Nucleon average quark momentum fraction with $N_f = 2+1$ Wilson fermions

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Outline

Theory: Definitions and operators

Nucleon quark momentum fraction is the first momentum of unpolarized quark distribution:

$$\langle x \rangle_q = \int_0^1 x \cdot [q(x) + \bar{q}(x)] \,. \tag{1}$$

Similarly, for the first moment of helicity distribution (quark fields Δq):

$$\langle \Delta x \rangle_{\Delta q} = \int_0^1 x \cdot [\Delta q(x) + \Delta \bar{q}(x)] \,, \tag{2}$$

and for the first moment of transversity distribution (quark fields δq):

$$\langle \delta x \rangle_{\delta q} = \int_0^1 x \cdot [\delta q(x) + \delta \bar{q}(x)] \,. \tag{3}$$

On the lattice they require one-derivative (twist-2) operators:

$$\mathcal{O}^{vD}_{\mu\nu} = \bar{q}\gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}} q \,, \quad \mathcal{O}^{aD}_{\mu\nu} = \bar{q}\gamma_{\{\mu}\gamma_5 \overleftrightarrow{D}_{\nu\}} q \,, \quad \mathcal{O}^{tD}_{\mu\nu\rho} = \bar{q}\sigma_{[\mu\{\nu]} \overleftrightarrow{D}_{\rho\}} q \,. \tag{4}$$

Theory: FF decomposition

For $\mathcal{O}^V_{\mu\nu}$ the form factor decomposition of the nucleon matrix element reads

$$\left\langle N(p',s') \left| \mathcal{O}^{vD}_{\mu\nu} \right| N(p,s) \right\rangle = \bar{u}(p',s') \left[\gamma_{\{\mu} \bar{P}_{\nu\}} A_{20}(Q^2) - \frac{\sigma_{\{\mu\alpha} Q_\alpha Q_{\nu\}}}{2m_N} B_{20}(Q^2) + \frac{1}{m_N} Q_{\{\mu} Q_{\nu\}} C_{20}(Q^2) \right] u(p,s) \,. \tag{5}$$

Spin-projecting with $\Gamma_0 = \frac{1}{2}(1+\gamma_0)$ and $\Gamma_z = \Gamma_0(1+i\gamma_5\gamma_3)$ and considering zero momentum transfer $Q = 0$ we compute the ratio

$$R^{vD}(t_f,t,t_i) \equiv \frac{C^{\mu\mu}_{\rm 3pt}(\vec{q}=0,t_f,t_i,t;\Gamma_z)}{C_{\rm 2pt}(\vec{q}=0,t_f-t_i;\Gamma_0)} \to \begin{cases} -\frac{3}{4} m \langle x \rangle_{u\pm d} & \text{for } \mu = 0 \\ +\frac{1}{4} m \langle x \rangle_{u\pm d} & \text{for } \mu = 1,2,3 \end{cases} , \tag{6}$$

for $t_f - t \gg 1,\ t - t_i \gg 1$ and where $\langle x \rangle_{u\pm d} \equiv A_{20}(0)$.

Similar relations from GPDF decompositions hold for $\mathcal{O}^{aD}_{\mu\nu}$, $\mathcal{O}^{tD}_{\mu\nu\rho}$: *Phys. Lett. B594 (2004) 164-170*

$$R^{aD}(t_f,t,t_i) \to -\frac{i}{2} m \langle x \rangle_{\Delta u \pm \Delta d} \quad \text{for } \mu = 3, \nu = 0 \,, \tag{7}$$

$$R^{tD}(t_f,t,t_i) \to +\frac{i}{4} m (2\delta_{0\rho} - \delta_{0\mu} - \delta_{0\nu}) \langle x \rangle_{\delta u \pm \delta d} \quad \neq 0 \text{ for e.g. } \mu = 0, \nu = 1, \rho = 2 \,. \tag{8}$$

Setup

Computations are performed on **CLS ensembles**:

ID	β	$a/\,\mathrm{fm}$	$M_\pi\,/\,\mathrm{MeV}$	$M_\pi L$	L/a	$t_{\mathrm{sep}}/\,\mathrm{fm}$	N_{meas}
H105	3.40	0.087	280	3.9	32	1.0, 1.2, 1.4	48912
N200	3.55	0.064	280	4.4	48	1.0, 1.2, 1.3, 1.4	20364
D200	3.55	0.064	200	4.2	64	1.0, 1.2, 1.3, 1.4	32672
J303	3.70	0.050	260	4.1	64	1.0, 1.1, 1.2, 1.3	5856

- $N_f = 2+1$ flavors of non-perturbatively improved Wilson clover fermions. *JHEP 1502 (2015) 043*
- Lüscher-Weisz gauge action *Commun.Math.Phys. 97 (1985)*
- Exceptional configurations are prevented by a twisted mass regulator. *PoS LATTICE2008 (2008) 049*
- Generated with open boundary conditions in time. *Comput. Phys. Commun. 184 (2013)*
- Subset of the ensembles used for EM form factors and g_A

Computation of 2pt and 3pt functions

We use the truncated solver method:

$$\langle \mathcal{O} \rangle = \langle \frac{1}{N_{LP}} \sum_{i=1}^{N_{LP}} \mathcal{O}_n^{LP} \rangle + \langle \mathcal{O}_{\mathrm{bias}} \rangle , \quad \mathcal{O}_{\mathrm{bias}} = \frac{1}{N_{HP}} \sum_{i=1}^{N_{HP}} (\mathcal{O}_n^{HP} - \mathcal{O}_n^{LP}) \tag{9}$$

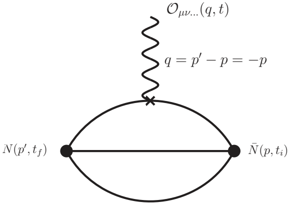
Comput.Phys.Commun. 181 (2010) 1570-1583
Phys.Rev. D91 (2015) no.11, 114511

Typically, per configuration:

- $N_{HP} = 1$ high-precision inversion(s)
- $N_{LP} = 16...48$ low-precision inversions

→ **Gain of factor 2-3 in compute time**

- For 3pt functions we use sequential inversions through the sink, setting $p' = 0$.
- Isovector matrix elements require only quark-connected 3pt functions
- For isoscalar matrix elements we plan to add disconnected diagrams



Renormalization

Non-perturbative renormalization has been performed for the **two smallest values of** β using the Rome-Southampton method:

