Four-Fermi-Theories in 3 dimensions: Critical flavour number of the massless Thirring model

Björn H. Wellegehausen

with Daniel Schmidt and Andreas Wipf

Institut für Theoretische Physik, JLU Giessen and
Theoretisch-Physikalisches Institut, FSU Jena (GRK 1523)

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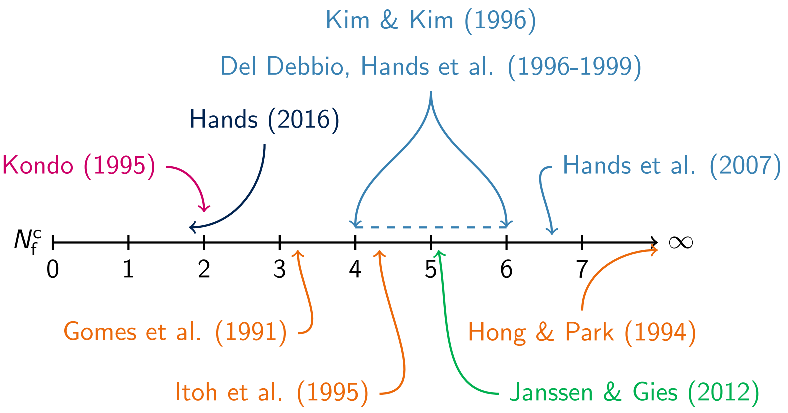
Results for N_f^c from Schwinger-Dyson equations, $\frac{1}{N_f}$-expansion, functional renormalization group, Lattice simulation with staggered and domain wall fermions:



Thirring model in the irreducible representation

$$S_{\text{eff}} = N_f^{\text{ir}} \left(\lambda \int d^3x \, V_\mu(x)^2 - \ln\det\left(\mathrm{i}\partial\!\!\!/ + \gamma_\mu V_\mu\right) \right) \quad \text{with} \quad \lambda = \frac{1}{2g^2}$$

- Vector field V_μ is invariant under chiral transformations
  $\Rightarrow$ no order parameter for chiral symmetry breaking left on the lattice
- Free of fermion sign problem for $N_f^{\text{ir}} > 1$

Fierz transformation

$$\left(\bar{\psi}_\alpha \gamma_\mu \psi_\alpha\right)^2 = -2\left(\bar{\psi}_\alpha \psi_\beta\right)\left(\bar{\psi}_\beta \psi_\alpha\right) - \left(\bar{\psi}\psi\right)^2$$

$$S_{\text{eff}} = N_f^{\text{ir}} \left(\lambda \int d^3x \left(\frac{1}{2} \operatorname{tr} T^2 + \phi^2 \right) - \ln\det\left(\mathrm{i}\partial\!\!\!/ + \mathrm{i}\, T + \mathrm{i}\, \phi\right) \right)$$

- $T = T^\dagger$ and real scalar field ϕ with $T \to U\, T\, U^\dagger$
  T is an order parameter for chiral symmetry breaking
- Severe fermion sign problem

Chiral effective potential

Solution: Combine both formulations

1. Parametrize the scalar fields

$$T(x) = \underbrace{T^{c}(x)}_{\text{Cartan subalgebra}} + \underbrace{T^{r}(x)}_{\text{rest}}$$

2. Calculate the effective potential

$$V_{\text{eff}}(T^{c}) = -\frac{1}{V}\ln\int \mathcal{D}T\,\mathcal{D}\phi\, e^{-S_{\text{eff}}(T,\phi)}\,\delta(T^{c} - T^{c}(x_0))$$
$$= -\frac{1}{V}\ln\sum_{n=0}^{2N_f^{\text{ir}}}\sum_{i=1}^{N_f^{\text{ir}}} a_{n,i}\,(t_i)^n\,, \quad T^{c} = t_i H^i$$

3. Relate the chiral invariant coefficients $a_{n,i}$ to observables $O_{n,i}$ in the vector field formulation of the Thirring model

$$a_{n,i} = \langle O_{n,i}\rangle_{V_\mu}$$

- Dirac operator $\underbrace{\not\partial}_{\text{free fermions}} + \underbrace{T + \phi}_{\text{interaction}} \Rightarrow$ expand interaction
- Partition function as a sum over *spin*-configurations $k_{x\,i}^{\alpha\beta} \in \{0, 1\}$ with

$$\mathbf{n}_\alpha(k_x) = \sum_{i=1}^{2} k_{x\,i}^{\alpha\alpha} \in \{0, 1, 2\}$$

$$Z(\lambda) = C(\lambda) \sum_{\{k_{x\,i}^{\alpha\beta}\}} \lambda^{-\frac{k}{2}} \underbrace{\det\left(\not\partial[k]\right)}_{\text{reduced determinant}} \prod_x \underbrace{\delta_{\text{constraint}}(k_x)}_{\text{local constraint function}} \underbrace{w_o(k_x)\, W_{\mathbf{n}(k_x)}}_{\text{gaussian integrals}}$$

- *Lattice filling factor*

$$k = \sum_{i=1}^{2} \sum_{\alpha,\beta=1}^{N_f^{ir}} \sum_{x=1}^{V} k_{x\,i}^{\alpha\beta} \in \{0\,, \ldots\,, 2\,V\,N_f^{ir}\}$$

counts how many fermions take part in the non-trivial interaction, i.e.

