K_L} - M_{K_S} = 2\text{Re}[M_{0\bar{0}}]$$

$$\epsilon_K = e^{i\phi_{\epsilon}} \sin(\phi_{\epsilon}) \left[\frac{\text{Im}[M_{0\bar{0}}]}{\Delta M_K} + \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right], \quad \phi_{\epsilon} = \arctan \frac{-2\Delta M_K}{\Delta \Gamma_K} \approx 45^{\circ}$$

Long-distance contribution to ΔM_K and ϵ_K

- $\Delta M_K \Rightarrow \text{Re}[M_{00}] \Rightarrow CP$ conserving part of $K^0-\bar{K}^0$ mixing



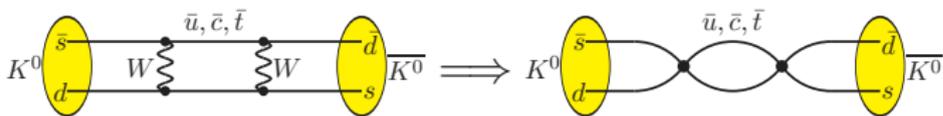
Dominant contribution from charm-charm loop:

$$\lambda_c^2 \frac{m_c^2}{M_W^2} \gg \lambda_t^2 \frac{m_t^2}{M_W^2}, \quad \text{where } \lambda_q = V_{qd} V_{qs}^*, \text{ for } q = u, c, t$$

\Rightarrow historically led to the predication of the mass scale of charm quark

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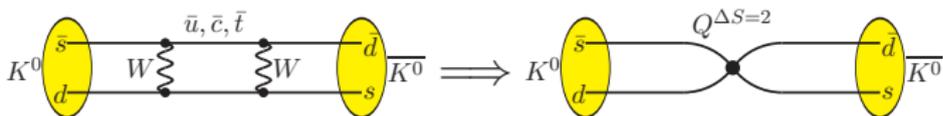


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- $\epsilon_K \Rightarrow \text{Im}[M_{00}] \Rightarrow CP$ violating part of $K^0-\bar{K}^0$ mixing



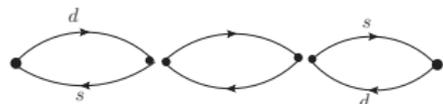
Top-top, top-charm and charm-charm loops compete in size

\Rightarrow important top-top loop, thus ϵ_K is sensitive to λ_t (determined from V_{cb})

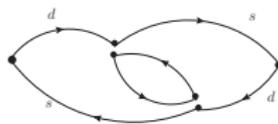
Status for ΔM_K

Use $32^3 \times 64$ ensemble: $a^{-1} = 1.38$ GeV, $m_\pi = 170$ MeV, $m_c = 750$ MeV
[Preliminary results from Z. Bai, for RBC-UKQCD]

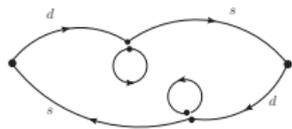
- Results based on 120 configurations



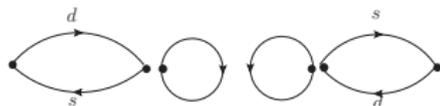
Type 1



Type 2



Type 3



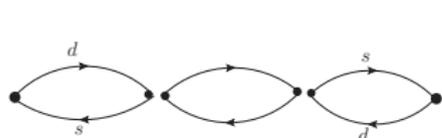
Type 4

	ΔM_K [10^{-12} MeV]
Type 1-4	3.85(46)
Type 1-2	4.49(16)
η	0
π	0.39(15)
$\pi\pi_{I=0}$	-0.06(2)
$\pi\pi_{I=2}$	$-6.25(11) \times 10^{-4}$
FV	0.024(11)
Exp	3.483(6)

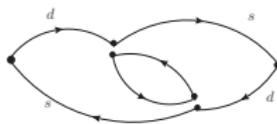
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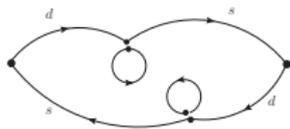
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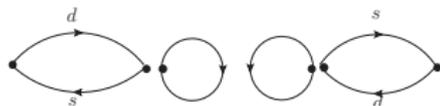
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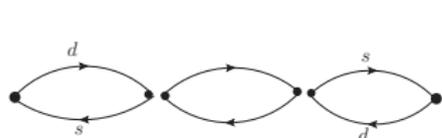
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- Double GIM cancellation \Rightarrow No SD divergence, but significantly rely on m_c

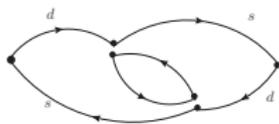
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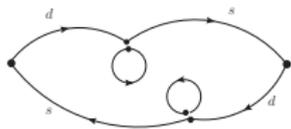
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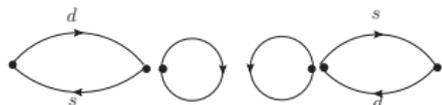
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New project: $64^3 \times 128$, $a^{-1} = 2.36$ GeV, $m_c = 1.2$ GeV, $m_\pi = 136$ MeV

- Based on 59 configurations: $\Delta M_K = 5.5(1.7) \times 10^{-12}$ MeV

[C. Sachrajda's talk, Wednesday 12:30@Seminarios 6+7]

SM predication vs Exp measurement [summarized by W. Lee @ Kaon 2016]

$$|\epsilon_K^{\text{SM}}| = 1.69(17) \times 10^{-3} \quad \text{using Exclusive } V_{cb} \text{ (Lattice QCD)}$$

$$|\epsilon_K^{\text{SM}}| = 2.10(17) \times 10^{-3} \quad \text{using Inclusive } V_{cb} \text{ (QCD sum rule)}$$

$$|\epsilon_K^{\text{Exp}}| = 2.228(11) \times 10^{-3}$$

- 3.2 σ deviation between SM (exclusive V_{cb}) and experiment
- SM uncertainty is $\sim 10\%$, dominated by V_{cb}
- Also important to determine LD contribution to ϵ_K (a few %)

