Extracting observables from lattice data in the three-particle sector

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Plan

- Introduction
- Non-relativistic EFT and dimer picture
- Finite-volume spectrum
- Independence from the off-shell effects
- Comparison with other approaches
- Applications: three-body bound state beyond the unitary limit
- Conclusions, outlook

Extraction of the observables on the lattice

Three-particle sector: continuum

- bound states
- elastic scattering, rearrangement reactions
- breakup...

Three-particle sector: finite volume

- energy levels below and above three-particle threshold

$\hookrightarrow$ Roper resonance

$\hookrightarrow$ Three-particle decays

$\hookrightarrow$ Nuclear physics on the lattice

The history

K. Polejaeva and AR, EPJA 48 (2012) 67

Finite volume energy levels determined solely by the S-matrix

M. Hansen and S. Sharpe, PRD 90 (2014) 116003; PRD 92 (2015) 114509

Quantization condition

R. Briceno and Z. Davoudi, PRD 87 (2013) 094507

Dimer formalism, quantization condition

P. Guo, PRD 95 (2017) 054508

Quantization condition in the 1+1-dimensional case

S. Kreuzer and H.-W. Hammer, PLB 694 (2011) 424; EPJA 43 (2010) 229; PLB 673 (2009) 260; S. Kreuzer and H. W. Grießhammer, EPJA 48 (2012) 93

Dimer formalism, numerical solution

- ↪ Complicated, not well suited for the analysis of the lattice data
- ↪ What is the convenient set of observables to be extracted from data?

NREFT: dimer picture in the two-particle sector

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi + \mathcal{L}_2 + \mathcal{L}_3$$

$$\mathcal{L}_2 = -\frac{C_0}{2}\,\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{2}\,(\psi^\dagger\nabla^2\psi^\dagger\psi\psi + \text{h.c.}) + \cdots$$

$C_0, C_2, \ldots$ matched to $p\cot\delta(p) = -\frac{1}{a} + \frac{r}{2}\,p^2 + \cdots$

dimer:

$$\mathcal{L}_2^{\text{dimer}} = \sigma T^\dagger T + \left(T^\dagger \left[f_0\psi\psi + f_1\psi\nabla^2\psi + \cdots \right] + \text{h.c.} \right)$$

- Two frameworks algebraically equivalent
- Higher partial waves can be included: dimers with arbitrary spin $T_{i_1\cdots i_{2k}}$
- Can be generalized to the non-rest frames

NREFT: three-particle sector

$$
\begin{aligned}
\mathcal{L}_3 &= -\frac{D_0}{6}\,\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi - \frac{D_2}{12}\,(\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\psi\psi\psi + \text{h.c.}) \\
&+ \frac{D_4}{12}\,(\psi^\dagger\psi^\dagger\nabla^4\psi^\dagger\,\psi\psi\psi + \text{h.c.}) \\
&+ \frac{D_4'}{12}\,(\psi^\dagger\psi^\dagger\nabla^4\psi^\dagger\,\psi\psi\psi + 2\psi^\dagger\nabla^2\psi^\dagger\nabla^2\psi^\dagger\,\psi\psi\psi + 3\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\,\psi\psi\nabla^2\psi + \text{h.c.}) \\
&+ \frac{D_4''}{12}\,(\psi^\dagger\psi^\dagger\nabla^4\psi^\dagger\,\psi\psi\psi + 2\psi^\dagger\nabla^2\psi^\dagger\nabla^2\psi^\dagger\,\psi\psi\psi - 3\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\,\psi\psi\nabla^2\psi + \text{h.c.}) + \cdots
\end{aligned}
$$