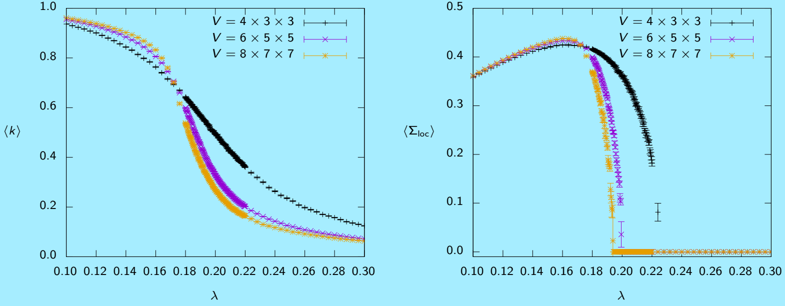
$k = 0 \quad \Leftrightarrow \quad$ free theory $\quad \lambda \to \infty\,(g \to 0)$

$k = 2\,N_f^{ir}\,V \quad \Leftrightarrow \quad$ strong coupling limit $\quad \lambda \to 0\,(g \to \infty)$

Remarks on this form of the partition function

- This formulation of the partition function leads to a *fermion-bag* algorithm
- It does not solve the sign problem introduced by the Fierz transformation
- For $N_f^{ir} = 1$ we can solve the sign problem by resummation of certain weights such that the remaining effective weights are always positive.

Results for k and the chiral condensate for $N_f^{ir} = 1$ with Slac fermions



Effective potential

$$V_{\text{eff}}(T^c) = \frac{1}{2}\,\text{tr}\,T^{c2} + \frac{1}{4}(\text{tr}\,T^c)^2 - \ln\left(\sum_I a_I \sum_{\mathbf{n}\in P(I)} \prod_{\alpha=1}^{N_f^{\text{ir}}} \left({T^c}_{\alpha\alpha} + \frac{1}{2}\,\text{tr}\,T^c\right)^{\mathbf{n}_\alpha}\right)$$

- Sum over $I = (n_0, n_1, n_2)$ with $n_i \in \mathbb{N}$ and $n_0 + n_1 + n_2 = N_f^{\text{ir}}$
- $P(I)$ is the permutation group of all N_f^{ir}-tuples with n_0 elements 0, n_1 elements 1 and n_2 elements 2
- Observables: $\sigma_{\mathbf{d}} = \lambda^{-\frac{|\mathbf{d}|}{2}} \left\langle \prod_{\alpha=1}^{N_f^{\text{ir}}} \left(\bar{\psi}_\alpha \psi_\alpha\right)^{\mathbf{d}_\alpha} \right\rangle_{V_\mu}$

$$\vec{a} = K^{-1}\,\vec{\sigma} \quad \text{with} \quad K_{IJ} = \frac{\text{Vol}(P(J))}{\text{Vol}(P(I))} \sum_{\mathbf{d}\in P(I)} W_{\mathbf{n}(J),\mathbf{d}}$$

with invertable matrix K and moments of the gaussian integrals $W_{\mathbf{n},\mathbf{d}}$

- Minima of the effective potential at

$$T^{\mathrm{c}}_{\min} = \frac{2\,x}{N_{\mathrm{f}}^{\mathrm{ir}}}\,\mathrm{diag}\,(\mathbf{z}) \quad \text{with} \quad \mathbf{z} \in \mathbb{Z}_2^{N_{\mathrm{f}}^{\mathrm{ir}}} = \{(\pm 1, \pm 1, \ldots, \pm 1)\}$$

- Equivalent solutions:

$x \to -x$ and permutations, i.e. the Weyl group of $U(N_{\mathrm{f}}^{\mathrm{ir}})$

- Distinct solutions are characterized by the trace

$$\mathrm{tr}\, T^{\mathrm{c}}_{\min} = n\,\frac{2\,x}{N_{\mathrm{f}}^{\mathrm{ir}}} \quad \text{with} \quad n = \begin{cases} 0, 2, 4, \ldots, N_{\mathrm{f}}^{\mathrm{ir}} & N_{\mathrm{f}}^{\mathrm{ir}} \quad \text{even} \\ 1, 3, 5, \ldots, N_{\mathrm{f}}^{\mathrm{ir}} & N_{\mathrm{f}}^{\mathrm{ir}} \quad \text{odd} \end{cases}$$