β	$Z_{v2a}^{\overline{\mathrm{MS}}}$	$Z_{v2b}^{\overline{\mathrm{MS}}}$	$Z_{r2a}^{\overline{\mathrm{MS}}}$	$Z_{r2b}^{\overline{\mathrm{MS}}}$	$Z_{h1a}^{\overline{\mathrm{MS}}}$	$Z_{h1b}^{\overline{\mathrm{MS}}}$
3.40	1.0885(01)	1.0684(01)	1.1118(01)	1.0561(01)	1.0996(01)	1.1156(01)
3.55	1.1388(01)	1.1237(01)	1.1601(01)	1.1130(01)	1.1612(01)	1.1756(01)
3.70	1.1850(11)	1.1745(11)	1.2045(11)	1.1653(11)	1.2178(11)	1.2307(11)

- Each of the derivative operators falls into two different irreps of $H(4)$. *Phys. Rev. D54 (1996) 5705* *Phys. Rev. D82 (2010) 114511*
- Matrix elements agree in the continuum limit.
- Blue irreps are required for the vD, aD and tD operators actually used in our calculation.
- Values at $\beta = 3.70$ extrapolated (for now).
- (Relative) effects of renormalization are of $\mathcal{O}(10\%)$; similar size as found in other studies.
- Errors are statistical only; irrelevant for total error budget.
- Results are given in $\overline{\mathrm{MS}}$ at $Q^2 = 4\ \mathrm{GeV}^2$.

Excited States

Nucleon structure calculations notoriously hampered by **excited state contaminations**:

- Need large t_{sep} →for plateau method

  →signal-to-noise problem

- Summation method

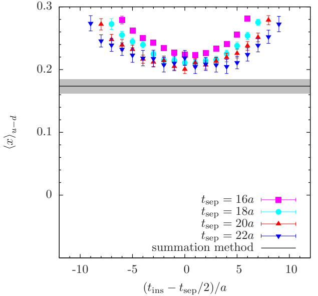
$$\sum_{t=t_i+2}^{t_f-2} R(t_f, t, t_i) \sim c + t_f \cdot \mathcal{M}_0 + \mathcal{O}\left(e^{-\Delta E(t_f - t_i)}\right)$$

  → Excited state suppressed by $(t_f - t_i)$

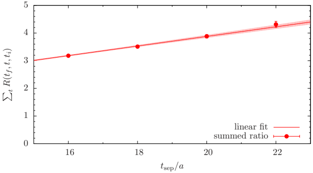
  → But large errors

- Two- / multi-state fits (not yet implemented)
- Generalized pencil of functions

  → Use GEVP to obtain ground state ...



Plateaux and summation method on N200 ($V = 128 \times 48^3$, $M_\pi = 280$ MeV, $a = 0.064$ fm)



Linear fit for summation method on N200

Generalized pencil-of-functions (GPOF)

Instead of extending the actual operator basis, make use of the fact that

$$\mathcal{O}_{\Delta t}(t) \equiv \mathcal{O}(t+\Delta t) = \exp(H\Delta t)\mathcal{O}(t)\exp(-H\Delta t)\,, \tag{10}$$

is a new, and **linearly independent** interpolating operator. *AIP Conf.Proc. 1374 (2011)*

For 2pt functions construct $(n+1)\times(n+1)$ correlation function matrix for fixed Δt, $t \equiv t_f - t_i$:

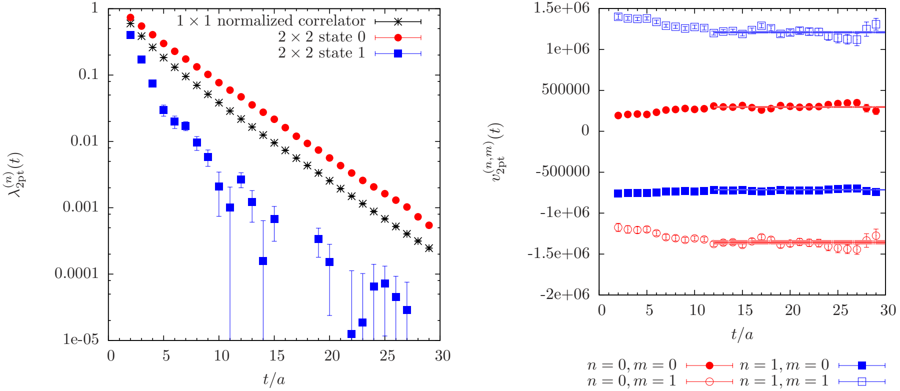
$$\mathcal{C}_{\rm 2pt}(t) = \begin{pmatrix} \langle \mathcal{O}_{0\cdot\Delta t}(t_f)\mathcal{O}^\dagger(t_i)\rangle & \dots & \langle \mathcal{O}_{0\cdot\Delta t}(t_f)\mathcal{O}^\dagger_{n\cdot\Delta t}(t_i)\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathcal{O}_{n\cdot\Delta t}(t_f)\mathcal{O}^\dagger(t_i)\rangle & \dots & \langle \mathcal{O}_{n\cdot\Delta t}(t_f)\mathcal{O}^\dagger_{n\cdot\Delta t}(t_i)\rangle \end{pmatrix} = \begin{pmatrix} C_{\rm 2pt}(t) & \dots & C_{\rm 2pt}(t+n\cdot\Delta t) \\ \vdots & \ddots & \vdots \\ C_{\rm 2pt}(t+n\cdot\Delta t) & \dots & C_{\rm 2pt}(t+2n\cdot\Delta t) \end{pmatrix}. \tag{11}$$

$\rightarrow$ GEVP gives eigenvalues $\lambda^{(n)}(t,t_0)$ and matrix of eigenvectors $V \equiv (\vec{v}^0(t,t_0),\dots,\vec{v}^n(t,t_0))$.

Similarly, for 3pt functions:

$$\mathcal{C}_{\rm 3pt}(t_f,t,t_i) = \begin{pmatrix} C(t_f,t,t_i) & \dots & C(t_f+n\cdot\Delta,t+n\cdot\Delta t,t_i) \\ \vdots & \ddots & \vdots \\ C(t_f+n\cdot\Delta t,t,t_i) & \dots & C(t_f+2n\cdot\Delta t,t+n\cdot\Delta t,t_i)\,. \end{pmatrix} \tag{12}$$

$\rightarrow$ **This matrix is not symmetric but can be diagonalized using V from 2pt case.**



Eigenvalues and eigenvectors for J303 ($V = 192 \times 64^3$, $M_\pi = 260$ MeV, $a = 0.050$ fm)

- Clear reduction of excited states for ground state principal correlator from 2×2 GEVP.
- Stat. err. on nucleon mass reduced by typically $\mathcal{O}(50\%)$.
- Some residual excited state effects remains (as expected).
- Fits to eigenvectors are stable; typically $\chi^2_{\text{red}} \approx 1$.
- Actual choice of fitrange has little impact on final result.