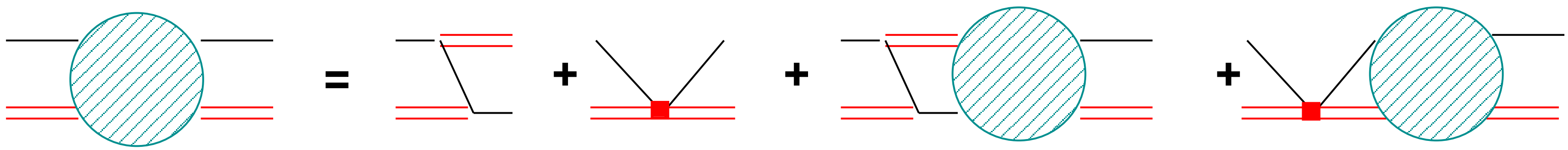
Off-shell term does not contribute to the S-matrix – no-scale integrals

$$
\begin{aligned}
\mathcal{L}_3^{\text{dimer}} &= h_0 T^\dagger T\psi^\dagger\psi + h_2 T^\dagger T(\psi^\dagger\nabla^2\psi + \text{h.c.}) \\
&+ h_4 T^\dagger T(\psi^\dagger\nabla^4\psi + \text{h.c.}) + h_4' T^\dagger T\nabla^2\psi^\dagger\nabla^2\psi + \cdots
\end{aligned}
$$

↪ No restriction on angular momentum: dimers with arbitrary spin

↪ Two couplings h_4, h_4': off-shell coupling D_4'' can be eliminated.

The scattering equation



$$\mathcal{M}(\mathbf{p},\mathbf{q};E)=Z(\mathbf{p},\mathbf{q};E)+\int_{\mathbf{k}}^{\Lambda}Z(\mathbf{p},\mathbf{q};E)\tau(\mathbf{k};E)\mathcal{M}(\mathbf{k},\mathbf{q};E)$$

$$Z(\mathbf{p},\mathbf{q};E)=\frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{pq}-mE}+H_0+H_2(\mathbf{p}^2+\mathbf{q}^2)+\cdots$$

$H_0, H_2, \ldots$ are related to the couplings $h_0, h_2, \ldots$

$$\tau^{-1}(\mathbf{k};E)=k^*\cot\delta(k^*)+\underbrace{\sqrt{\frac{3}{4}\mathbf{k}^2-mE}}_{=k^*}$$

Finite volume

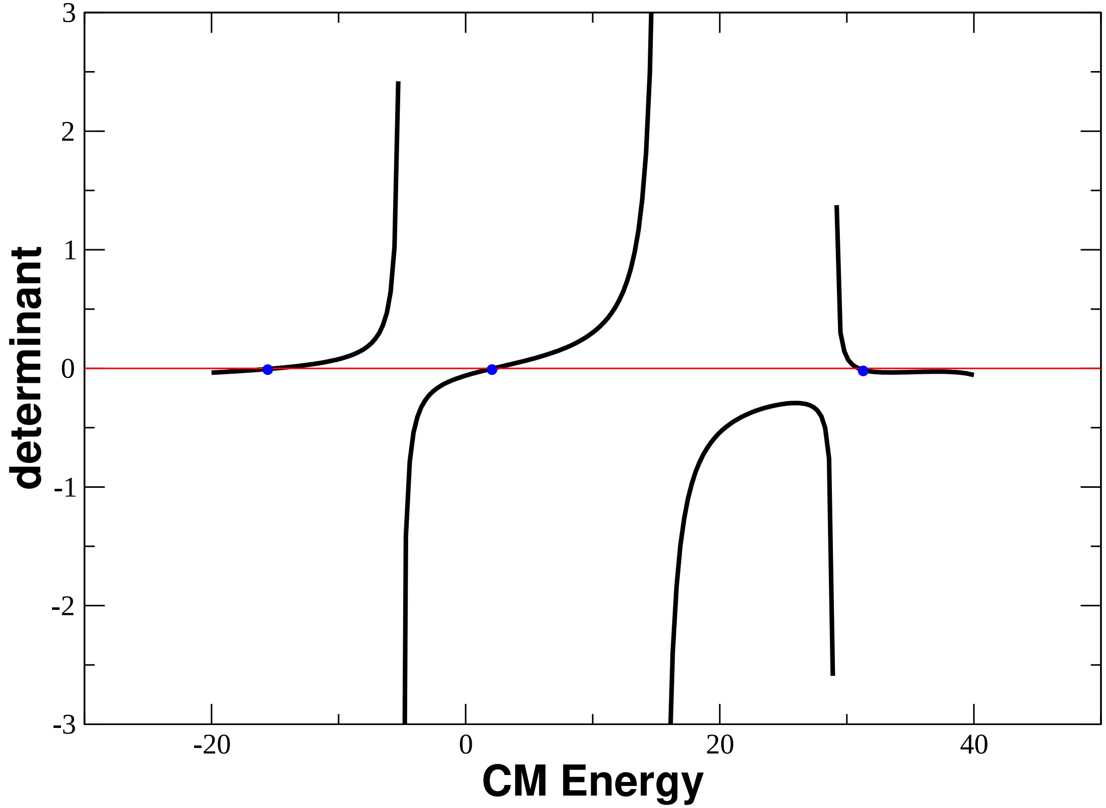
$$\mathbf{k}=\frac{2\pi}{L}\,\mathbf{n}\,,\quad \mathbf{n}\in\mathbb{Z}^3\,,\qquad\qquad \int_{\mathbf{k}}^{\Lambda}\to\frac{1}{L^3}\sum_{\mathbf{k}}^{\Lambda}$$

