

*Accurate Simulation of the Finite Density
Lattice Thirring Model*

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June 22nd 2017

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Motivation

- Calculations at finite density of fermions encounter sign problems
- New ideas are being explored to alleviate or solve sign problems
- 2D lattice Thirring model is an excellent toy model for a sign problem similar to QCD
- This sign problem is solvable using world line methods in the massless limit with open boundary conditions. Accurate simulations are possible.
- With periodic and anti-periodic boundary conditions the sign problem remains but is alleviated in the fermion bag approach.

Introduction

We study the massless lattice Thirring model with the action

$$S = \sum_{x,y} \bar{\chi}_x D_{x,y}^{KS}(\mu) \chi_y - U \sum_{x,\nu} \bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu}$$
$$D_{x,y}^{KS} = \frac{1}{2} \eta_{x,\nu} e^{\mu \delta_{\nu,0}} \delta_{y,x+\nu} - \frac{1}{2} \eta_{x,\nu}^\dagger e^{-\mu \delta_{\nu,0}} \delta_{y,x-\nu}$$

Similar to QCD

- Asymptotically safe
- Dynamically generated fermion mass
- Massless boson but no chiral symmetry breaking ¹
- Sign problem like in QCD

¹E. Witten, Nucl. Phys. B **145**, 110 (1978).

Fermion Bag Representation

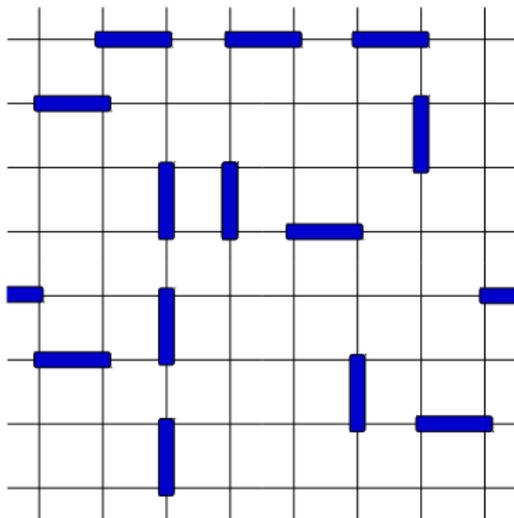
- Expand partition function

$$\begin{aligned} Z &= \int d\bar{\chi} d\chi e^{-\sum_{x,y} \bar{\chi}_x D_{x,y}^{KS}(\mu) \chi_y} \prod_{x,\nu} (1 + U \bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu}) \\ &= \sum_{[d]} U^{N_d} \int d\bar{\chi} d\chi (\bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu})^{d_{x,\nu}} e^{-\sum_{x,y \in [f]} \bar{\chi}_x D_{x,y}^{KS} \chi_y} \\ &= \sum_{[d]} U^{N_d} \det(W([f], \mu)) \end{aligned}$$

- The matrix W defined on free sites $[f]$

Fermion Bag Representation

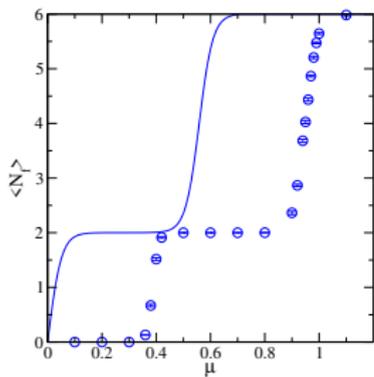
- New dimer variable $d_{x,\nu}$
- Determinant only over free sites
- Disconnected groups of sites: fermion bags
- Local updates, calculate change in determinant