General procedure (sample-wise):

1. Solve GEVP for corresponding 2pt problem.
2. Fit eigenvectors in plateau region $\rightarrow V$ (time-independent)
3. Diagonalize $\mathcal{C}_{\mathrm{3pt}}(t_f, t, t_i)$:

$$\mathcal{C}_{\mathrm{3pt}}(t_f, t, t_i) \rightarrow V^T \mathcal{C}_{\mathrm{3pt}}(t_f, t, t_i) V \equiv \Lambda_{\mathrm{3pt}}(t_f, t, t_i) = \mathrm{diag}(\Lambda^{(0)}, \ldots, \Lambda^{(n)})(t_f, t, t_i) \,. \quad (13)$$

4. Replace "standard" ratio by new, optimized ratio

$$\frac{C_{\mathrm{3pt}}(\vec{q}=0, t_f, t_i, t; \Gamma_z)}{C_{\mathrm{2pt}}(\vec{q}=0, t_f - t_i; \Gamma_0)} \rightarrow \frac{\Lambda^{(0)}_{\mathrm{3pt}}(\vec{q}=0, t_f, t_i, t; \Gamma_z)}{\lambda^{(0)}_{\mathrm{2pt}}(\vec{q}=0, t_f - t_i; \Gamma_0)} \,. \quad (14)$$

Advantages:

- Method is straightforward to implement and does not require model assumptions.
- Uses existing data.
- Errors similar to plateau method at max $t_{\rm sep}$; typically smaller than summation method.

A few restrictions in practice:

- We always have $t_i = 0$ and three or four values for t_f.
- Need $n+1$ equidistant values of $t_{\rm sep} \equiv t_f - t_i = t_f$ to build an $n \times n$ 3pt function matrix.

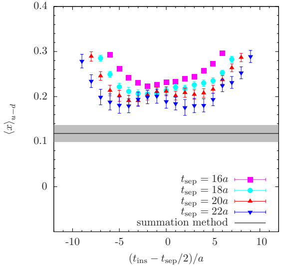
$\Rightarrow$ We are restricted to 2×2 problems. (for now...)

- t_f values are spaced by $\Delta t = 2a$ (e.g. $t_f/a = 16, 18, 20, 22$)

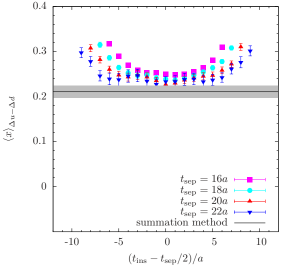
$\Rightarrow$ We are restricted to $\Delta t = 2a$ in the operator construction for the 3pt case.

- Excited state removal certainly not perfect for 2x2 GPOF
- For too many operators GPOF tends to become degenerate / singular

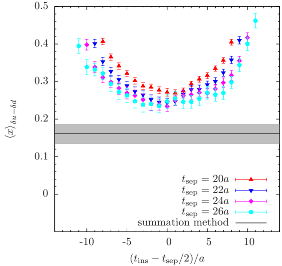
Results – Plateaux and summation method



$\langle x \rangle_{u-d}$ on D200
($M_\pi = 200$ MeV, $a = 0.064$ fm)



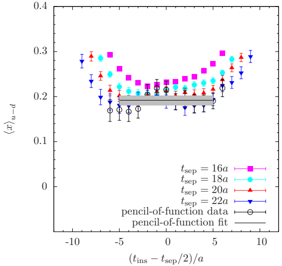
$\langle x \rangle_{\Delta u-\Delta d}$ on N200
($M_\pi = 280$ MeV, $a = 0.064$ fm)



$\langle x \rangle_{\delta u-\delta d}$ on J303
($M_\pi = 260$ MeV, $a = 0.050$ fm)

- We observe excited state contamination on all ensembles
- $t_{\text{sep}} = 1.0 \ldots 1.4$ fm
- Larger M_π reduces excited state effects (D200 vs N200)
- Values from summation method 0…50% different from plateau method.

Results – Plateaux and generalized pencil of functions



$\langle x \rangle_{u-d}$ on D200
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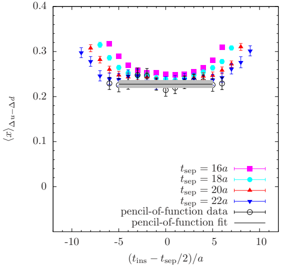
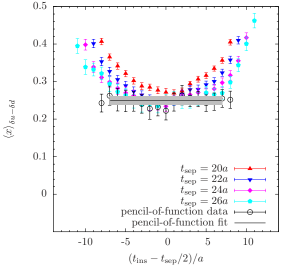


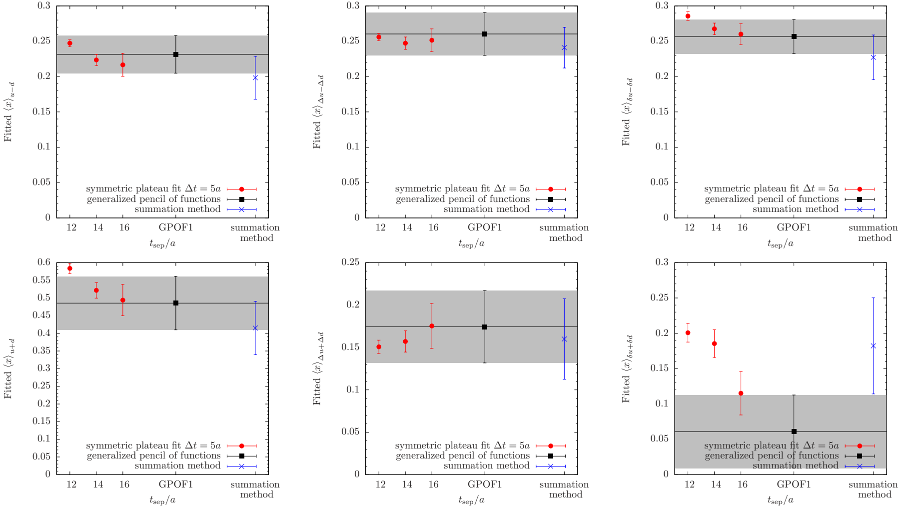
$\langle x \rangle_{\Delta u-\Delta d}$ on N200
($M_\pi = 280$ MeV, $a = 0.064$ fm)



$\langle x \rangle_{\delta u-\delta d}$ on J303
($M_\pi = 260$ MeV, $a = 0.050$ fm)