- This solutions are associated to a breaking pattern

$$U(N_{\mathrm{f}}^{\mathrm{ir}}) \to U(n_+) \otimes U(n_-) \quad \text{with} \quad n_\pm = \frac{N_{\mathrm{f}}^{\mathrm{ir}} \pm n}{2}.$$

$$V_{\text{eff}}(T^c) \to V_{\text{eff}}(x, n)$$

$n = N_f^{\text{ir}}$: *Gross-Neveu* type potential

- $x_{\text{min}} \neq 0 \quad \Leftrightarrow \quad$ Parity symmetry broken
- Flavour symmetry unbroken
- No spontaneous breaking of a continuous chiral symmetry

$n = 0$: *Thirring* type potential

- Only possible for even N_f^{ir} $\Rightarrow$ Reducible Thirring model
- $x_{\text{min}} \neq 0 \quad \Leftrightarrow \quad$ Flavour symmetry broken

$$U(N_f^{\text{ir}}) \to U(N_f^{\text{ir}}/2) \otimes U(N_f^{\text{ir}}/2)$$

- Parity symmetry unbroken

For $N_f^{ir} = 2$ we obtain

- Observables

$$\sigma_{2,0,0} = 1\,, \quad \sigma_{0,2,0} = \lambda^{-1}\left\langle (\bar{\psi}_1\psi_1)\,(\bar{\psi}_2\psi_2)\right\rangle_{V_\mu}$$

$$\sigma_{1,0,1} = \lambda^{-1}\left\langle (\bar{\psi}_1\psi_1)^2\right\rangle_{V_\mu}\,, \quad \sigma_{0,0,2} = \lambda^{-2}\left\langle (\bar{\psi}_1\psi_1)^2\,(\bar{\psi}_2\psi_2)^2\right\rangle_{V_\mu}$$

- Coefficients

$$\begin{pmatrix} a_{2,0,0} \\ a_{0,2,0} \\ a_{1,0,1} \\ a_{0,0,2} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & \frac{11}{16} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \sigma_{2,0,0} \\ \sigma_{0,2,0} \\ \sigma_{1,0,1} \\ \sigma_{0,0,2} \end{pmatrix}$$

- Gross-Neveu type potential

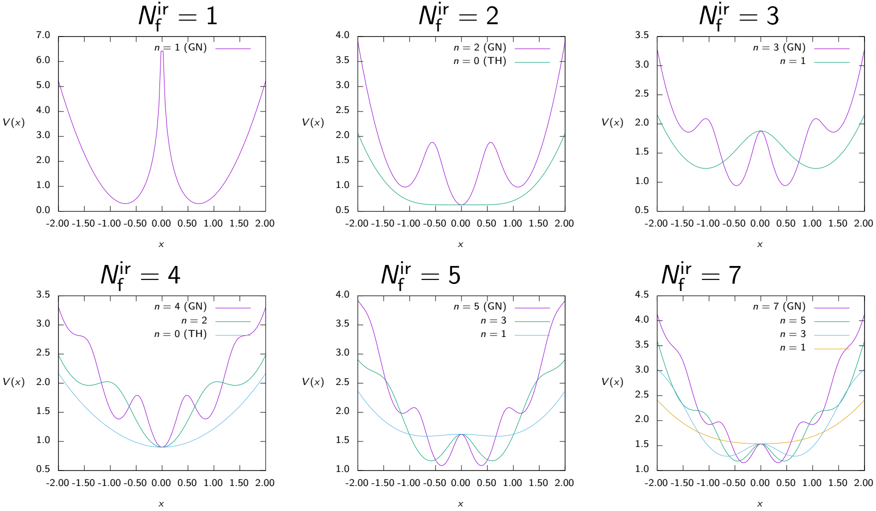
$$V_{\text{eff}}^{\text{GN}}(x) = 2\,x^2 - \ln\left(a_{2,0,0} + 4\,a_{0,2,0}\,x^2 + 8\,a_{1,0,1}\,x^2 + 16\,a_{0,0,2}\,x^4\right)$$

- Thirring type potential

$$V_{\text{eff}}^{\text{TH}}(x) = x^2 - \ln\left(a_{2,0,0} - a_{0,2,0}\,x^2 + 2\,a_{1,0,1}\,x^2 + a_{0,0,2}\,x^4\right)$$