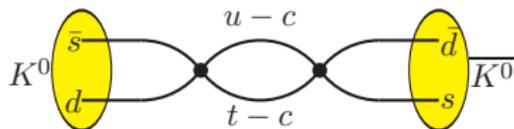
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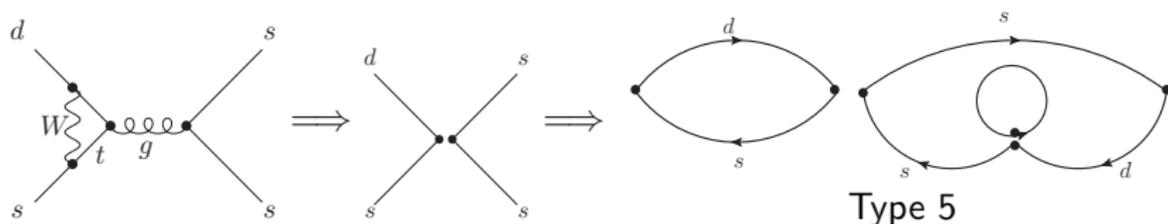


GIM subtraction of charm: $\lambda_u \times (u - c)$ and $\lambda_t \times (t - c)$

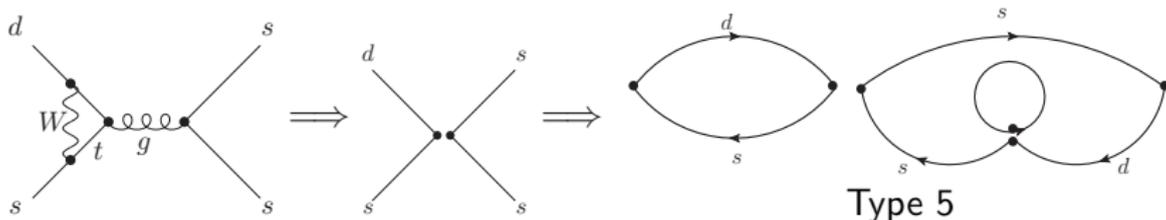
- Three terms:

$\underbrace{\lambda_u^2}$	$\underbrace{\lambda_t^2}$	$\underbrace{\lambda_u \lambda_t}$
irrelevant for ϵ_K	SD dominated	need LQCD

- In the $\lambda_u \lambda_t$ contribution, the top quark field has been integrated out
 - leaving QCD penguin operator and extra Type 5 quark contraction



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 - leaving QCD penguin operator and extra Type 5 quark contraction



- Without top quark in the lattice QCD calculation, logarithmic divergence

$$\langle \{Q_A Q_B\}^{\text{RI}} \rangle \Big|_{p_i^2 = \mu_{\text{RI}}^2} = \text{Diagram with loop } (u-c) - X(\mu_{\text{RI}}, a) \times \text{Diagram with } Q^{\Delta S=2} = 0$$

The equation shows the renormalization of the bilocal operator $\langle \{Q_A Q_B\}^{\text{RI}} \rangle$ at the scale μ_{RI} . The left-hand side is represented by a diagram with external momenta p_1, p_2, p_3, p_4 and internal quark lines u and c . The right-hand side shows the subtraction of a lattice cutoff effect $X(\mu_{\text{RI}}, a)$ multiplied by a diagram representing the $Q^{\Delta S=2}$ operator, resulting in zero.

- Define the bilocal operator in the RI/SMOM scheme
- Subtract $X(\mu_{\text{RI}}, a) Q^{\Delta S=2}$ to remove the lattice cutoff effects

[from Z. Bai, for RBC-UKQCD]

- Use $24^3 \times 64$ lattice with DWF + Iwasaki gauge action

$$a^{-1} = 1.78 \text{ GeV}, \quad m_\pi = 340 \text{ MeV}, \quad m_K = 590 \text{ MeV}, \quad m_c = 970 \text{ MeV}$$

- All Type 1-5 diagrams are evaluated
- Preliminary results based on 200 configurations

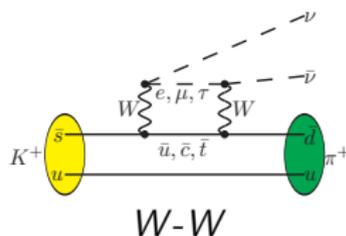
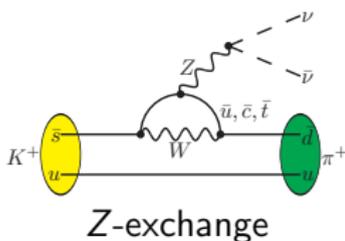
μ_{RI}	$\text{Im } M_{\bar{0}0}^{ut,RI}$	$\text{Im } M_{\bar{0}0}^{ut,RI \rightarrow MS}$	$\text{Im } M_{\bar{0}0}^{ut,ld corr}$	$\epsilon_K^{ut,ld corr}$
1.54 GeV	-0.75(39)	0.28	-0.46(39)	$0.091(76) \times 10^{-3}$
1.92 GeV	-0.91(39)	0.38	-0.53(39)	$0.104(76) \times 10^{-3}$
2.11 GeV	-0.99(39)	0.43	-0.55(39)	$0.108(76) \times 10^{-3}$
2.31 GeV	-1.05(39)	0.49	-0.57(39)	$0.111(77) \times 10^{-3}$
2.56 GeV	-1.12(39)	0.55	-0.57(39)	$0.111(77) \times 10^{-3}$