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E)=Z(\mathbf{p},\mathbf{q};E)+\frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda}Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{k};E)\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$\tau_L^{-1}(\mathbf{k};E)=k^*\cot\delta(k^*)-\frac{4\pi}{L^3}\sum_{\mathbf{l}}\frac{1}{\mathbf{k}^2+\mathbf{l}^2+\mathbf{kl}-mE}$$

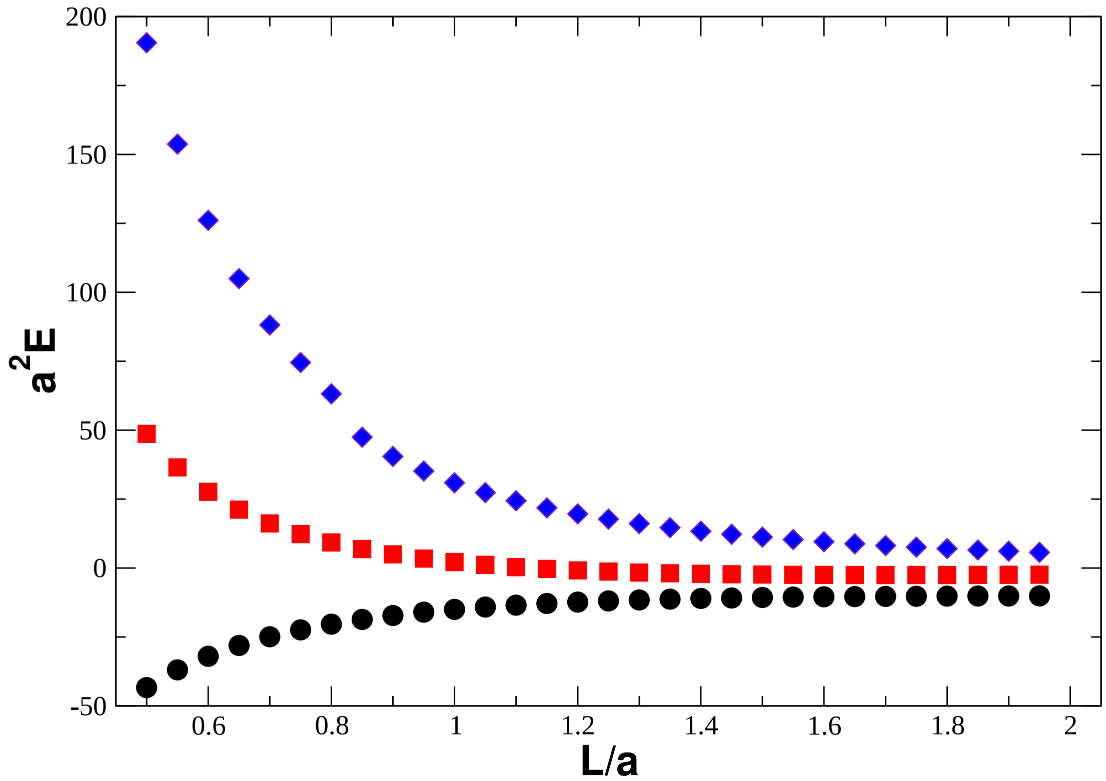
- Poles of $\mathcal{M}_L$ $\quad\rightarrow\quad$ finite-volume energy spectrum
- $k^*\cot\delta(k^*)$ fitted in the two-particle sector; $H_0,H_2,\ldots$should be fitted to the three-particle energies
- S-matrix in the infinite volume $\quad\rightarrow\quad$ equation with $H_0,H_2,\ldots$
- The “off-shell” couplings like D_4'' do not appear in $\mathcal{M}_L$ – the finite volume spectrum is determined by the S-matrix!