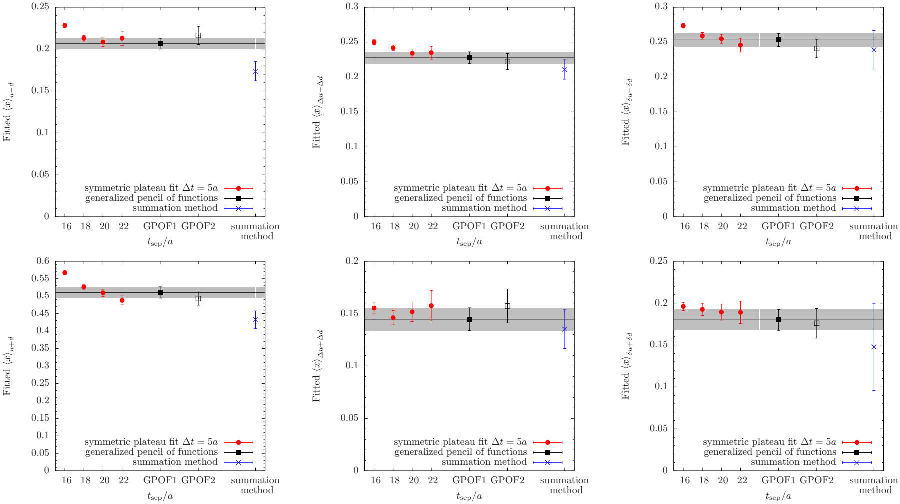
- Values from GPOF closer to plateau method.
- Smaller errors than summation method as expected.
- Observe decent plateau for GPOF on e.g. J303; good fits for $t \in \left[3, \ldots, t_f^{min} - 3\right]$.
- Still, value differs from summation method by more than 3σ.

Comparison of methods for H105 ($M_\pi = 280\,\mathrm{MeV}$, $a = 0.087\,\mathrm{fm}$)



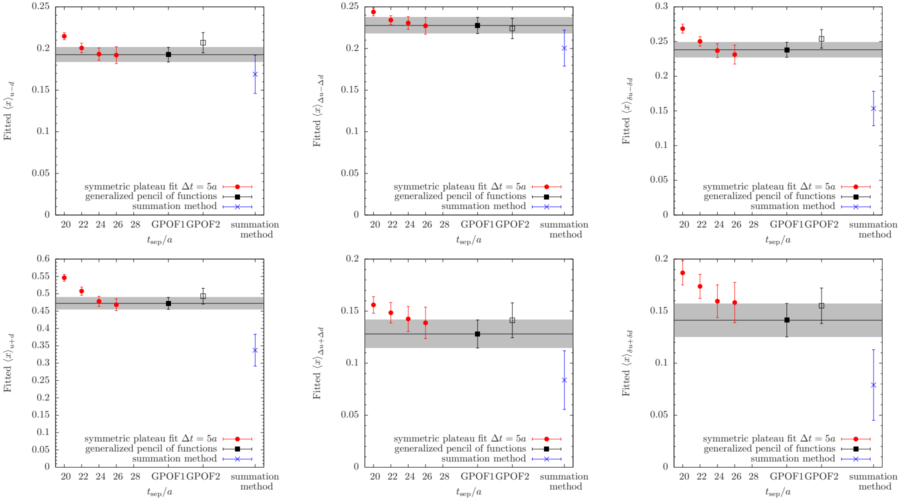
- Coarsest lattice spacing has largest stat. errors
- Methods yield compatible results.

Comparison of methods for N200 ($M_\pi = 280\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$)



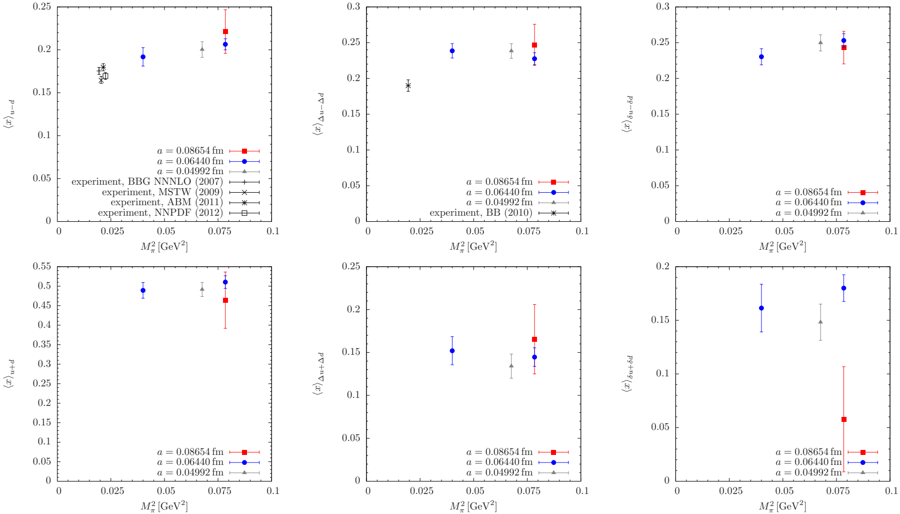
- Ensemble with highest statistics / smallest errors
- Results from summation method and GPOF compatible apart from $\langle x \rangle_{u\pm d}$
- No trend visible for $t_{f,min} = 16a$ vs $t_{f,min} = 18a$ for GPOF.

Comparison of methods for J303 ($M_\pi = 260\,\mathrm{MeV}$, $a = 0.050\,\mathrm{fm}$)



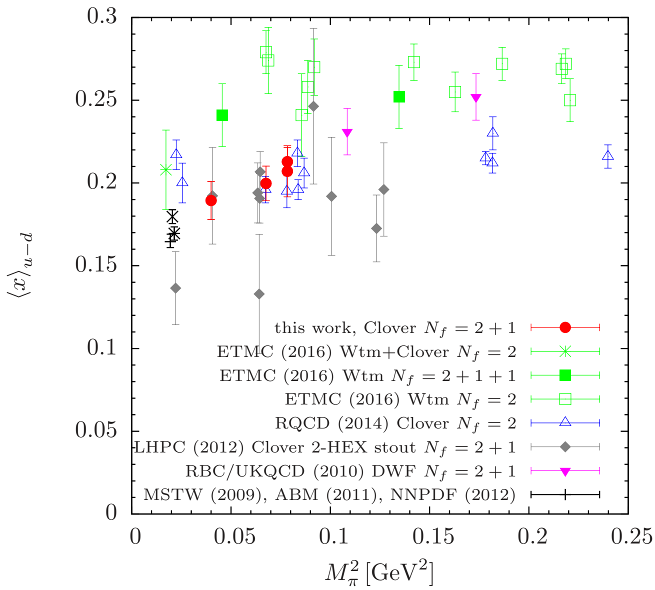
- Smallest lattice spacing
- Surprisingly small errors given that J303 has $\sim 12\%$ of the statistics of H105!
- Similar to D200 and N200, summation methods favors (much) smaller values in some cases.