Experimental value for $|\epsilon_K| = 2.228(11) \times 10^{-3}$

- LD correction to ϵ_K is about 5% at unphysical kinematics

Rare Kaon decays

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: Experiment vs Standard model



$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: largest contribution from top quark loop, thus theoretically clean

$$\mathcal{H}_{\text{eff}} \sim \frac{G_F}{\sqrt{2}} \cdot \underbrace{\frac{\alpha_{\text{EM}}}{2\pi \sin^2 \theta_W} \lambda_t X_t(x_t)}_{\mathcal{N} \sim 2 \times 10^{-5}} \cdot (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}$$

Probe the new physics at scales of $\mathcal{N}^{-\frac{1}{2}} M_W = O(10 \text{ TeV})$

Past experimental measurement is 2 times larger than SM prediction

$$\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})_{\text{exp}} = 1.73_{-1.05}^{+1.15} \times 10^{-10} \quad [\text{BNL E949, '08}]$$

$$\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})_{\text{SM}} = 9.11 \pm 0.72 \times 10^{-11} \quad [\text{Buras et. al., '15}]$$

but still consistent with > 60% exp. error

New generation of experiment: NA62 at CERN

- aims at observation of $O(100)$ events [2014-2018]
- 10%-precision measurement of $\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})$



Latest results reported at FPCP 2017

- Detector installation completed in 09.2016
- 5% of 2016 data \Rightarrow no event yet
- Full 2016 data $\Rightarrow O(1)$ events

2nd-order weak interaction and bilocal matrix element

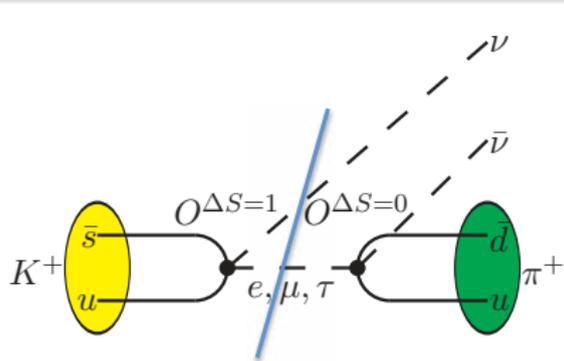
Hadronic matrix element for the 2nd-order weak interaction

$$\int_{-T}^T dt \langle \pi^+ \nu \bar{\nu} | T [Q_A(t) Q_B(0)] | K^+ \rangle$$
$$= \sum_n \left\{ \frac{\langle \pi^+ \nu \bar{\nu} | Q_A | n \rangle \langle n | Q_B | K^+ \rangle}{M_K - E_n} + \frac{\langle \pi^+ \nu \bar{\nu} | Q_B | n \rangle \langle n | Q_A | K^+ \rangle}{M_K - E_n} \right\} (1 - e^{(M_K - E_n)T})$$

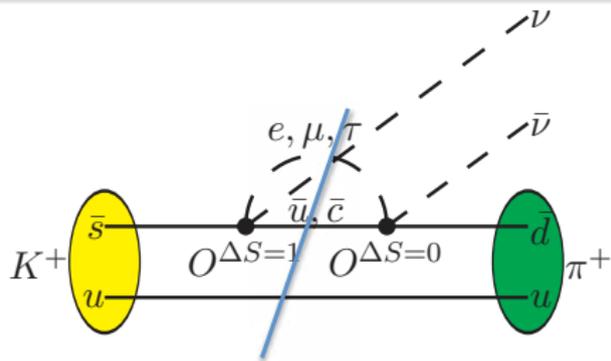
- For $E_n > M_K$, the exponential terms exponentially vanish at large T
- For $E_n < M_K$, the exponentially growing terms must be removed
- \sum_n : principal part of the integral replaced by finite-volume summation
 - possible large finite volume correction when $E_n \rightarrow M_K$

[Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510]

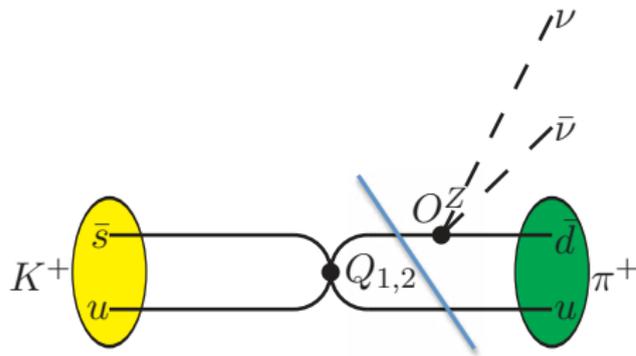
Low lying intermediate states



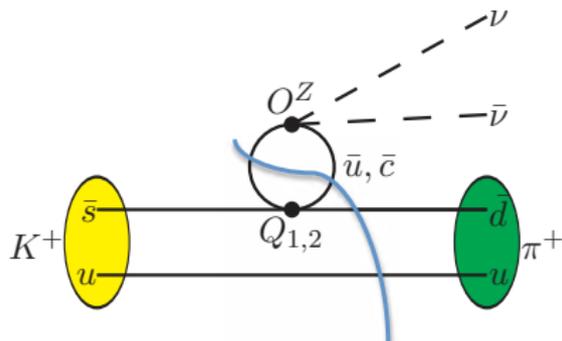
$$K^+ \rightarrow l^+ \nu \quad \& \quad l^+ \rightarrow \pi^+ \bar{\nu}$$



$$K^+ \rightarrow \pi^0 l^+ \nu \quad \& \quad \pi^0 l^+ \rightarrow \pi^+ \bar{\nu}$$



$$K^+ \xrightarrow{H_W} \pi^+ \quad \& \quad \pi^+ \xrightarrow{V_\mu} \pi^+$$



$$K^+ \xrightarrow{H_W} \pi^+ \pi^0 \quad \& \quad \pi^+ \pi^0 \xrightarrow{A_\mu} \pi^+$$