Solving the equation numerically *(preliminary)*



- The determinant for a given box size L
- No derivative couplings

The finite-volume spectrum *(preliminary)*



- The spectrum both below and above the three-particle threshold is given

Comparison with known approaches

Hansen & Sharpe, Polejaeva & AR, Briceno & Davoudi

- Hansen & Sharpe introduce "smooth" cutoff on spectator momenta. For the momenta above the cutoff, infinite-volume limit can be performed
- Polejaeva & AR split infinite- and finite-volume contributions for all momenta, no cutoff on spectator momenta
- In Briceno & Davoudi, the cutoff is determined by a dimer with the lowest binding energy in a finite volume.

$\hookrightarrow$ In all approaches, an attempt to *split* the infinite- and finite-volume contributions generates complicated expressions (unconventional K-matrix elements,...)

$\hookrightarrow$ It is convenient to use the parameters $H_0, H_2, \ldots$ to fit the spectrum

Example: three-particle bound state

Using Poisson's formula. . .

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + 8\pi \int_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\hat{\tau}_L(\mathbf{k};E)\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$\hat{\tau}_L(\mathbf{k};E) = \frac{1 + \sum_{\mathbf{n}\neq\mathbf{0}} e^{iL\mathbf{n}\mathbf{k}}}{\tau^{-1}(\mathbf{k};E) + \underbrace{\Delta_L(\mathbf{k};E)}_{\text{zeta-function}}} = \tau(\mathbf{k};E) + \sum_{\mathbf{n}\neq\mathbf{0}} e^{iL\mathbf{n}\mathbf{k}}\tau(\mathbf{k};E) + \cdots$$

$$\hookrightarrow \quad \Delta E = 8\pi \int_{\mathbf{k}}^{\Lambda} \big[\Psi(\mathbf{k})\big]^2 \sum_{\mathbf{n}\neq\mathbf{0}} e^{iL\mathbf{n}\mathbf{k}}\tau(\mathbf{k};E) + \cdots$$

See also M. T. Hansen and S. R. Sharpe, PRD 95 (2017) 034501

Normalization condition

$$-8\pi\int_{\mathbf{p}}^{\Lambda}\big[\Psi(\mathbf{p})\big]^2\frac{\partial\tau(\mathbf{p};E)}{\partial E}-(8\pi)^2\int_{\mathbf{p}}^{\Lambda}\int_{\mathbf{q}}^{\Lambda}\Psi(\mathbf{p})\tau(\mathbf{p};E)\frac{\partial Z(\mathbf{p},\mathbf{q};E)}{\partial E}\tau(\mathbf{q};E)\Psi(\mathbf{q})=1$$

Faddeev-Minlos solution: $\Lambda\to\infty$ and $H(\Lambda)=0$.

$$\Psi_0(p)=iN_0\frac{\kappa}{p}\,\sin(s_0u)\,,\qquad u=\ln\left(\frac{\sqrt{3}}{2}\,\frac{p}{\kappa}+\sqrt{\frac{3p^2}{4\kappa^2}+1}\right),\quad E=\frac{\kappa^2}{m}$$