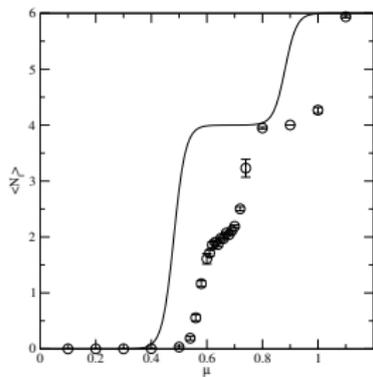
Average quark number

$$\langle N_f \rangle = \left\langle \sum_{x \in S} \frac{\eta_{x,\alpha}}{2} [e^{\mu} \bar{\psi}_x \psi_{x+\alpha} - e^{-\mu} \bar{\psi}_{x+\alpha} \psi_x] \right\rangle$$

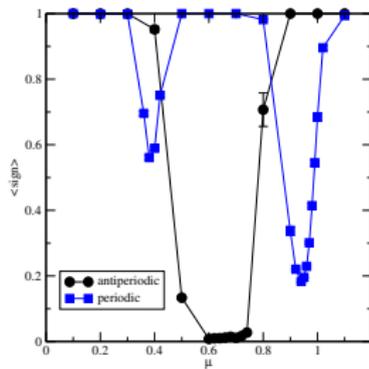
Results



Periodic



Anti-periodic



Average sign

Measurements with $U = 0.3$, $L_X = 6$ and $L_T = 48$. The solid lines shows the value of $\langle N_f \rangle$ at $U = 0$.

World Line Representation

Sign problem absent with open boundary conditions

$$\bar{\chi}_x = \chi_x = 0 \text{ when } x_1 = 0, L_X$$

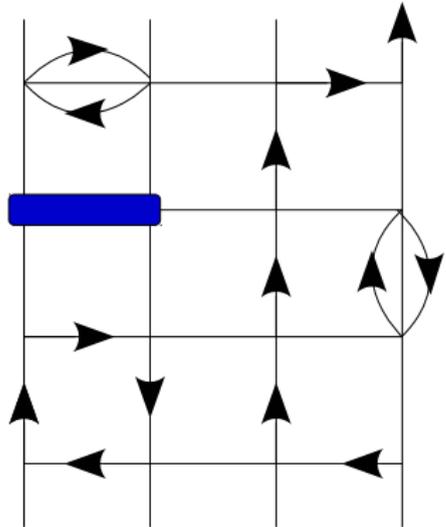
We show this in the world line formalism

$$\begin{aligned} \det(W([f], \mu)) &= \\ &\prod_{x \in [f]} \left(\int d\bar{\chi}_x d\chi_x \right) e^{-\sum_{x \in [f]} \left(\frac{1}{2} \eta_{x,\nu} e^{\mu\delta_{\nu,0}} \bar{\chi}_x \chi_{x+\nu} - \frac{1}{2} \eta_{x,\nu}^\dagger e^{-\mu\delta_{\nu,0}} \bar{\chi}_{x+\nu} \chi_x \right)} \\ &= \sum_{[f]} \prod_{loop \in I} \left(- \prod_{x, \alpha \in loop} e^{\pm \mu \delta_{\pm \alpha, \hat{t}} \frac{s_\alpha \eta_{x, \alpha}}{2}} \right) \end{aligned}$$

- $s_\alpha = +1$ (-1) for positive (negative) directions α

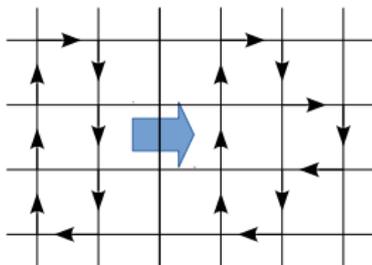
World Line Representation

- Fermion world line variable $l_{x,\nu}$
- Closed loops of fermion world lines
- Update with a worm algorithm

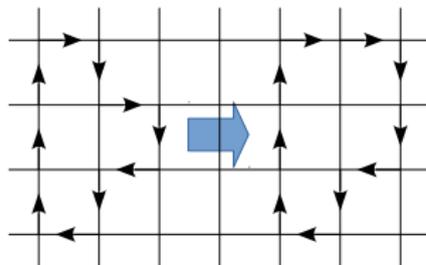


World Line Representation

- Loops can be deformed into each other using two operations:



(a) does not change the sign or the volume



(b) changes the sign and the enclosed volume

World Line Representation



Enumerate all possible loops

Two classes:

- Time wrapping loops
- Non-wrapping loops

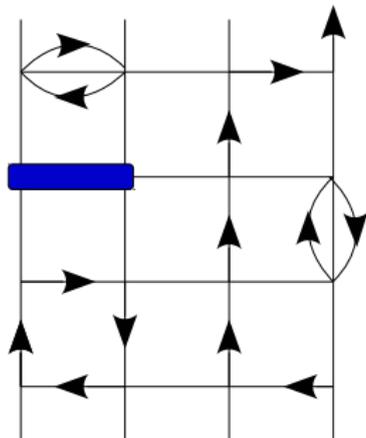
Only even volumes

can be enclosed

→ only positive signs

World Line Representation

- With other boundary conditions signs arise from wrapping loops
- No sign problem in the $N_f = 0$ sector
- Mass allows filling an odd volume
→ Sign problem with any boundary conditions



Results

- Fermion number

$$\begin{aligned}\langle N_f \rangle &= \left\langle \sum_{x \in S} \frac{\eta_{x,\alpha}}{2} [e^{\mu} \bar{\psi}_x \psi_{x+\alpha} - e^{-\mu} \bar{\psi}_{x+\alpha} \psi] \right\rangle \\ &= \left\langle \sum_{x \in S} l_{x,\hat{t}} - l_{x+\hat{t},-\hat{t}} \right\rangle\end{aligned}\tag{1}$$

- Fermion mass m_f
 - Value of μ at first jump in $\langle N_f \rangle$ on an anisotropic lattice
 - On a large square lattice $\langle N_f \rangle = \max(a(\mu - m_f), 0)$

Results

- Fermion number susceptibility

$$\chi = \frac{U}{V} \sum_{x,y} \langle \bar{\psi}_x \psi_x \bar{\psi}_y \psi_y \rangle \quad (2)$$

which scales as

$$\chi = AL^{2-\eta}, \quad (3)$$

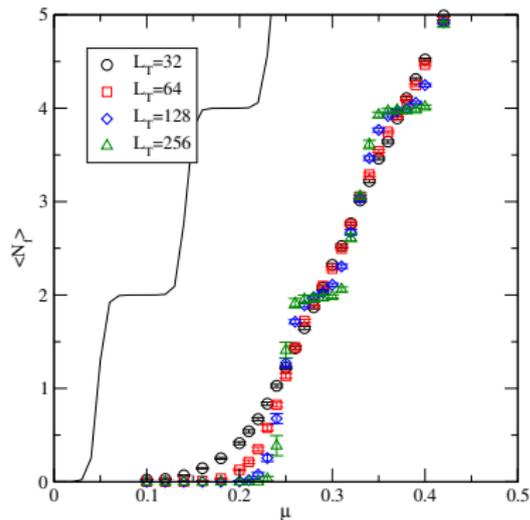
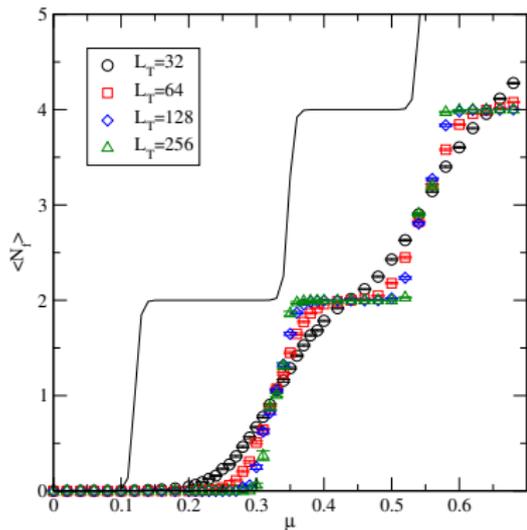
- Boson mass