Results for charm quark contribution

Charm quark contribution P_c

$$P_c = P_c^{\text{SD}} + \delta P_{c,u}$$

NNLO QCD [Buras, Gorbahn, Haisch, Nierste, '06]:

$$P_c^{\text{SD}} = 0.365(12)$$

Phenomenological ansatz [Isidori, Mescia, Smith, '05]

$$\delta P_{c,u} = 0.040(20)$$

Lattice results @ $m_\pi = 420$ MeV, $m_c = 860$ MeV

[RBC-UKQCD, arXiv:1701.02858]

$$P_c = 0.2529(\pm 13)_{\text{stat}}(\pm 32)_{\text{scale}}(-45)_{\text{FV}}$$

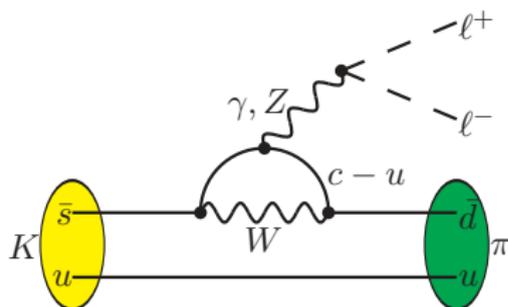
$$P_c - P_c^{\text{SD}} = 0.0040(\pm 13)_{\text{stat}}(\pm 32)_{\text{scale}}(-45)_{\text{FV}}$$

- As a smaller m_c is used, P_c is also smaller
- Cancellation in W - W and Z -exchange diag. leads to small $P_c - P_c^{\text{SD}}$
- Important to perform the calculation at physical m_π and m_c

$K \rightarrow \pi l^+ l^-$: CP conserving channel

CP conserving decay: $K^+ \rightarrow \pi^+ l^+ l^-$ and $K_S \rightarrow \pi^0 l^+ l^-$

- Involve both γ - and Z-exchange diagram, but γ -exchange is much larger



- Unlike Z-exchange, the γ -exchange diagram is LD dominated
 - By power counting, loop integral is quadratically UV divergent
 - EM gauge invariance reduces divergence to logarithmic
 - $c - u$ GIM cancellation further reduces log divergence to be UV finite

Focus on γ -exchange

- Hadronic part of decay amplitude is described by a form factor

$$\begin{aligned} T_{+,S}^\mu(p_K, p_\pi) &= \int d^4x e^{iqx} \langle \pi(p_\pi) | T \{ J_{em}^\mu(x) \mathcal{H}^{\Delta S=1}(0) \} | K^+ / K_S(p_K) \rangle \\ &= \frac{G_F M_K^2}{(4\pi)^2} V_{+,S}(z) [z(k+p)^\mu - (1-r_\pi^2)q^\mu] \end{aligned}$$

with $q = p_K - p_\pi$, $z = q^2/M_K^2$, $r_\pi = M_\pi/M_K$

The target for lattice QCD is to calculate the form factor $V_{+,S}(z)$

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The target for lattice QCD is to calculate the form factor $V_{+,S}(z)$

- Lattice calculation strategy (I): [RBC-UKQCD, PRD92 (2015) 094512]
 - Use conserved vector current to protect the EM gauge invariance
 - Use charm as an active quark flavor to maintain GIM cancellation

First exploratory calculation on $K^+ \rightarrow \pi^+ \ell^+ \ell^-$

Use $24^3 \times 64$ ensemble, $N_{\text{conf}} = 128$
[RBC-UKQCD, PRD94 (2016) 114516]

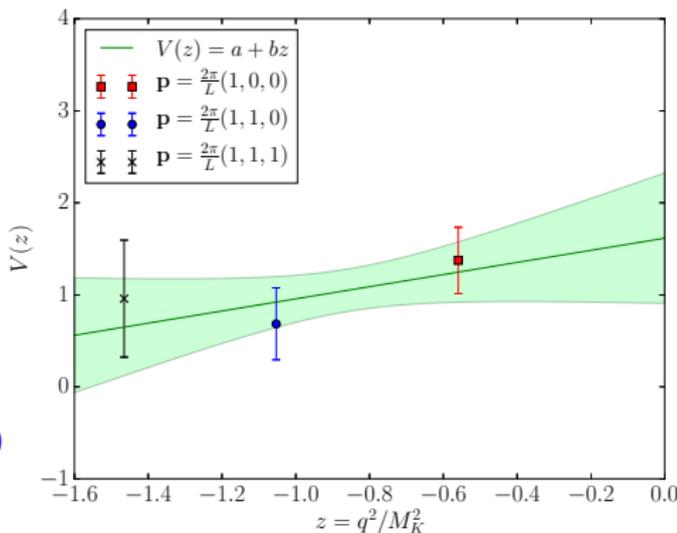
$a^{-1} = 1.78 \text{ GeV}$, $m_\pi = 430 \text{ MeV}$