Asymptotic normalization coefficient

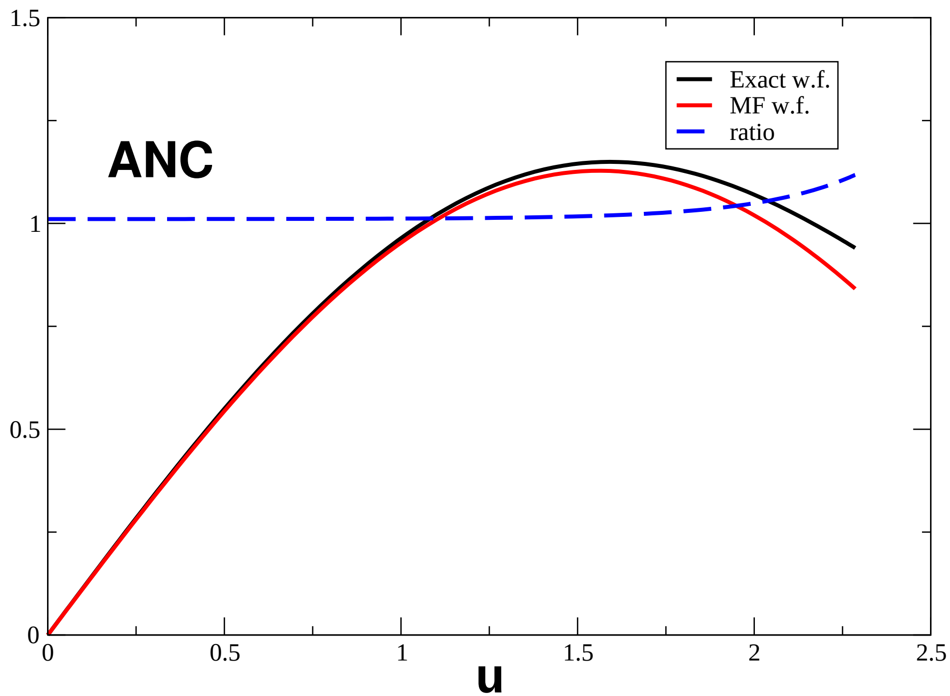
$$\mathcal{A}=\lim_{p\to 0}\Psi(p)/\Psi_0(p)$$

No derivative couplings: $\mathcal{A}=1+O(\kappa/\Lambda)$

Derivative couplings: $\mathcal{A}\neq 1$

Energy shift in the unitary limit

U.-G. Meißner, G. Rios and AR, PRL 114 (2015) 091602



$$\frac{\Delta E}{|E|} = c(\kappa L)^{-3/2}\mathcal{A}^2 \exp\left(-\frac{2\kappa L}{\sqrt{3}}\right)$$

$\mathcal{A} = 1 + O(\kappa/\Lambda)$ in the absence of derivative couplings

Going beyond unitary limit

$$\Delta E \propto \int_{\mathbf{p}}^{\Lambda} \frac{\big[\Psi(\mathbf{p})\big]^2 e^{iL\mathbf{n}\mathbf{p}}}{-a^{-1} + \sqrt{\frac{3}{4}\,\mathbf{p}^2 + \kappa^2}}\,, \qquad |\mathbf{n}| = 1$$

$\Psi(\mathbf{p})$ is only weakly singular in the low-momentum region $\to$ const.

$$\Delta E = \frac{\#}{aL}\exp\left(-\frac{2}{\sqrt{3}}\sqrt{\kappa^2 - \frac{1}{a^2}}\,L\right) + \frac{\#}{(\kappa L)^{3/2}}\exp\left(-\frac{2\kappa L}{\sqrt{3}}\right) + \cdots$$

1. Lüscher equation, bound state of a particle and a dimer
2. Three-particle bound state in the unitary limit

Conclusions, outlook

- An EFT formalism in a finite volume is proposed to analyze the data in the three-particle sector
- The low-energy couplings $H_0, H_2, \ldots$ are fitted to the spectrum; S-matrix is obtained through the solution of equations
- A systematic approach: higher partial waves, derivative couplings, two$\to$three transitions, relativistic kinematics,$\ldots$
- Equivalent to other known approaches, much easier to use: allows to go beyond the unitary limit
- Other EFT's in a finite volume could be used (e.g., the chiral EFT for nucleons)