$$\chi(L_T) = \chi(\infty) - ce^{-m_b L_T} \quad (4)$$

- Chiral charge susceptibility

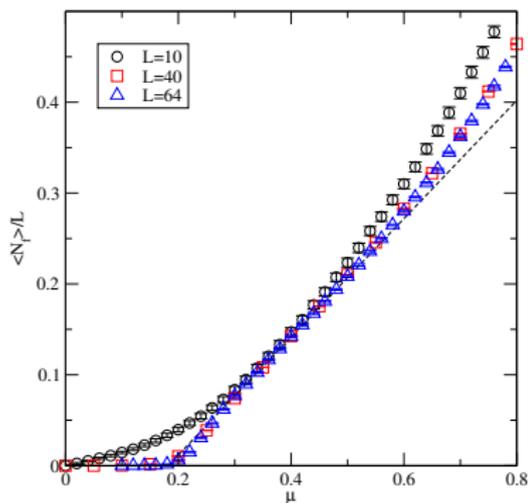
$$\langle Q_\chi^2 \rangle = \left\langle \left(\sum_x \epsilon_x \bar{\psi}_x \psi_{x+\hat{t}} - \epsilon_x \bar{\psi}_{x+\hat{t}} \psi_x \right)^2 \right\rangle \quad (5)$$

Results

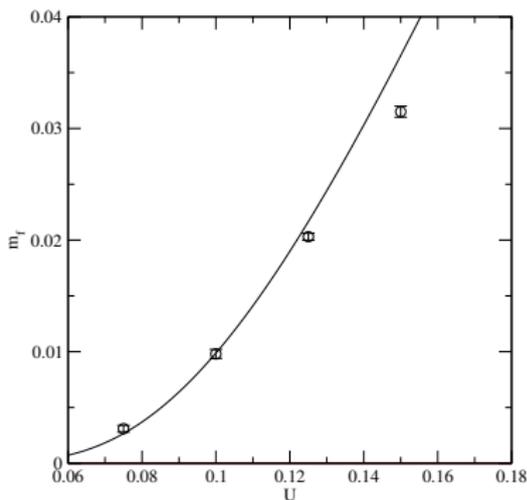


Fermion number at $U = 0.3$ and $L_X = 12$ (left) and 32 (right)

Results



Fermion number density on square lattices



The fermion mass with small U with one-loop continuum scaling

$$m_f = C \exp\left(\frac{-2\pi}{b_0 U}\right), \quad b_0 = 16, C = 0.49$$

Results

U	$2 - \eta$	$\langle Q_x^2 \rangle$	m_f	$m_b^{L_x=32}$	$m_f^{L_x=32}$
0	0	0	0	0.067393	0.045(5)
0.1	0.90(1)	0.499(7)	0.0098(4)	0.061(4)	0.0705(6)
0.2	1.201(4)	0.61(1)	0.081(1)	0.06(1)	0.1397(3)
0.3	1.303(4)	0.780(8)	0.183(1)	0.066(7)	0.247(2)
0.4	1.371(7)	0.895(4)	0.290(1)	0.060(3)	0.356(1)
0.5	1.393(3)	0.972(3)	0.395(3)	0.057(2)	0.465(1)
0.6	1.423(4)	1.024(3)	0.491(1)	0.049(5)	0.557(2)
1.0	1.467(4)	1.128(2)	0.793(1)	0.050(4)	0.842(2)
∞	1.5	1.208(8)	∞	0.0476(9)	∞

Conclusions

- Efficient simulation of the 2D lattice Thirring model
- Significant sign problem with anti-symmetric boundaries
- Sign problem absent with
 - Open boundary conditions
 - Zero fermion mass
- Allows a fast worm algorithm