$m_K = 625 \text{ MeV}$, $m_c = 530 \text{ MeV}$

Momentum dependence of $V_+(z)$

$$V_+(z) = a_+ + b_+ z$$

$$\Rightarrow a_+ = 1.6(7), b_+ = 0.7(8)$$



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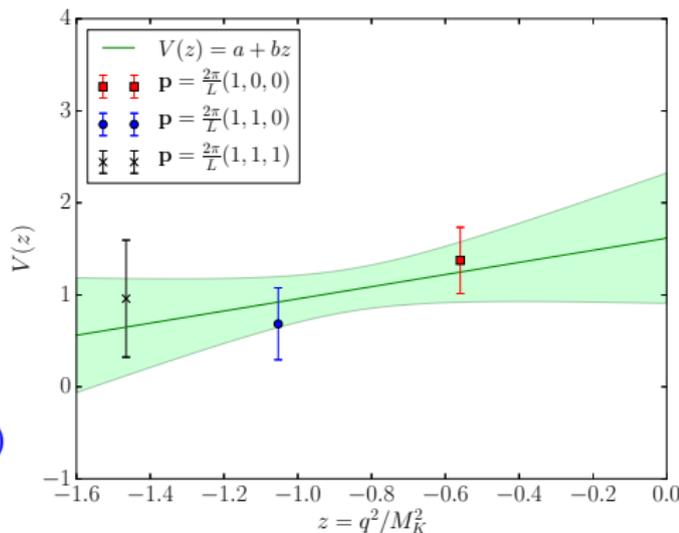
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$$\Rightarrow a_+ = 1.6(7), b_+ = 0.7(8)$$



$K^+ \rightarrow \pi^+ e^+ e^-$ data + phenomenological analysis: $a_+ = -0.58(2)$, $b_+ = -0.78(7)$
 [Cirigliano, et. al., Rev. Mod. Phys. 84 (2012) 399]

$$V_j(z) = a_j + b_j z + \underbrace{\frac{\alpha_j r_\pi^2 + \beta_j (z - z_0)}{G_F M_K^2 r_\pi^4}}_{K \rightarrow \pi\pi} \underbrace{\left[1 + \frac{z}{r_V^2} \right]}_{F_V(z)} \underbrace{\left[\phi(z/r_\pi^2) + \frac{1}{6} \right]}_{\text{loop}}, \quad j = +, S$$

- Experimental data only provide $\frac{d\Gamma}{dz} \Rightarrow$ square of form factor $|V_+(z)|^2$
- Need phenomenological knowledge to determine the sign for a_+ , b_+

Lattice calculation strategy (II)

Important to have a **physical point simulation**, however

- physical m_π requires large lattice volume to control FV effects
- physical m_c requires ultra-fine lattice spacing
 - ⇒ very high demanding on computer resources

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[M. Tomii's talk, Friday 18:50@Seminarios 6+7]

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$K_L \rightarrow \pi^0 \ell^+ \ell^-$ decay contains important CPV information

- Indirect CPV : $K_L \xrightarrow{\epsilon} K_+^0 \rightarrow \pi^0 \gamma^* \rightarrow \pi^0 \ell^+ \ell^-$
- Direct + indirect CPV contribution to branching ratio

[Cirigliano et. al., Rev. Mod. Phys. 84 (2012) 399]

$$\text{Br}(K_L \rightarrow \pi^0 e^+ e^-)_{CPV} = 10^{-12} \times \left[15.7 |a_S|^2 \pm 6.2 |a_S| \left(\frac{\text{Im } \lambda_t}{10^{-4}} \right) + 2.4 \left(\frac{\text{Im } \lambda_t}{10^{-4}} \right)^2 \right]$$

- $\text{Im } \lambda_t$ -term from direct CPV , $\lambda_t \approx 1.35 \times 10^{-4}$
- $|a_S|$ -term from indirect CPV , $a_S = V_S(0)$
- \pm arises due to the unknown sign of a_S

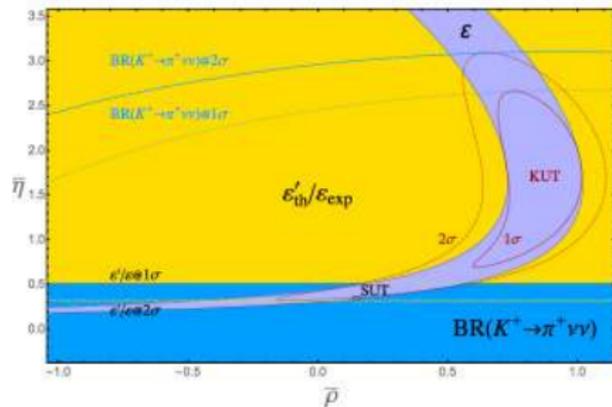
Even a determination of the sign of a_S from lattice is desirable

Outlook: K -unitarity triangle

[C. Lehner, E. Lunghi, A. Soni, PLB759 (2016) 82]

Advances in lattice QCD simulation + experiments

⇒ construction of a unitarity triangle purely from Kaon physics?

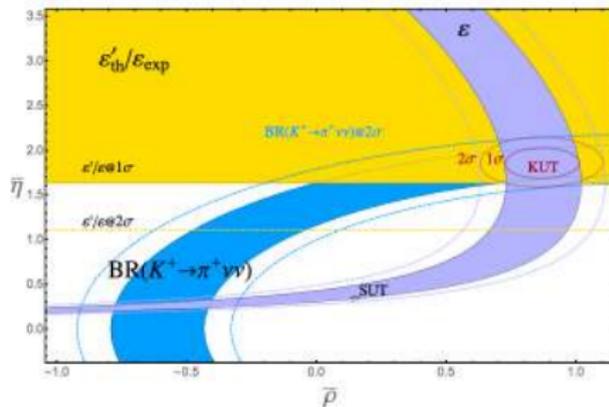


Current status

$$\delta[A_2] = 12\%$$

$$\delta[A_0] = 86\%$$

$$\delta[\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})] = 64\%$$



Future scenario

$$\delta[A_2] = 5\%$$

$$\delta[A_0] = 18\%$$

$$\delta[\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})] = 10\%$$

When will the future scenario become true?