Overview of results and chiral behavior



- Chiral behavior mild; $\langle x \rangle_{u-d}$ seems to reproduce experimental value.
- Lattice artifacts are small compared to stat. errors.
- Isoscalar observables are less well determined

Comparison with other studies – $\langle x \rangle_{u-d}$



Summary and Outlook

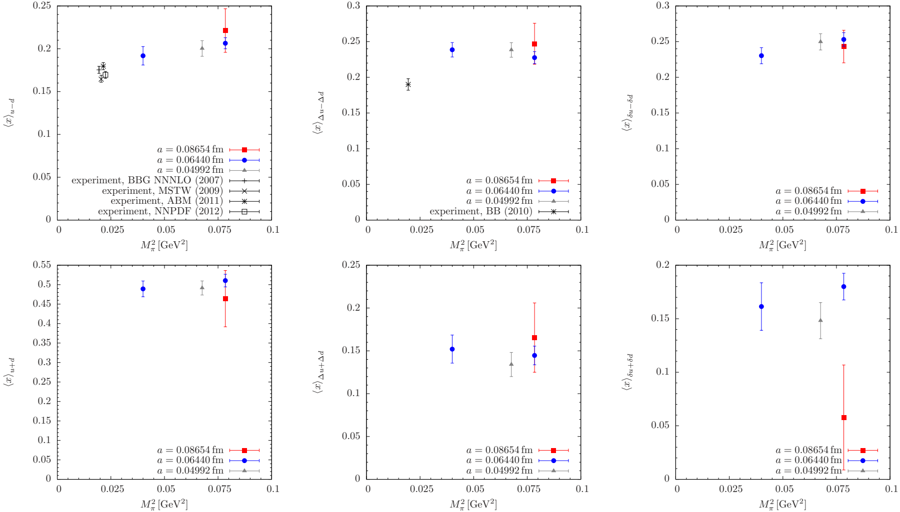
- Computed $\langle x \rangle_{u-d}$ at three lattice spacings and for $M_\pi = 200 \ldots 300\,\mathrm{MeV}$.
- Found only mild chiral extrapolation and lattice artifacts for $\langle x \rangle_{u-d}$.
- GPOF has smaller errors than summation method.
- It favors values closer to plateau at max. t_{sep}.
  $\rightarrow$ Need more data for definite conclusion
- Large t_{sep} (and small M_π) are important

Ongoing work / future plans:

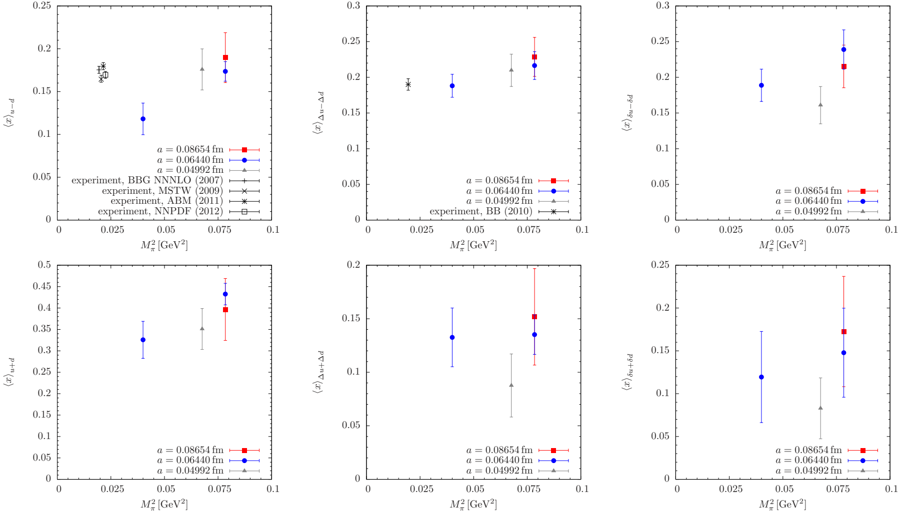
- More ensembles for chiral and continuum extrapolation
- Two-state fits
- Additional source-sink separation to test $\Delta t = 4a$ or larger basis for GPOF
- Renormalization at finest lattice spacing
- Inclusion of quark disconnected diagrams
- Q^2-dependence of GPDFs

Backup slides

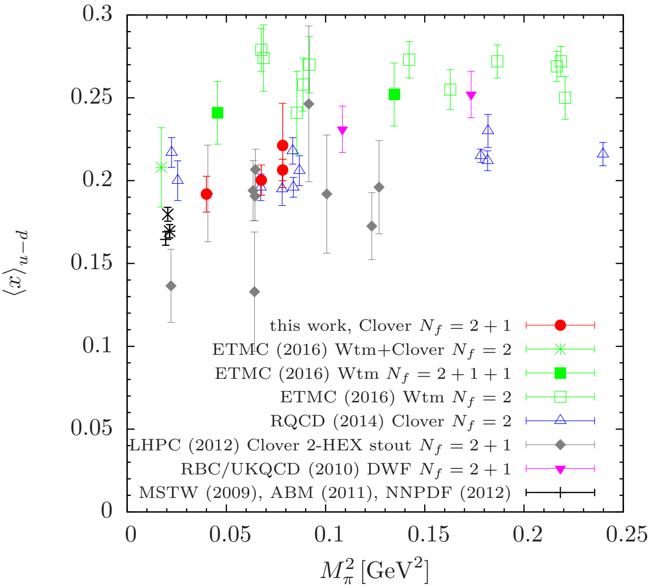
Chiral and scaling behavior – GPOF



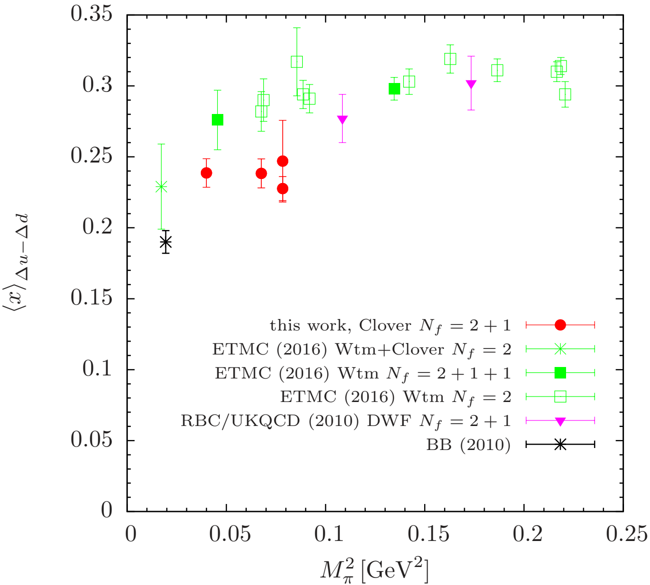
Chiral and scaling behavior – summation method