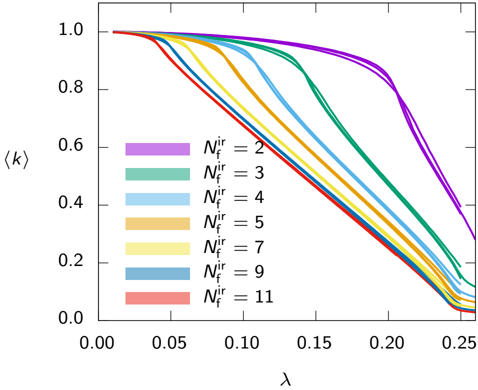
Strong coupling expansion

- Observables $\sigma_{\mathbf{d}}$ do not depend on λ (infinite volume)
- $\langle k \rangle = 1 \quad \Rightarrow \quad$ every lattice site is occupied by a fermion
- Strong coupling phase / lattice artefact phase



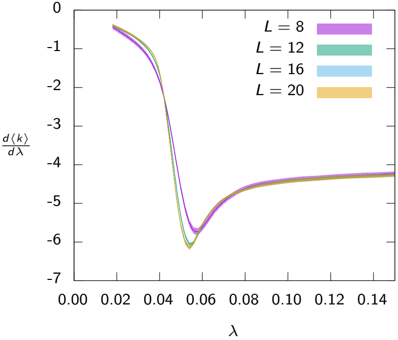
Monte-Carlo simulation results

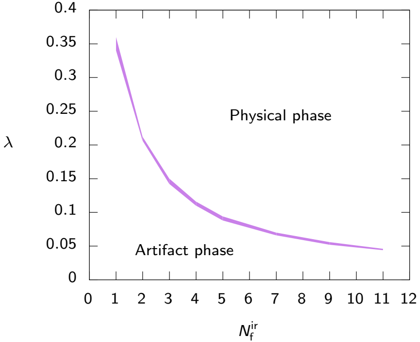
- Slac derivative, exact chiral symmetry at finite lattice spacing
- $200 \times N_f^{ir}$ stochastic estimators for the observables
- Lattice sizes from 8×7^2 to 20×19^2
- 1000 to 10000 configurations



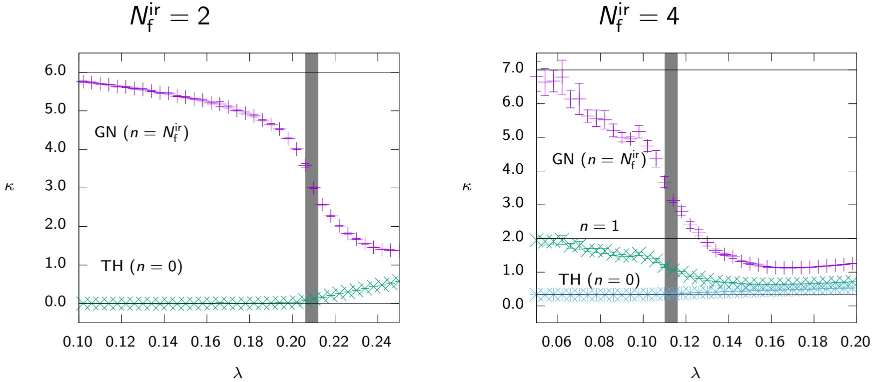
Lattice filling factor $k \in \{0, 1\}$

Derivate of the lattice filling factor for $N_f^{ir} = 9$ and strong coupling transition



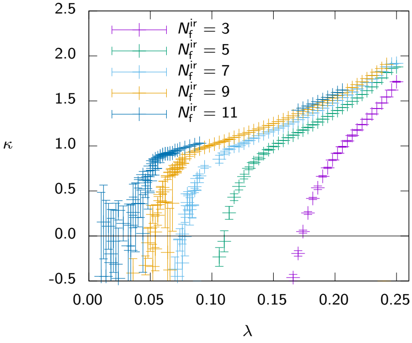


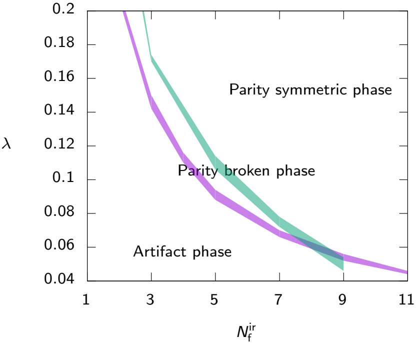
Curvature of the effective potential $\kappa = \left.\frac{d^2 V_{\text{eff}}(x)}{dx^2}\right|_{x=0}$



No spontaneous chiral symmetry breaking for even flavour numbers
(in the reducible Thirring model)

Curvature and phase diagram for odd flavour numbers





Parity breaking for odd flavour numbers $N_f^{ir} \leq 9$

Conclusions

- We found no spontaneous symmetry breaking in the reducible Thirring model
- For odd flavour numbers $N_f^{ir} \leq N_f^{ir^c} = 9$ the irreducible Thirring model shows spontaneous breaking of parity symmetry
- For $N_f^{ir} = 1$ (irreducible Gross-Neveu model) we can solve the sign problem with dual variables and a *fermion bag* algorithm for the Slac operator