- For “standard” quantities such as f_K/f_π , $f_+(0)$ and B_K

	N_f	FLAG average	Frac. Err.
f_K/f_π	2 + 1 + 1	1.1933(29)	0.25%
$f_+(0)$	2 + 1 + 1	0.9706(27)	0.28%
\hat{B}_K	2 + 1	0.7625(97)	1.27%

lattice QCD calculations play important role in precision flavor physics

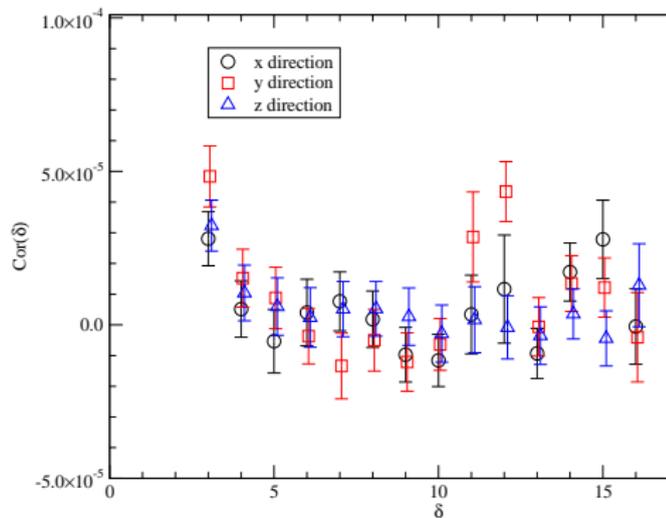
- It's time to go beyond “standard”
 - $K \rightarrow \pi\pi$ and ϵ'
 - ΔM_K and ϵ_K
 - rare kaon decays: $K \rightarrow \pi\nu\bar{\nu}$ and $K \rightarrow \pi\ell^+\ell^-$
- Lattice QCD is now capable of first-principles calculation of the above “beyond-standard” quantities
- Realistic calculation of some of these quantities may require the next generation of super-computers

Backup slides

$K \rightarrow \pi\pi$: Error in ensemble generation

Duplicated RNG seeds used in quark forces \Rightarrow unphysical correlation

- Such correlation is observed in plaquettes separated by 12 in y-direction
- Its size is only $\sim 5 \times 10^{-5}$
- Unlikely affect A_2 , A_0 strongly, whose errors are ~ 1000 times larger



Average plaquette

- Correct ensemble 0.512239(3)(7)
- Incorrect ensemble 0.512239(6)

Systematic error breakdown for $\text{Re } A_2$ and $\text{Im } A_2$

[RBC-UKQCD, PRD91 (2015) 074502]

Systematic errors	$\text{Re } A_2$	$\text{Im } A_2$
NPR (nonperturbative)	0.1%	0.1%
NPR (perturbative)	2.9%	7.0%
Finite-volume corrections	2.4%	2.6%
Unphysical kinematics	4.5%	1.1%
Wilson coefficients	6.8%	10%
Derivative of the phase shift	1.1%	1.1%
Total	9%	12%

Systematic error for individual operator contributions to $\text{Re}(A_0)$, $\text{Im}(A_0)$

[RBC-UKQCD, PRL115 (2015) 212001]

Description	Error	Description	Error
Finite lattice spacing	12%	Finite volume	7%
Wilson coefficients	12%	Excited states	$\leq 5\%$
Parametric errors	5%	Operator renormalization	15%
Unphysical kinematics	$\leq 3\%$	Lellouch-Lüscher factor	11%
Total (added in quadrature)			27%

ΔM_K and ϵ_K : Removal of the exponentially growing terms

- Determine the hadronic matrix element for all low-lying intermediate states

$$\frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{M_K - E_n} (1 - e^{(M_K - E_n)T})$$

- Change of weak operator $H_W \rightarrow H_W + c_s \bar{s}d + c_p \bar{s}\gamma_5 d$ does not affect the physical amplitude
 - Apply the chiral Ward identity

$$\begin{aligned}\partial_\mu \bar{s}\gamma_\mu d &= (m_s - m_d)\bar{s}d \\ \partial_\mu \bar{s}\gamma_\mu\gamma_5 d &= (m_s + m_d)\bar{s}\gamma_5 d\end{aligned}$$

- K^0 - \bar{K}^0 transition amplitude is given by

$$\int d^4x \langle \bar{K}^0 | T[H_W(x)H_W(0)] | K^0 \rangle$$

$\partial_\mu \bar{s}\gamma_\mu d$ and $\partial_\mu \bar{s}\gamma_\mu\gamma_5 d$ do not contribute to the $\int d^4x$ integral

- Choose appropriate c_s and c_p , e.g.

$$\begin{aligned}\langle 0 | H_W + c_p \bar{s}\gamma_5 d | K^0 \rangle &= 0 \\ \langle \eta | H_W + c_s \bar{s}d | K^0 \rangle &= 0\end{aligned}$$

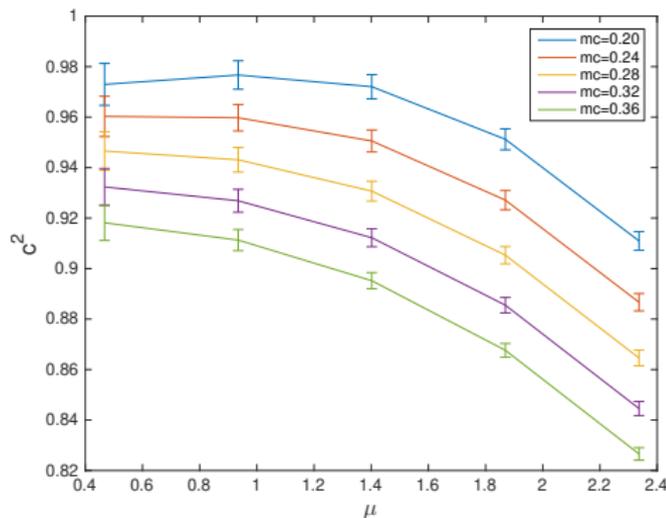
ΔM_K : Lattice artifacts with physical charm quark mass