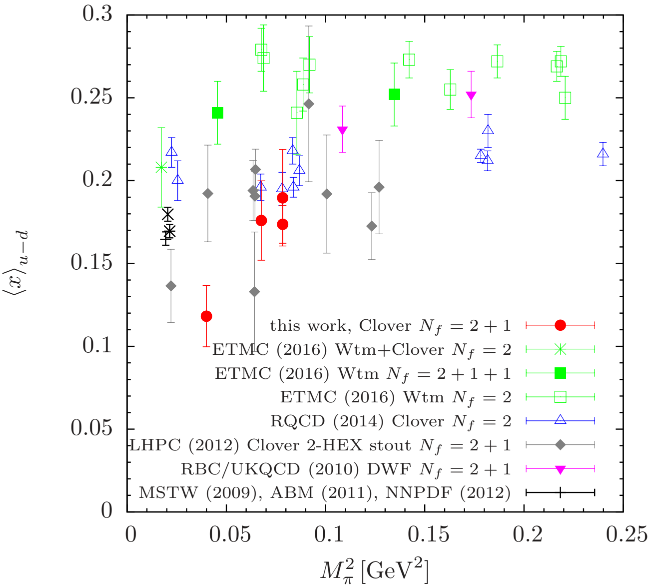
Comparison with other studies – $\langle x \rangle_{u-d}$ (GPOF)



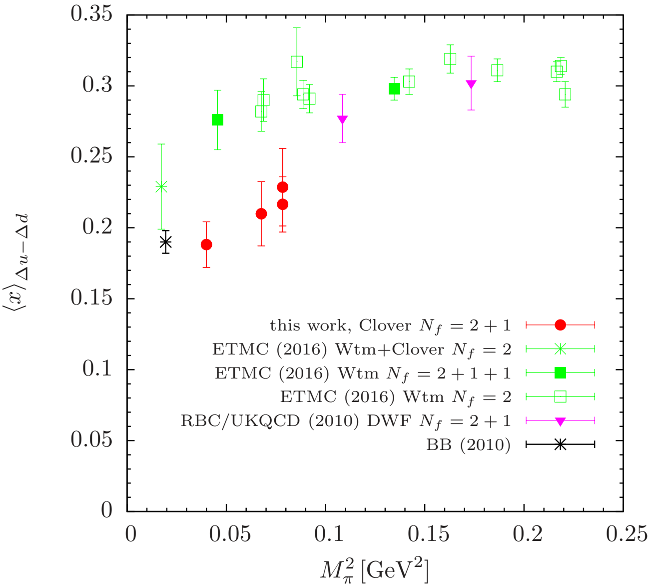
Comparison with other studies – $\langle x \rangle_{\Delta u - \Delta d}$ (GPOF)



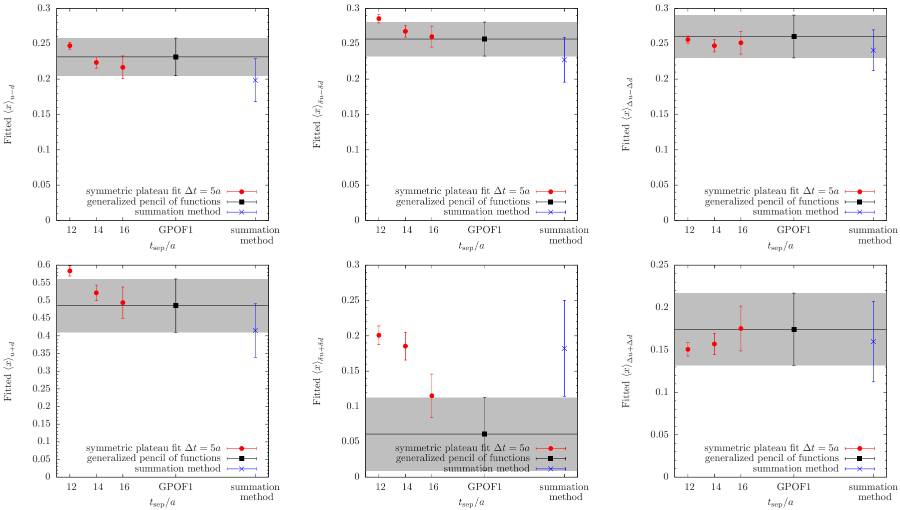
Comparison with other studies – $\langle x \rangle_{u-d}$ (summation method)



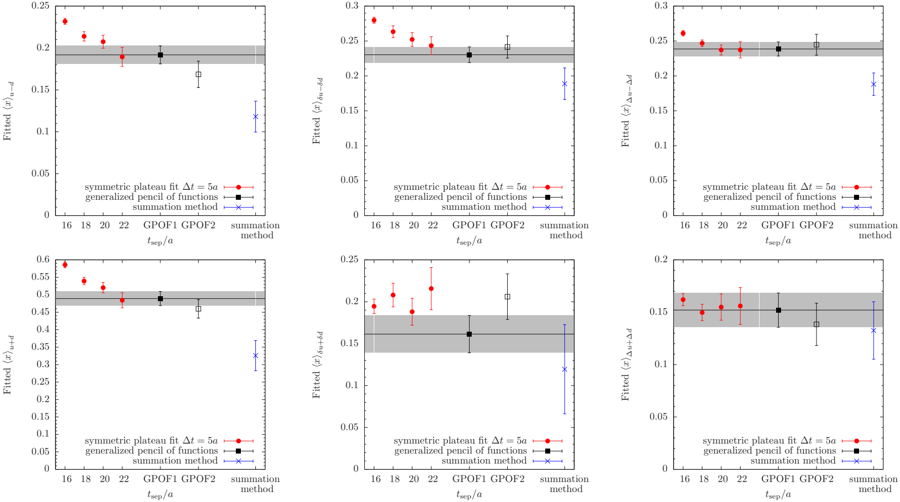
Comparison with other studies – $\langle x \rangle_{\Delta u - \Delta d}$ (summation method)



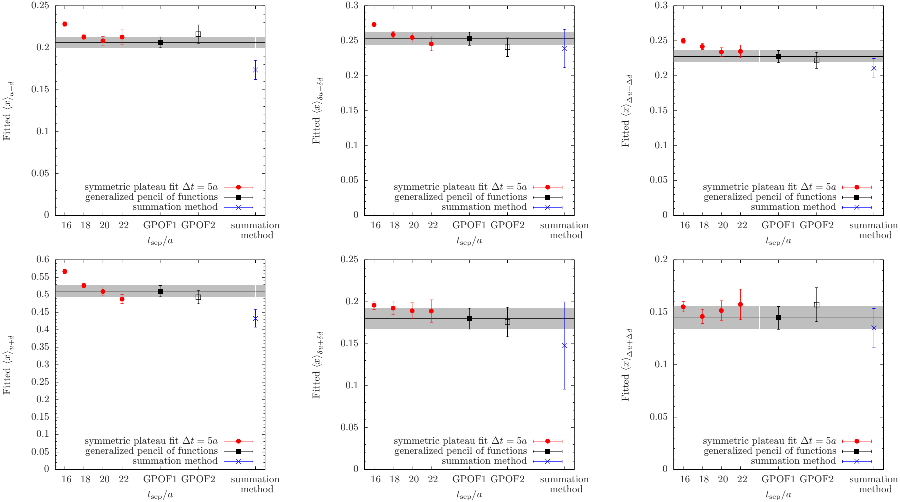
Fit results vs fitrange for H105



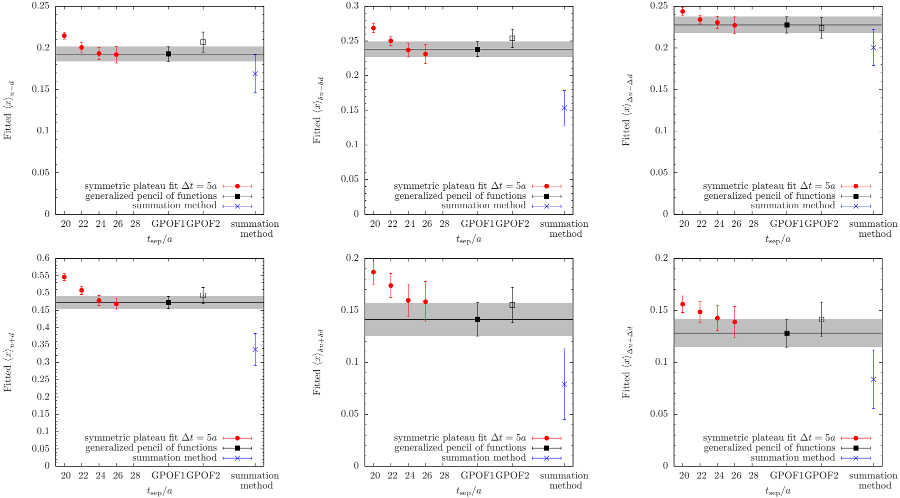
Fit results vs fitrange for D200



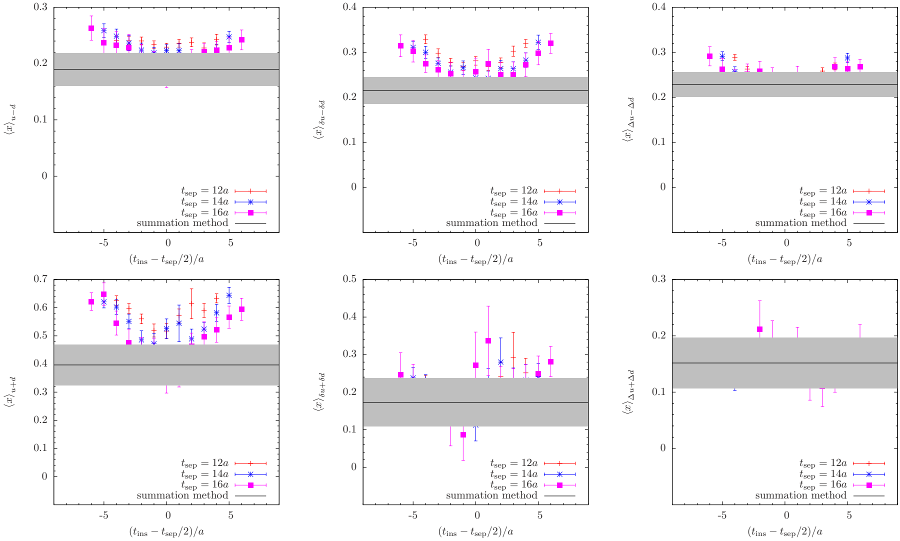
Fit results vs fitrange for N200



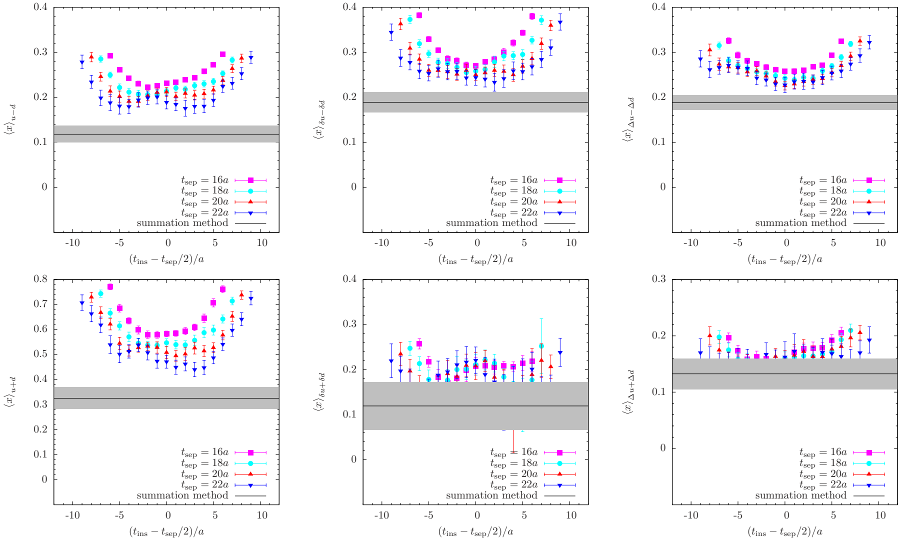
Fit results vs fitrange for J303



Plateaux and summation method for H105



Plateaux and summation method for D200