Naive estimate of lattice artifacts $\sim (m_c^{\overline{\text{MS}}}(2 \text{ GeV})a)^2 = 25\%$ with $a^{-1} = 2.36$ GeV

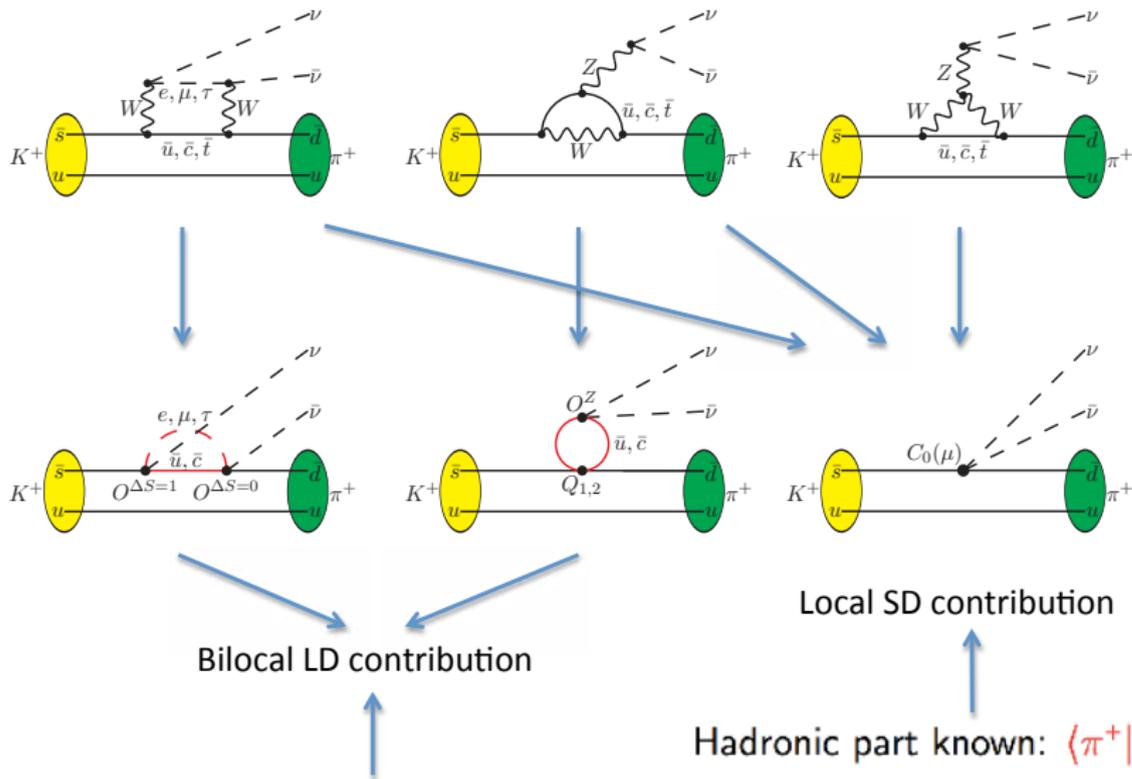
D meson dispersion relation

$$c^2 = \frac{E^2 - m^2}{p^2}, \quad \mu^2 = p^2$$

- The physical charm quark mass is related to bare mass $m_c a = 0.32$
- c^2 value deviate from 1 by $\sim 10\%$



$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: OPE to separate SD and LD parts

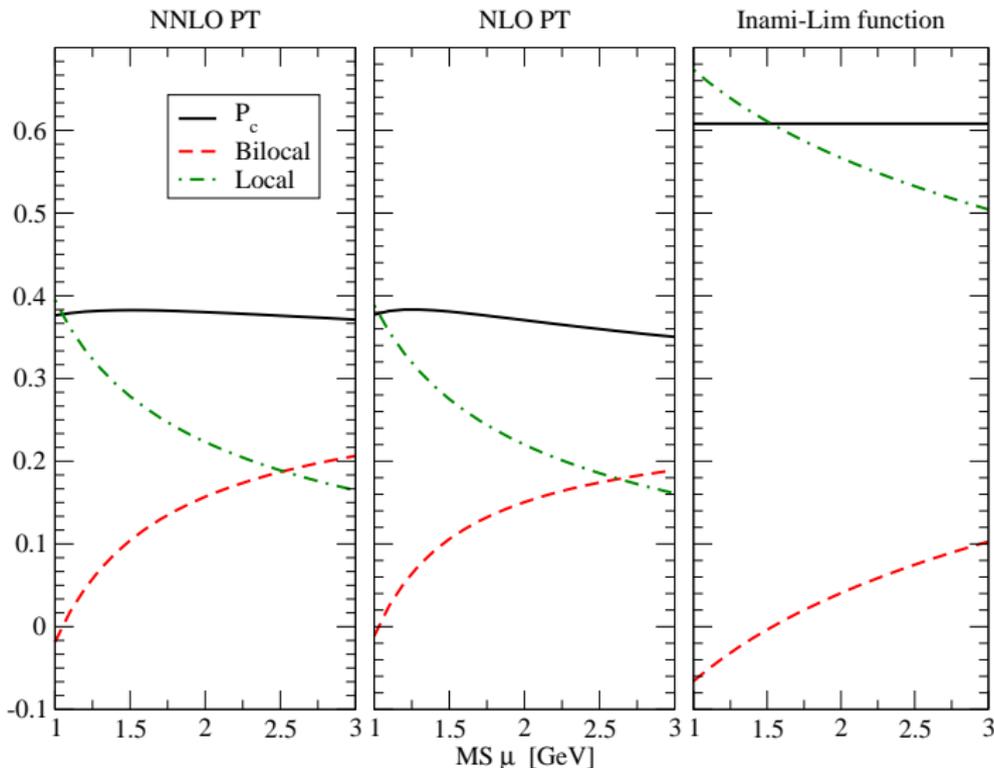


$\langle \pi^+ \nu \bar{\nu} | Q_A(x) Q_B(0) | K^+ \rangle$: need lattice QCD

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: Bilocal contribution vs local contribution