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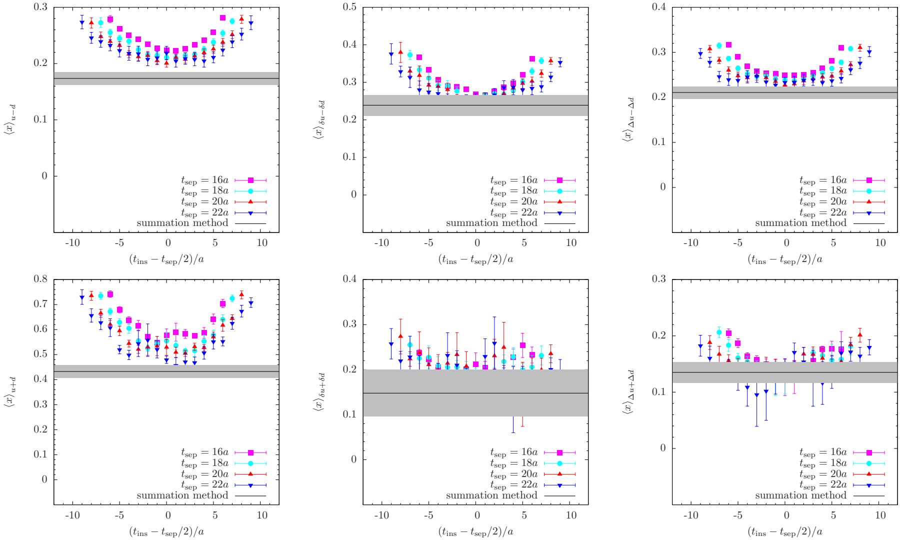
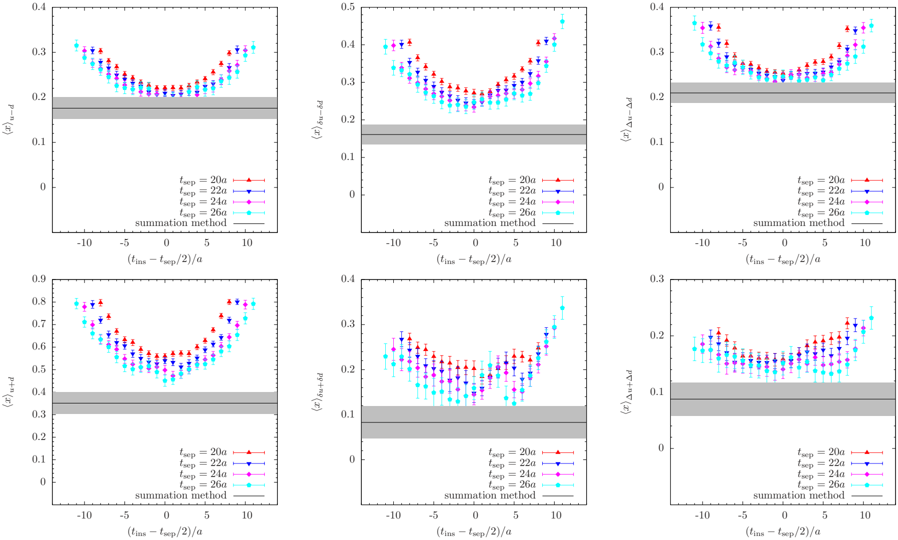
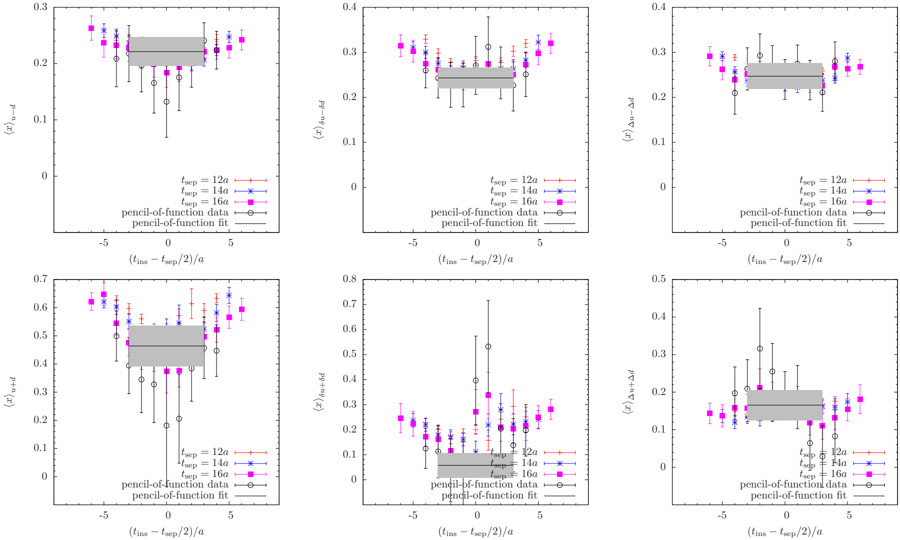


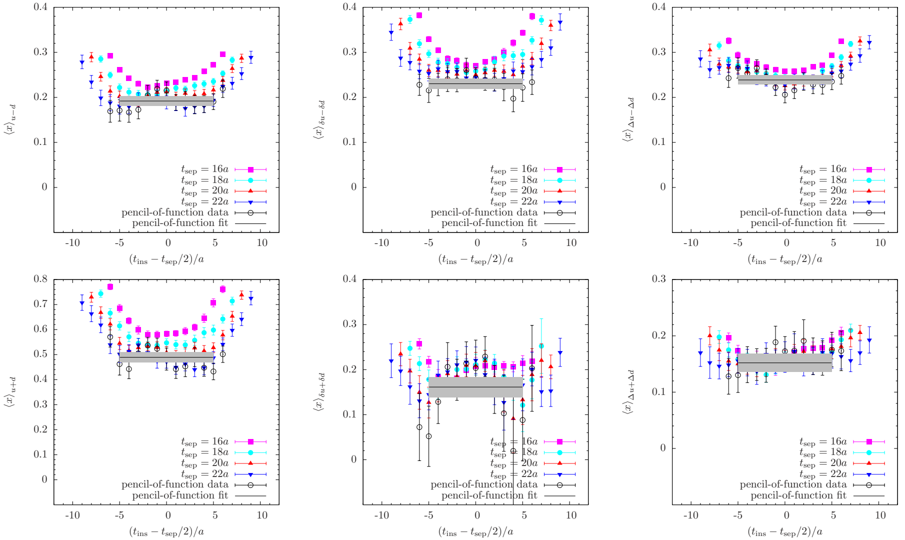
Plateaux and summation method for J303



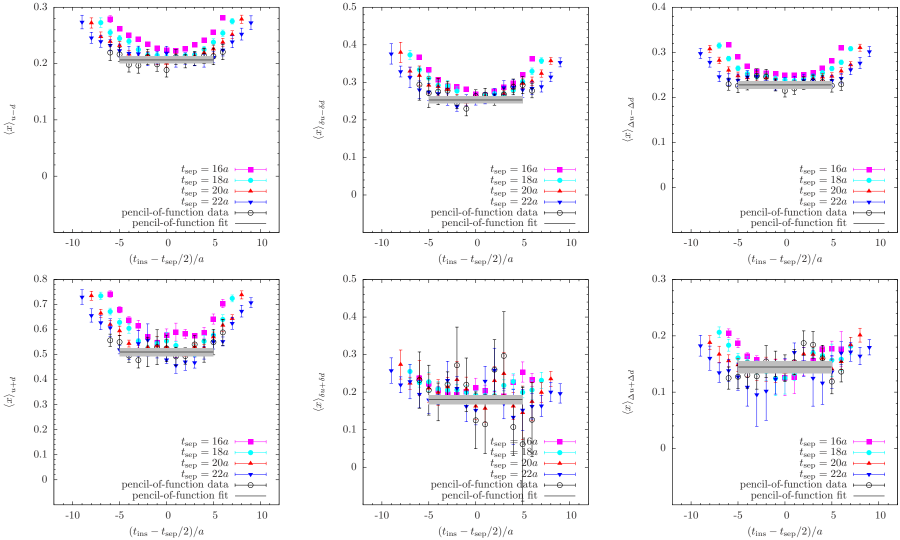
Plateaux and GPOF for H105



Plateaux and GPOF for D200



Plateaux and GPOF for N200



Plateaux and GPOF for J303