Bilocal $C_A^{\overline{MS}}(\mu) C_B^{\overline{MS}}(\mu) r_{AB}^{\overline{MS}}(\mu)$ vs Local $C_0^{\overline{MS}}(\mu)$

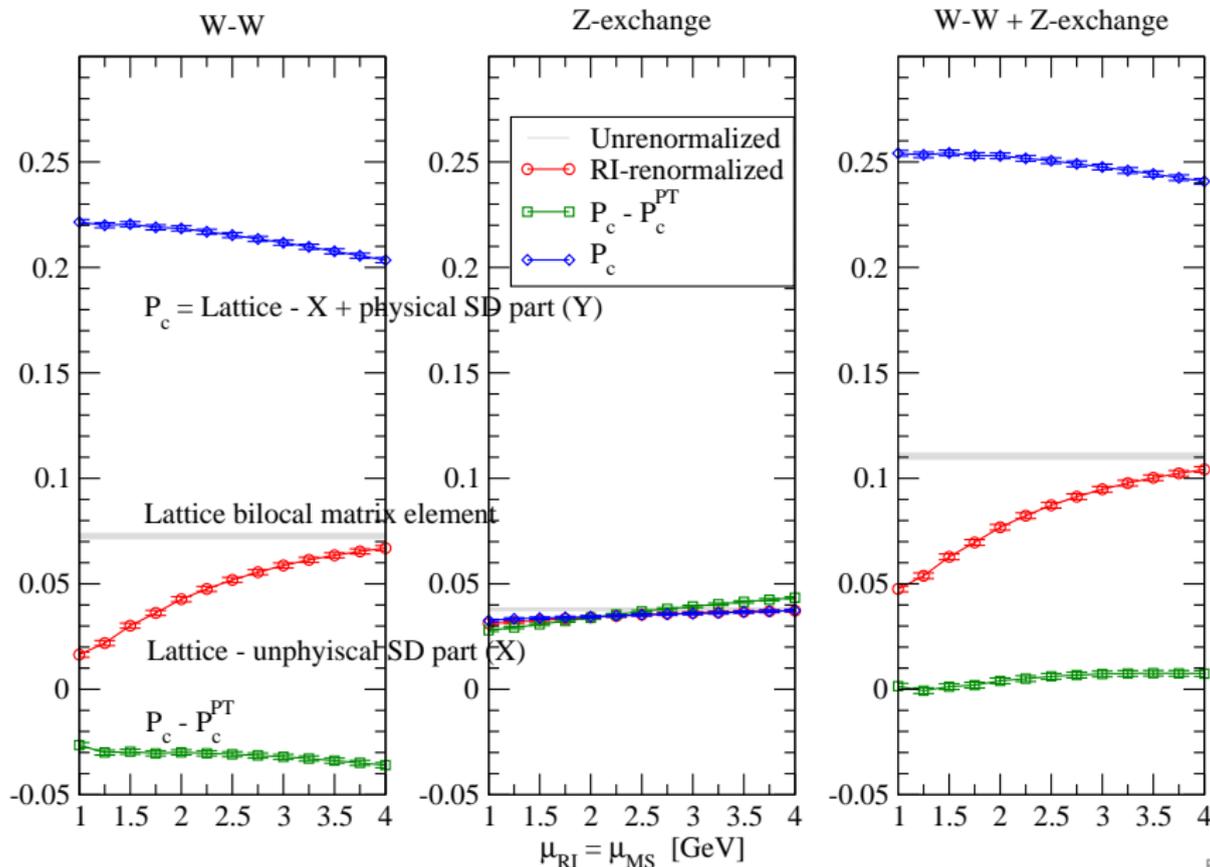
[Buras, Gorbahn, Haisch, Nierste, '06]



At $\mu = 2.5$ GeV, 50% charm quark contribution from bilocal term

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: Lattice results

Use $m_\pi = 420$ MeV, $m_c = 860$ MeV [RBC-UKQCD, arXiv:1701.02858]



$K \rightarrow \pi \ell^+ \ell^-$ in 3-flavor theory

[A. Lawson's talk, Wednesday 12:50@Seminarios 6+7]

From 4-flavor to 3-flavor theory

$$C^{N_f=4}(\mu_c) \underbrace{\langle H_W^{N_f=4}(\mu_c) J^\mu \rangle}_{\text{UV finite}} = C^{N_f=3}(\mu_c) \underbrace{\langle H_W^{N_f=3}(\mu_c) J^\mu \rangle}_{\text{log divergent}} + \sum_i C_i(\mu_c) \underbrace{\langle Q_i^\mu(\mu_c) \rangle}_{\text{counter term}}$$

- The local counter term is mainly given by the penguin operator

$$Q_a^\mu = (\delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) \bar{s} \gamma_\nu (1 - \gamma_5) d$$

Use **NPR** to convert bare lattice bilocal operator to RI/SMOM scheme

$$\langle \{H_W J^\mu\}^{\text{RI}} \rangle \Big|_{p_i^2 = \mu_{\text{RI}}^2} = \text{Diagram 1} - X(\mu_{\text{RI}}, a) \times \text{Diagram 2} = 0$$

Use **PT** to convert RI/SMOM bilocal operator to $\overline{\text{MS}}$ scheme

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